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# TEPPER'S IDENTITY

$$n! = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a-k)^n,\ a \in \mathbb{R}$$

*Proof by Ravi Prakash-New Delhi-India*

***We have***

$$(a-k)^n = \sum_{r=0}^{n} (-1)^r \binom{n}{r} a^{n-r} k^r \Rightarrow$$

$$\Rightarrow \sum_{k=0}^{n} (-1)^k \binom{n}{k} (a-k)^n =$$

$$= \sum_{k=0}^{n} (-1)^k \binom{n}{k} \left[\sum_{r=0}^{n} (-1)^r \binom{n}{r} a^{n-r} k^r\right]$$

$$= \sum_{r=0}^{n} \binom{n}{r} a^{n-r} (-1)^r \left\{\sum_{k=0}^{n} (-1)^k \binom{n}{k} k^r\right\}$$

***We have***

$$(1+x)^n = \sum_{k=0}^{n} \binom{n}{k} x^k \Rightarrow n(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} x^{k-1} k$$

$$\Rightarrow nx(1+x)^{n-1} = \sum_{k=1}^{n} \binom{n}{k} x^k k$$

$$n(1+x)^{n-1} + n(n-1)x(1+x)^{n-2} = \sum_{k=0}^{n} k^2 \binom{n}{k} x^{k-1}$$

$$\Rightarrow nx(1+x)^{n-1} + n(n-1)x^2(1+x)^{n-2} = \sum_{k=1}^{n} k^2 \binom{n}{k} x^k$$

$$\Rightarrow n(1+x)^{n-1}+3n(n-1)x(1+x)^{n-2}+n(n-1)(n-2)x^2(1+x)^{n-3}$$

$$=$$

$$=\sum_{k=1}^{n} k^3\binom{n}{k}x^{k-1}$$

***Repeating above procedure $r$ times, we get***

$$\sum_{k=1}^{n}(-1)^k\,(k^r)\binom{n}{k}=0;1\le r\le n-1$$

$$\Rightarrow \sum_{k=0}^{n}(-1)^k\,k^r\binom{n}{k}=0;1\le r\le n-1$$

***Also,*** $\sum_{k=1}^{n}(-1)^k\,k^n\binom{n}{k}=(-1)^n(n!)$

***And*** $\sum_{k=0}^{n}(-1)^k\binom{n}{k}=0$

***Thus,***

$$\sum_{k=0}^{n}(-1)^k(1-k)^n\binom{n}{k}=\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}\left[\sum_{k=0}^{n}(-1)^k k^r\binom{n}{k}\right]$$

$$+(-1)^n\binom{n}{n}\sum_{k=0}^{n}(-1)^k k^n\binom{n}{k}$$

$$=\sum_{r=0}^{n-1}(-1)^r\binom{n}{r}(0)+(-1)^n(1)(-1)^n n!=n!$$