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Find:

$$\Omega = \int_0^\infty \frac{x\, ln^2 x}{e^x - 1} dx$$

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$$\Rightarrow \int_0^\infty \frac{xe^{-x}\ln^2(x)}{1 - e^{-x}} dx \Rightarrow \sum_{n=0}^{\infty} \int_0^\infty x \ln^2(x)\, e^{-(n+1)x}\, dx$$

$$\Rightarrow \sum_{n=0}^{\infty} \frac{\partial^2}{\partial p^2}\left[\int_0^\infty x^p e^{-(n+1)x}\, dx\right]_{p=1} \Rightarrow \sum_{n=0}^{\infty} \frac{\partial^2}{\partial p^2}\left[\frac{\Gamma(\mathrm{p}+1)}{(n+1)^{p+1}}\right]_{p=1} \Rightarrow \sum_{n=1}^{\infty} \frac{\partial^2}{\partial p^2}\left[\frac{\Gamma(p+1)}{n^{p+1}}\right]_{p=1}$$

$$\Rightarrow \frac{\partial^2}{\partial p^2}[\Gamma(p+1)\tau(p+1)]_{p=1} \Rightarrow \frac{\partial}{\partial p}[\Gamma(p+1)\tau'(p+1) + \tau(p+1)\Gamma'(p+1)]_{p=1}$$

$$\Rightarrow [\Gamma(p+1)\tau''(p+1) + \tau'(p+1)\Gamma'(p+1) + \Gamma'(p+1)\tau'(p+1) + \tau(p+1)\Gamma''(p+1)]_{p=1}$$

$$\Rightarrow \Gamma(2)\tau''(2) + \tau'(2)\Gamma'(2) + \Gamma'(2)\tau'(2) + \tau(2)\Gamma''(2)$$

$$\Rightarrow \tau''(2) + \frac{\pi^2}{3}(1-\gamma)[\gamma + \ln(2\pi) - 12\ln(A)] + \frac{\pi^2}{6}\left[(1-\gamma)^2 + \frac{\pi^2}{6} - 1\right]$$

$$\Rightarrow \Gamma''(2) + \left(\frac{\pi^2}{3} - \frac{\pi^2}{3}\gamma\right)[\gamma + \ln(2\pi) - 12\ln(A)] + \frac{\pi^2}{6}\left[1 + \gamma^2 + \gamma + \frac{\pi^2}{6} - 1\right]$$

$$\Rightarrow \tau''(2) + \frac{\pi^2}{3}\gamma + \frac{\pi^2}{3}\ln(2\pi) - 4\pi^2\ln(A) - \frac{\pi^2}{3}\gamma^2 - \frac{\pi^2}{3}\gamma\ln(2\pi) + 4\pi^2\gamma\ln(A) +$$

$$+ \frac{\pi^2}{6}\gamma^2 - \frac{\pi^2}{3}\gamma + \frac{\pi^4}{36}$$

(OR)

$$I = \tau''(2) + \frac{\pi^2}{3}\ln(2\pi)[1-\gamma] + \frac{\pi^4}{36} - \frac{\pi^2}{6}\gamma^2 - 4\pi^2\ln(A)[1-\gamma]$$