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If a,b,c are positive real numbers such that $abc=1$ then:

$$\frac{1}{a^2+2bc}+\frac{1}{b^2+2ca}+\frac{1}{c^2+2ab}\le\frac{3(ab+bc+ca)}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}$$

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$$\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2\le 3\sum a\Rightarrow\frac{3\sum ab}{\left(\sqrt{a}+\sqrt{b}+\sqrt{c}\right)^2}\overset{(1)}{\ge}\frac{\sum ab}{\sum a}$$

$$LHS=\frac{\sum(b^2+2ca)(c^2+2ab)}{(a^2+2bc)(b^2+2ca)(a^2+2ab)}\overset{(2)}{=}\frac{\sum a^2b^2+2\sum a^3b+2\sum ab^3+4abc(\sum a)}{9a^2b^2c^2+2\sum a^3b^3+4abc(\sum a^3)}$$

(1), (2) ⇒ it suffices to prove:

$$\left(\sum ab\right)\left(9a^2b^2c^2+2\sum a^3b^3+4abc\left(\sum a^3\right)\right)\ge$$

$$\ge abc\left(\sum a\right)\left[\sum a^2b^2+2\sum a^3b+2\sum ab^3+4abc\left(\sum a\right)\right]$$

$$[\because abc=1]$$

$$\Leftrightarrow 2abc(\sum a^4b+\sum ab^4)+2\sum a^4b^4\ge abc(\sum a^3b^2+\sum a^2b^3)+4a^2b^2c^2(\sum a^2)\quad \textit{(a)}$$

Now, $\sum a^4b+\sum ab^4=\sum ab(a^3+b^3)\ge\sum a^2b^2(a+b)=\sum a^3b^2+\sum a^2b^3$

$$\Rightarrow abc\left(\sum a^4b+\sum ab^4\right)\overset{(3)}{\geq} abc\left(\sum a^3b^2+\sum a^2b^3\right)$$

(3) $\Rightarrow abc(\sum a^4b+\sum ab^4)\geq abc(\sum a^3b^2+\sum a^2b^3)$

$$=abc\left[\sum a^3(b^2+c^2)\right]\underset{(4)}{\overset{A-G}{\geq}} abc\left(\sum a^3\cdot 2bc\right)=2a^2b^2c^2\left(\sum a^2\right)$$

$$2\sum a^4b^4\overset{(5)}{\geq}2(a^2b^2\cdot b^2c^2+b^2c^2\cdot c^2a^2+c^2a^2\cdot a^2b^2)=2a^2b^2c^2\left(\sum a^2\right)$$

(3)+(4)+(5) ⇒ (a) is true (proved)