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Solve for real numbers:

$$\begin{cases} a, b, c > 0 \\ abc = 1 \\ a^4b + b^4c + c^4a = ab + bc + ca \end{cases}$$

Proposed by Carmen Chirfot-Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solution 2 by Nguyen Thanh Nho-Vietnam

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a^4b + b^4c + c^4a = \frac{a^4}{ac} + \frac{b^4}{ab} + \frac{c^4}{bc} \overset{Berström}{\geq} \frac{(a^2 + b^2 + c^2)}{ac + ab + bc} \geq$$

$$\geq \frac{(ac+bc+ab)^2}{ac+bc+ab} \geq ac + bc + ab \quad (1)$$

$$ab + bc + ca = a^4b + b^4c + c^4a \geq \frac{(a^2 + b^2 + c^2)}{ac + ab + bc} \Rightarrow$$

$$\Rightarrow ac + ab + bc \geq a^2 + b^2 + c^2, \text{ but } (1)$$

$$\Rightarrow a^2 + b^2 + c^2 \geq ac + ab + bc \Rightarrow$$

$$\Rightarrow a^2 + b^2 + c^2 = ab + ac + bc$$

$$\text{“=”}\ a = b = c \Rightarrow abc = 1 \Rightarrow a^3 = 1, a = 1, b = 1, c = 1$$

$$(a, b, c) = (1, 1, 1)$$

Solution 2 by Nguyen Thanh Nho-Vietnam

$$a, b, c > 0; abc = 1$$

$$\left(\frac{a^3}{c} + ca\right) + \left(\frac{b^3}{a} + ab\right) + \left(\frac{c^3}{b} + bc\right) \geq 2a^2 + 2b^2 + 2c^2 \geq 2(ab + bc + ca)$$

$$\Rightarrow \frac{a^3}{c} + \frac{b^3}{a} + \frac{c^3}{b} \geq ab + bc + ca$$

$$\Rightarrow abc\left(\frac{a^3}{c} + \frac{b^3}{a} + \frac{c^3}{b}\right) \geq 1(ab + bc + ca)$$

$$\Rightarrow a^4b + b^4c + c^4a \ge ab + bc + ca$$ ***Done!***

$$“=” \Leftrightarrow \begin{cases} a = b = c \\ abc = 1 \end{cases} \Leftrightarrow a = b = c = 1$$