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Find:

$$\Omega = \int_{n+1}^{n+2} \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \left(\frac{1}{x-j} \right) \right) dx$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

$$\sum_{i=1}^{n} \prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{x-j} = \frac{(x-1)+(x-2)+\cdots+(x-n)}{(x-1)(x-2)\dots(x-n)}$$

$$= \frac{nx - \frac{1}{2}n(n+1)}{(x-1)(x-2)\dots(x-n)} = \sum_{k=1}^{n} \left(n(k) - \frac{1}{2}n(n+1) \right) \frac{a_k}{x-k}$$

where

$$a_k = \frac{1}{\underbrace{(k-1)(k-2)\dots(1)}_{(k-1)\ factors}\underbrace{(-1)(-2)\dots(k-n)}_{(n-k)\ factors}}$$

$$= \frac{1}{(k-1)!\,(-1)^{n-k}(n-k)!} = \frac{(-1)^{n-k}}{(n-1)!}\binom{n-1}{k-1}$$

Thus,

$$\int_{n+1}^{n+2} \sum_{i=1}^{n} \left(\prod_{\substack{j=1 \\ j\neq i}}^{n} \frac{1}{x-j} \right)$$

$$= \sum_{k=1}^{n} \frac{nk - \frac{1}{2}n(n+1)}{(n-1)!} (-1)^k \binom{n-1}{k-1} \ln\left(\frac{n+2-k}{n+1-k} \right)$$