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In $\Delta\ ABC$ the following relationship holds:

$$\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}+\frac{3}{m_am_b+m_bm_c+m_cm_a}\ge\frac{16}{9R^2}$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution 1 by Mehmet Sahin-Ankara-Turkey, Solution 2 by Soumava Chakraborty-Kolkata-India

Solution 1 by Mehmet Sahin-Ankara-Turkey

$$\text{Let } T=\frac{1}{m_am_b}+\frac{1}{m_bm_c}+\frac{1}{m_cm_a}$$

Using Bergström inequality

$$T\ge\frac{(1+1+1)^2}{m_am_b+m_bm_c+m_cm_a}=\frac{9}{m_am_b+m_bm_c+m_cm_a}\quad (1)$$

with Cauchy – Scwarz inequality

$$(m_am_b+m_bm_c+m_cm_a)^2\le\left(m_a^2+m_b^2+m_c^2\right)^2$$

$$\Rightarrow m_am_b+m_bm_c+m_cm_a\le m_a^2+m_b^2+m_c^2=\frac{3}{4}(a^2+b^2+c^2)$$

$$\Rightarrow m_am_b+m_bm_c+m_cm_b\le\frac{3}{4}\cdot 9R^2=\frac{27R^2}{4}\quad (2)$$

Also,

$$T+\frac{3}{m_am_b+m_bm_c+m_cm_a}\ge\frac{9\cdot 4}{27R^2}+\frac{3\cdot 4}{27R^2}=\frac{16}{9r^2}$$

proof is completed.

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\text{In any } \Delta\ ABC, \sum\frac{1}{m_am_b}+\frac{3}{\sum m_am_b}\ge\frac{16}{9R^2}$$

$$\sum\frac{1}{m_am_b}+\frac{3}{\sum m_am_b}\overset{\text{Berström}}{\ge}\frac{9}{\sum m_am_b}+\frac{3}{\sum m_am_b}=\frac{12}{\sum m_am_b}$$

$$\overset{m_a m_b \le \frac{2c^2+ab}{4}}{\ge} \frac{48}{\sum(2c^2+ab)} = \frac{48}{4(s^2-4Rr-r^2)+(s^2+4Rr+r^2)}$$

$$= \frac{48}{5s^2-12Rr-3r^2} \overset{?}{\ge} \frac{16}{9R^2} \Leftrightarrow 5s^2-12Rr-3r^2 \overset{?}{\le} 27R^2$$

$$\Leftrightarrow 5s^2 \overset{?}{\le} 27R^2+12Rr+3r^2 \to \textbf{\textit{(1)}}$$

Now, LHS of (1) $\overset{Gerretsen}{\le} 20R^2+20Rr+15r^2 \overset{?}{\le} 27R^2+12Rr+3r^2$

$\Leftrightarrow 7R^2-8Rr-12r^2 \Leftrightarrow (7R+6r)(R-2r) \ge 0 \to$ ***true (Euler)***

(proved)