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If $x,y,z>0$ then:

$$x^2y^2z^2\left(\frac{x+y}{x^5+y^5}+\frac{y+z}{y^5+z^5}+\frac{z+x}{z^5+x^5}\right)\le x^2+y^2+z^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India, Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam, Solution 3 by Lazaros Zachariadis-Thessaloniki-Greece, Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by Ravi Prakash-New Delhi-India

$$x^3y^2+x^2y^3-x^5-y^5=(x^3-y^3)(y^2-x^2)\le 0$$

$$\Rightarrow \frac{x^3y^2+x^2y^3}{x^5+y^5}\le 1\Rightarrow x^2y^2z^2\left(\frac{x+y}{x^5+y^5}\right)\le z^2$$

$$\Rightarrow x^2y^2z^2\left(\frac{x+y}{x^5+y^5}+\frac{y+z}{y^5+z^5}+\frac{z+x}{z^5+x^5}\right)\le x^2+y^2+z^2$$

Solution 2 by Hoang Le Nhat Tung-Hanoi-Vietnam

We have $x^5+y^5-x^2y^2(x+y)$

$$=x^3(x^2-y^2)-y^3(x^2-y^2)=(x^3-y^3)(x^2-y^2)\ge 0$$

$$\Rightarrow x^5+y^5\ge x^2y^2(x+y)$$

$$\Rightarrow x^2y^2z^2\left(\sum\frac{x+y}{x^5+y^5}\right)\le x^2y^2z^2\sum\frac{x+y}{x^2y^2(x+y)}=\sum x^2$$

Solution 3 by Lazaros Zachariadis-Thessaloniki-Greece

$$\forall x,y>0\ \ x^5+y^5\ge (xy)^2(x+y)$$

$$\Rightarrow \frac{1}{x^5+y^5}\le\frac{1}{(xy)^2(x+y)}$$

$$\Rightarrow \frac{x+y}{(x^5+y^5)}\le\frac{x+y}{(xy)^2(x+y)}=\frac{1}{(xy)^2}$$

$$LHS=(xyz)^2\left(\frac{x+y}{x^5+y^5}+\frac{y+z}{y^5+z^5}+\frac{z+x}{z^5+x^5}\right)$$

$$\leq (xyz)^2\left[\frac{1}{(xy)^2}+\frac{1}{(yz)^2}+\frac{1}{(zx)^2}\right]$$

$$=z^2+x^2+y^2=RHS$$

Solution 4 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$x^2y^2z^2\sum\frac{x+y}{x^5+y^5}\overset{AM\geq GM}{\leq} z^2\cdot\frac{x^4+y^4}{2}\sum\frac{x+y}{x^5+y^5}=$$

$$=z^2\sum\frac{1}{2}\frac{(x^4+y^4)(x+y)}{x^5+y^5}\leq$$

$$\overset{Chebyshev}{\leq}\sum\frac{z^2\cdot\frac{1}{2}\left(x^5+y^5\right)\cdot 2}{x^5+y^5}=\sum x^2$$

$$(x=y=z)$$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

$$\textbf{Iff}\ \frac{x+y}{x^5+y^5}+\frac{y+z}{y^5+z^5}+\frac{z+x}{z^5+x^5}\leq\frac{1}{x^2y^2}+\frac{1}{y^2z^2}+\frac{1}{z^2x^2}$$

$$\textbf{If}\ \frac{x+y}{(x+y)(x^4+y^4+x^2y^2-x^2y-xy^2)}+\frac{y+z}{(y+z)(y^4+z^4+y^2z^2-y^2z-yz^2)}+\frac{z+y}{(z+y)(x^4+y^4+z^2x^2-z^2x-zx^2)}$$

$$=\frac{1}{(x^4+y^4+x^2y^2-x^2y+xy^3)}+\frac{1}{(y^4+z^4+y^2z^2-y^2z+yz^3)}+\frac{1}{(z^4+x^4+z^2x^2-z^2x-zx^2)}\leq$$

$$\leq\frac{1}{x^2y^2}+\frac{1}{y^2z^2}+\frac{1}{z^2x^2}$$

and if it to be true

$$\textbf{Because}\ x^4+y^4+x^2y^2-x^3y-xy^3\geq x^2y^2$$

$$y^4+z^4+y^2z^2-y^3z-yz^2\geq y^2z^2$$

$$\textbf{and}\ z^5+x^4+z^2x^2-z^3x-zx^3\geq z^2x^2$$

Therefore it is to be true.