R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

If a, b, c are positive real numbers such that $a+b+c=3$, then:

$$\frac{1+b+c}{(a+b+ab)^2}+\frac{1+c+a}{(b+c+bc)^2}+\frac{1+a+b}{(c+a+ca)^2}\ge 1$$

Proposed by Le Minh Cuong-Ho Chi Minh-Vietnam

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece, Solution 2 by Sarah El-Kenitra-Morocco

Solution 1 by Lazaros Zachariadis-Thessaloniki-Greece

Let $b\le a\le c$ **then** $\frac{1}{(a+b+ab)^2}\ge\frac{1}{(b+c+bc)^2}\ge\frac{1}{(c+a+ca)^2}$ **and** $c\ge a\ge b$

$$\overset{Chebyshev}{\rightarrow}\frac{c}{(a+b+ab)^2}+\frac{a}{(b+c+bc)^2}+\frac{b}{(c+a+ca)^2}\ge$$

$$\ge\frac{a+b+c}{3}\left(\frac{1}{(\dots)^2}+\frac{1}{(\dots)^2}+\frac{1}{(\dots)^2}\right)\overset{a+b+c=3}{=}$$

$$\frac{1}{(\dots)^2}+\frac{1}{(\dots)^2}+\frac{1}{(\dots)^2}\ge\frac{1}{3}\quad (3)$$

$$=\frac{1}{(a+b+ab)^2}+\frac{1}{(b+c+bc)^2}+\frac{1}{(c+a+ca)^2}\underset{f\ convex}{\overset{f(x)=\frac{1}{x^2},x>0}{=}}$$

$$=f(a+b+ab)+f(b+c+bc)+f(c+a+ca)$$

$$\overset{Jensen}{\ge}3f\left(\frac{2(a+b+c)+ab+bc+ca}{3}\right)=3f\left(\frac{6+ab+bc+ca}{3}\right)$$

$$\ge 3f(3)=3\cdot\frac{1}{e^2}=\frac{1}{3}$$

$$\left[\begin{array}{c} ab+bc+ca\le\frac{(a+b+c)^2}{3}=3 \\ 6ab+bc+ca\le 9\Rightarrow \\ \frac{6+ab+bc+ca}{3}\le 3\Rightarrow f\searrow \\ f\left(\frac{6+ab+bc+ca}{3}\right)\ge f(3) \end{array}\right]$$

so $\frac{1}{(a+ab+b)^2}+\frac{1}{(b+bc+c)^2}+\frac{1}{(c+a+ca)^2}\ge\frac{1}{3}$ **(1)**

$$\frac{b}{(a+b+ab)^2}+\frac{c}{(b+bc+c)^2}+\frac{a}{(c+a+ca)^2}\overset{Rearangement}{\ge}\frac{c}{(a+b+ab)^2}+$$

$$+\frac{a}{(b+c+bc)^2}+\frac{b}{(c+a+ca)^2}\overset{(1)}{\geq}\frac{1}{3}$$

so, $\frac{b}{(a+b+ab)^2}+\frac{c}{(b+bc+c)^2}+\frac{a}{(c+a+ac)^2}\geq\frac{1}{3}$ ***(2)***

$\overset{(1)+(2)+(3)}{\Rightarrow}$ ***the inequality given "*** $a=b=c=1$***"***

Solution 2 by Sarah El-Kenitra-Morocco

By C-S we have $(1+b+c)\left(a^2+b+\frac{a^2b^2}{c}\right)\geq(a+b+ab)^2$***. Then*** $\frac{1+b+c}{(a+b+ab)^2}\geq$

$\geq\frac{1}{a^2+b+\frac{a^2b^2}{c}}=\frac{c^2}{a^2c^2+bc^2+a^2b^2c}$***. One more C-S, LHS*** $\geq\sum\frac{c^2}{a^2c^2+bc^2+a^2b^2c}\geq\frac{(a+b+c)^2}{\sum(a^2c^2+bc^2+a^2b^2c)}=$

$=\frac{9}{a^2b^2+a^2c^2+b^2c^2+ab^2+bc^2+ca^2+abc(ab+ac+bc)}$***. But*** $ab+ac+bc\leq 3$***. Hence, it suffices to prove*** $X=a^2b^2+a^2c^2+b^2c^2+ab^2+bc^2+ca^2+3abc\leq 9$

- - If $a^2b^2+a^2c^2+b^2c^2\leq a^2b+b^2c+c^2a$***, then*** $X\leq(a+b+c)(ab+ac+bc)\leq 9$

- - If $a^2b^2+a^2c^2+b^2c^2\geq a^2b+b^2c+c^2a$ ***we have*** $(a^2b^2+a^2c^2+b^2c^2)^2\leq$

$\leq(a^2b+b^2c+c^2a)(ab^2+bc^2+ca^2)$***.***

Then $a^2b+b^2c+c^2a\leq a^2b^2+a^2c^2+b^2c^2\leq$

$\leq ab^2+bc^2+ca^2$***. Hence*** $X\leq 2(ab^2+bc^2+ca^2)+3abc$***. We can prove easily the famous inequality*** $ab^2+bc^2+ca^2+abc\leq\frac{4}{27}(a+b+c)^3=4$***.***

Therefore $X\leq 8+abc\leq 9$***.***