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PROBLEM 584 - INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2017

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1) In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}\geq\frac{3}{4}(p^2+r^2+4Rr).$$

Proposed by Boris Colakovic - Belgrade - Serbia

Proof.

Using Bergström inequality we obtain:

$$\sum\frac{m_a^4}{h_bh_c}\geq\frac{(\sum m_a^2)^2}{\sum h_bh_c}=\frac{(\frac{3}{4}\sum a^2)^2}{\frac{2rp^2}{R}}=\frac{\frac{9}{16}(\sum a^2)^2}{\frac{2rp^2}{R}}\geq\frac{9R(\sum bc)^2}{32rp^2}=$$

$$=\frac{9R(p^2+r^2+4Rr)^2}{32rp^2}\geq\frac{3}{4}(p^2+r^2+4Rr)$$

where the last inequality is equivalent with:

$$3R(p^2+r^2+4Rr)\geq 8rp^2\Leftrightarrow p^2(3R-8r)+3Rr(4R+r)\geq 0.$$

We distinguish the cases:

Case 1). If $3R-8r\geq 0$*, the inequality is obvious.*

Case 2). If $3R-8r<0$*, the inequality can be rewritten* $3Rr(4R+r)\geq p^2(8r-3R)$ *which is true from Gerretsen's inequality* $p^2\geq 16Rr-5r^2$*. It remains to prove that:*

$$3Rr(4R+r)\geq(16Rr-5r^2)(8r-3R)\Leftrightarrow 3R^2-2R^2r-5Rr^2-6r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(3R^2+4Rr+3r^2)\geq 0$$ *obviously from Euler's inequality* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1)** can be written:*

2) In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}\geq\frac{3}{4}(ab+bc+ca).}$$

Proof.

We use the identity $ab+bc+ca=p^2+r^2+4Rr$.

□

Remark.

Inequality 2) can be strengthened:

3) In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}\geq\frac{3}{4}(a^2+b^2+c^2)$$

Proof.

Using Bergström's inequality, we obtain:

$$\sum\frac{m_a^4}{h_bh_c}\geq\frac{(\sum m_a^2)^2}{\sum h_bh_c}=\frac{(\frac{3}{4}\sum a^2)^2}{\frac{2rp^2}{R}}=\frac{\frac{9}{16}(\sum a^2)^2}{\frac{2rp^2}{R}}\geq\frac{9R(\sum a^2)^2}{32rp^2}\geq\frac{3}{4\sum a^2}$$

where the last inequality is equivalent with:

$$3R\sum a^2\geq 8rp^2\Leftrightarrow 3R\cdot 2(p^2-r^2-4Rr)\geq 8rp^2\Leftrightarrow p^2(3R-4r)\geq 3Rr(4Rr+r)$$

which is true from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$(16Rr-5r^2)(3R-4r)\geq 3Rr(4R+r)\Leftrightarrow 18R^2-41Rr+10r^2\geq 0\Leftrightarrow (R-2r)(18R-5r)\geq 0$$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality 3) is stronger than inequality 2):

4) In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}\geq\frac{3}{4}(a^2+b^2+c^2)\geq\frac{3}{4}(ab+bc+ca).$$

Proof.

See inequality 3) and $a^2+b^2+c^2\geq ab+bc+ca$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

Inequality 3) can be also strengthened:

5) In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}\geq\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemmas:

Lemma 1.

6) In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}=\frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4R+r)^3}{8r^2p^2}$$

Proof.

$$\sum \frac{m_a^4}{h_b h_c} = \sum \frac{(m_a^2)^2}{\frac{2S}{b}\cdot\frac{2S}{c}} = \frac{1}{4S^2}\sum bc\left(\frac{2b^2+2c^2-a^2}{4}\right)^2 = \frac{1}{64S^2}\sum bc(E-3a^2)^2 =$$

$$= \frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4R+r)^3}{8r^2p^2},\ \textit{where } E = 2\sum a^2.$$

□

Lemma 2.
7) In ΔABC

$$\boldsymbol{\frac{m_a^4}{h_b h_c} + \frac{m_b^4}{h_c h_a} + \frac{m_c^4}{h_a h_b} \ge \frac{77R^3-112R^2r+25Rr^2-2r^3}{4R}.}$$

Proof.

*Using **Lemma 1** we obtain:*

$$\frac{m_a^4}{h_b h_c}+\frac{m_b^4}{h_c h_a}+\frac{m_c^4}{h_a h_b} = \frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4Rr+r)^3}{8r^2p^2} =$$

$$= \frac{1}{8r^2}\left[2p^4-p^2(23Rr+2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p^2}\right] =$$

$$= \frac{1}{8r^2}\left[p^2(2p^2-23Rr-2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p}\right] \ge$$

$$\ge \frac{1}{8r^2}\left[(16Rr-5r^2)\Big(2(16Rr-5r^2)-23Rr-2r^2\Big)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{\frac{R(4R+r)^2}{2(2R-r)}}\right] =$$

$$= \frac{77R^3-112R^2r+25Rr^2-2r^3}{4R},$$ *where the last inequality follows from*

Gerretsen's inequality $p^2 \ge 16Rr-5r^2$ *and Blundon's inequality* $p^2 \le \dfrac{R(4R+r)^2}{2(2R-r)}$.

□

*Let's pass to solving inequality **5)**.*

*Using **Lemma 2** and the identities* $a^3+b^3+c^3 = 2p(p^2-3r^2-6Rr)$ *and* $a+b+c = 2p$

It suffices to prove that $\dfrac{77R^3-112R^2r+25Rr^2-2r^3}{4R} \ge \dfrac{9}{4}\cdot\dfrac{2p(p^2-3r^2-6Rr)}{2p} \Leftrightarrow$

$$77R^3-112R^2r+25Rr^2-2r^3 \ge 9R(p^2-3r^2-6Rr)$$

which follows from Gerretsen's inequality $p^2 \le 4R^2+4Rr+3r^2$. *It remains to prove that:*

$$77R^3-112R^2r+25Rr^2-2r^3 \ge 9R(4R^2+4Rr+3r^2-3r^2-6Rr) \Leftrightarrow$$

$$41R^3-94R^2r+25Rr^2-2r^3 \ge 0 \Leftrightarrow (R-2r)(41R^2-12Rr+r^2) \ge 0$$

obviously from Euler's inequality $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **5)** is stronger than inequality **3)**:*

8) In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}\geq\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{3}{4}(a^2+b^2+c^2).$$

Proof.

*See inequality **5)** and*

$$\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq\frac{3}{4}(a^2+b^2+c^2)\Leftrightarrow a^3+b^3+c^3\geq\frac{1}{3}(a+b+c)(a^2+b^2+c^2)$$

true from Chebysev's inequality.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The following inequalities can be written:

9. In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}\geq\frac{77R^3-112R^2r+25Rr^2-2r^3}{4R}\geq\frac{9}{4}\cdot\frac{a^3+b^3+c^3}{a+b+c}\geq$$

$$\geq\frac{3}{4}(a^2+b^2+c^2)\geq\frac{3}{4}(ab+bc+ca)$$

Proof.

*See inequalities **7)**, **8)**, and **4)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an apposite sense.

10) In ΔABC

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_bh_a}\leq\frac{4R^4-37r^4}{r^2}.$$

Proof.

*Using **Lemma 1** we obtain:*

$$\frac{m_a^4}{h_bh_c}+\frac{m_b^4}{h_ch_a}+\frac{m_c^4}{h_ah_b}=\frac{2p^6-p^4(23Rr+2r^2)+p^2(10R^2r^2-19Rr^3-2r^4)+2r^3(4R+r)^3}{8r^2p^2}=$$

$$=\frac{1}{8r^2}\left[2p^4-p^2(23Rr+2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p^2}\right]=$$

$$=\frac{1}{8r^2}\left[p^2(2p^2-23Rr-2r^2)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{p^2}\right]\leq$$

$$\leq\frac{1}{8r^2}\left[(4R^2+4Rr+3r^2)\Big(2(4R^2+4Rr+3r^2)-23Rr-2r^2\Big)+10R^2r^2-19Rr^3-2r^4+\frac{2r^3(4R+r)^3}{\frac{r(4R+r)^2}{R+r}}\right]=$$

$$= \frac{16R^4 - 14R^3r - R^2r^2 - 19Rr^3 + 6r^4}{4r^2} \leq \frac{4R^4 - 37r^4}{r^2}$$ *where the last inequality follows from Euler's inequality* $R \geq 2r$ *and the penultimate from Gerretsen's inequality*

$$p^2 \leq 4R^2 + 4Rr + 3r^2 \text{ and } p^2 \geq \frac{r(4R+r)^2}{R+r}$$

true from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

11) In ΔABC

$$\frac{21R^3 + 48r^3}{4R} \leq \frac{m_a^4}{h_b h_c} + \frac{m_b^4}{h_c h_a} + \frac{m_c^4}{h_a h_b} \leq \frac{4R^4 - 37r^4}{r^2}.$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **10), 7)** and Euler's inequality* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

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