SP.096. Let ABC be a triangle and w_a, w_b, w_c are bisectors of ABC. Prove that:

$$\frac{1}{aw_a^2}+\frac{1}{bw_b^2}+\frac{1}{cw_c^2}\geq\frac{1}{R\Delta}$$

where R is the circumradius of ABC, Δ is area of ABC.

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Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

Solution 1 by Soumava Chakraborty-Kolkata-India

$$w_a^2=\frac{4b^2c^2}{(b+c)^2}\cdot\frac{s(s-a)}{bc}=\frac{4bcs(s-a)}{(b+c)^2}$$

$$\Rightarrow\frac{1}{aw_a^2}=\frac{(b+c)^2}{4abcs(s-a)}\quad(1)$$

Similarly, $\frac{1}{bw_b^2}\overset{(2)}{=}\frac{(c+a)^2}{4abcs(s-b)}$ & $\frac{1}{cw_c^2}\overset{(3)}{=}\frac{(a+b)^2}{4abcs(s-c)}$

$$(1)+(2)+(3)\Rightarrow LHS=\frac{1}{4s\cdot 4R\Delta}\sum\frac{(a+b)^2}{s-c}$$

$$=\frac{1}{16sR\Delta}\sum\frac{(s+s-c)^2}{s-c}=\frac{1}{16sR\Delta}\sum\frac{s^2+2s(s-c)+(s-c)^2}{s-c}$$

$$=\frac{1}{16sR\Delta}\left\{s^2\sum\frac{1}{s-c}+2s\sum(1)+\sum(s-c)\right\}$$

$$=\frac{1}{16sR\Delta}\left[\frac{s^3}{r^2s^2}\sum\{s^2-s(a+b)+ab\}+6s+(3s-2s)\right]$$

$$=\frac{1}{16sR\Delta}\left\{\frac{s}{r^2}(3s^2-4s^2+s^2+4Rr+r^2)+7s\right\}$$

$$=\frac{1}{16sR\Delta}\left\{\frac{s(4R+r)}{r}+7s\right\}=\frac{s(4R+8r)}{16sR\Delta r}$$

$$=\frac{R+2r}{4r\cdot R\Delta}\overset{Euler}{=}\frac{4r}{4r\cdot R\Delta}=\frac{1}{R\Delta}$$ ***(Proved)***

Proof 2: $w_a^2\le s(s-a)\Rightarrow aw_a^2\le as(s-a)\Rightarrow\frac{1}{aw_a^2}\ge\frac{1}{as(s-a)}$ ***(1)***

Similarly, $\frac{1}{bw_b^2}\overset{(2)}{\ge}\frac{1}{bs(s-b)}\ \&\ \frac{1}{cw_c^2}\overset{(3)}{\ge}\frac{1}{cs(s-c)}$

(1)+(2)+(3)⇒LHS $\ge\frac{1}{s}\sum\frac{1}{a(s-a)}$ ***(4)***

WLOG, we may assume $a\ge b\ge c$

Then $\frac{1}{a}\le\frac{1}{b}\le\frac{1}{c}$ ***and*** $\frac{1}{s-a}\ge\frac{1}{s-b}\ge\frac{1}{s-c}$

(4) $\Rightarrow$ ***LHS*** $\overset{Chebyshev}{\ge}\frac{1}{3s}\sum\frac{1}{a}\sum\frac{1}{s-a}$

$$=\frac{1}{3s}\left(\frac{\sum ab}{abc}\right)\frac{s}{r^2s^2}\left\{\sum(s-b)(s-c)\right\}$$

$$=\frac{(s^2+4r+r^2)}{3r^2s^2\cdot 4R\Delta}(3s^2-4s^2+s^2+4Rr+r^2)$$

$$=\frac{(s^2+4Rr+r^2)(4R+r)}{12rs^2R\Delta}\overset{?}{\ge}\frac{1}{R\Delta}$$

$$\Leftrightarrow(s^2+4Rr+r^2)(4R+r)\ge 12rs^2\quad\textbf{(5)}$$

Now, LHS of (5) $\overset{Gerretsen}{\ge}(20Rr-4r^2)(4R+r)$

& RHS of (5) $\overset{Gerretsen}{\le}12r(4R^2+4Rr+3r^2)$

∴ it suffices to prove:

$$(5R-r)(4R+r)\ge 3(4R^2+4Rr+3r^2)$$

$\Leftrightarrow 8R^2-11Rr-10r^2\ge 0\Leftrightarrow(R-2r)(8R+5r)\ge 0\rightarrow$ ***true***

$\because R\ge 2r$ ***(Euler)*** $\Rightarrow$ ***(5) is true (Proved)***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\sum \frac{1}{a \cdot w_a^2} \geq \frac{1}{R \cdot \Delta}$$

$$x = p - a$$
$$y = p - b \Rightarrow x + y + z = p$$
$$z = p - c$$

$$\mathbf{1)} \sum \frac{1}{a \cdot w_a^2} = \frac{1}{(y+z) \cdot \left(\frac{2}{2x+y+z} \cdot \sqrt{x(x+z)(y+x) \cdot \sum x}\right)^2} =$$

$$= \sum \frac{(2x+y+z)^2}{4x \prod(x+y) \cdot \sum x} \overset{Bergstrom}{\geq} \frac{\left(\sum(2xy+y+z)\right)^2}{4 \sum x \prod(x+y)} = \frac{16(x+y+z)^2}{4(x+y+z)^2 \cdot \prod(x+y)}$$

$$= \frac{4}{\prod(x+y)} = LHS$$

$$\mathbf{2)} \frac{1}{R \cdot \Delta} = \frac{1}{\frac{abc}{4\Delta} \cdot \Delta} = \frac{4}{abc} = \frac{4}{\prod(x+y)} = RHS$$

$$\mathbf{1), 2)} \sum \frac{1}{aw_a^2} \geq \frac{4}{\prod(x+y)} = \frac{1}{R \cdot \Delta}$$