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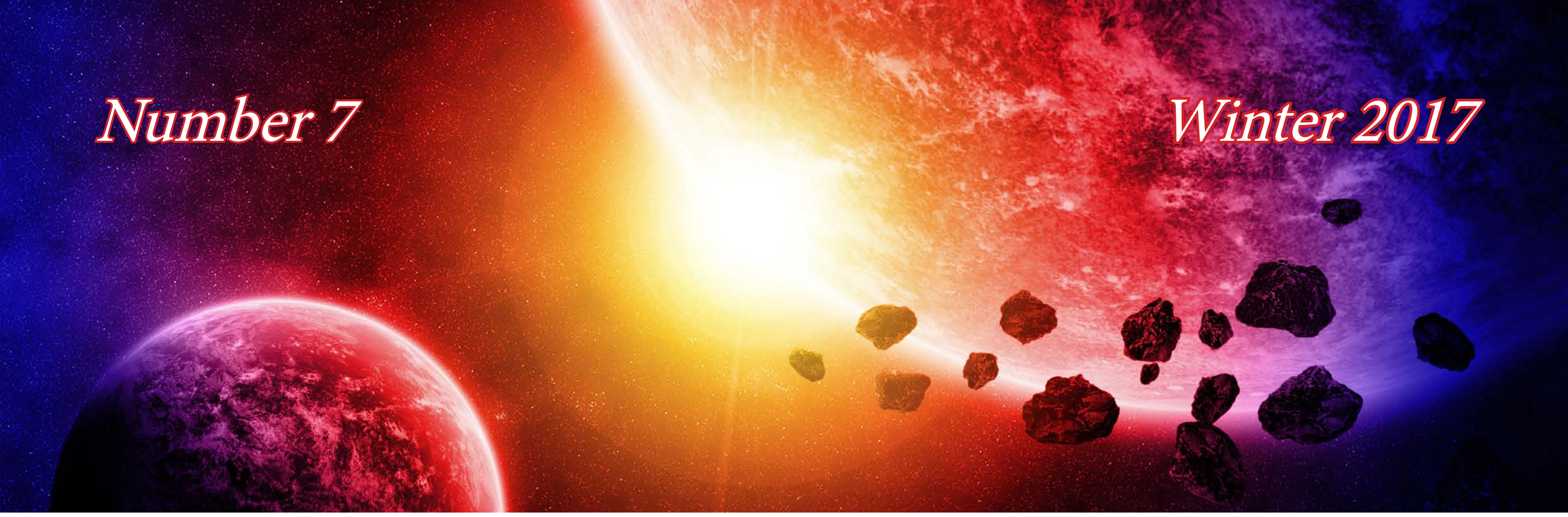

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PROBLEMS FOR JUNIORS

JP.091. Prove that the following inequalities hold for all positive real numbers

$$\text{a. } \frac{a^3}{ab+c^2}+\frac{b^3}{bc+a^2}+\frac{c^3}{ca+b^2}\geq\frac{3}{2}\cdot\frac{a^2+b^2+c^2}{a+b+c}$$

$$\text{b. } \frac{1}{a(b+c)}+\frac{1}{b(c+a)}+\frac{1}{c(a+b)}\geq\frac{3}{2}\cdot\frac{a+b+c}{a^3+b^3+c^3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.092.Prove that the following inequalities holds for all positive real numbers a,b,c

$$\text{a. } \frac{b}{a^2}+\frac{c}{b^2}+\frac{a}{c^2}\geq\frac{3(a+b+c)}{a^2+b^2+c^2}$$

$$\text{b. } \frac{b^3}{a^2}+\frac{c^3}{b^2}+\frac{a^3}{c^2}\geq\frac{3(a^2+b^2+c^2)}{a+b+c}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.093. Let a,b,c be positive real numbers such that $a+b+c=1$. Prove that

$$\text{a. } \frac{1}{a+bc}+\frac{1}{b+ca}+\frac{1}{c+ab}\leq\frac{1}{4abc}$$

$$\text{b. } \frac{\sqrt{a}}{a+\sqrt{bc}}+\frac{\sqrt{b}}{b+\sqrt{ca}}+\frac{\sqrt{c}}{c+\sqrt{ab}}\leq\frac{1}{2\sqrt{abc}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.094. Let a,b,c be positive real numbers such that $ab+bc+ca=1$. Prove that

$$bc\sqrt{a^2+2bc}+ca\sqrt{b^2+2ca}+ab\sqrt{c^2+2ab}\geq 1$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.095. Prove that for all positive real numbers a,b,c

$$\frac{a(b^2+c^2)}{2a^2+bc}+\frac{b(c^2+a^2)}{2b^2+ca}+\frac{c(a^2+b^2)}{2c^2+ab}\geq\frac{6abc}{ab+bc+ca}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

JP.096. Let a, b, c positive numbers such that $a^4 + b^4 + c^4 = 3$. Prove that

$$\left(\frac{a^3}{b^5} + \frac{b^3}{c^5} + \frac{c^3}{a^5}\right)\left(\frac{b^3}{a^5} + \frac{c^3}{b^5} + \frac{a^3}{c^5}\right) \geq 9$$

Proposed by Nguyen Ngoc Tu - Ha Giang – Vietnam

JP.097. Let $a, b, c > 0$ such that $(a+b)(b+c)(c+a) = 8$. Prove that

$$\frac{a}{a+1} + \sqrt{\frac{2b}{b+1}} + 2\sqrt[4]{\frac{2c}{c+1}} \leq \frac{7}{2}$$

Proposed by Nguyen Ngoc Tu - Ha Giang – Vietnam

JP.098. Let a, b, and c be the side lengths of a triangle ABC with incenter I. Prove that

$$\frac{1}{IA^2} + \frac{1}{IB^2} + \frac{1}{IC^2} \geq 3\left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right)$$

Proposed by George Apostolopoulos – Messolonghi – Greece

JP.099. Find the value of the following expression:

$$E = \left(\frac{x^2}{y^2} + \frac{y^2}{z^2}\right)^2 + \left(\frac{y^2}{z^2} + \frac{z^2}{x^2}\right)^2 + \left(\frac{z^2}{x^2} + \frac{x^2}{y^2}\right)^2$$

where $x = \tan 20, y = \tan 40, z = \tan 80$.

Proposed by Kevin Soto Palacios - Huarmey - Peru

JP.100. Let in triangle w_a, w_b, w_c be the angle bisectors and R, r the circumradius and inradius respectively. Prove the inequality:

$$\frac{3}{R+r} \leq \frac{1}{w_a} + \frac{1}{w_b} + \frac{1}{w_c} \leq \frac{1}{r}.$$

Proposed by D.M. Bătinețu – Giurgiu – Romania, Martin Lukarevski – Skopje

JP.101. Let x, y, z be positive real numbers with $xyz = 1$. Prove that:

$$\frac{\sqrt{x^4+1} + \sqrt{y^4+1} + \sqrt{z^4+1}}{x^2+y^2+z^2} \leq \sqrt{2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

JP.102. Let $x, y, z > 0$ be positive real numbers. Then

$$\frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \geq \frac{4\sqrt{3xyz(x+y+z)}}{(x+y)(y+z)(z+x)}$$

Proposed by D.M. Bătinețu – Giurgiu – Romania, Martin Lukarevski – Skopje

JP.103. Let $x, y, z > 0$ be positive real numbers. Then in triangle ABC with semiperimeter s and inradius r.

$$\frac{x}{y+z}\cot^2\frac{A}{2}+\frac{y}{z+x}\cot^2\frac{B}{2}+\frac{z}{x+y}\cot^2\frac{C}{2}\geq 18-\frac{s^2}{2r^2}$$

Proposed by D.M. Bătineţu – Giurgiu – Romania, Martin Lukarevski – Skopje

JP.104. Let r_a, r_b, r_c be the exradii, h_a, h_b, h_c the altitudes and m_a, m_b, m_c the medians of a triangle ABC with semiperimeter s, circumradius R and inradius r. Then

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{54r^2}{s^2-r^2-4Rr}$$

Proposed by D.M. Bătineţu – Giurgiu – Romania, Martin Lukarevski – Skopje

JP.105. Let $m > 0$ and F be the area of the triangle ABC. Then

$$\frac{a^{m+2}}{b^m+c^m}+\frac{b^{m+2}}{c^m+a^m}+\frac{c^{m+2}}{a^m+b^m}\geq 2\sqrt{3}F.$$

Proposed by D.M. Bătineţu – Giurgiu – Romania, Martin Lukarevski – Skopje

PROBLEMS FOR SENIORS

SP.091. Prove that for all positive real numbers a, b, c, d

$$\frac{a^2}{a+b+c}+\frac{b^2}{b+c+d}+\frac{c^2}{c+d+a}+\frac{d^2}{d+a+b}\geq$$
$$\geq\frac{a+b+c+d}{3}+\frac{4(2a+b-2c-d)^2}{27(a+b+c+d)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.092. Prove that for all positive real numbers a, b, c

a. $$\frac{a^2}{a+b}+\frac{b^2}{b+c}+\frac{c^2}{c+a}\geq\frac{a+b+c}{2}+\frac{(b-c)^2}{2(a+b+c)}$$

b. $$\frac{a^2}{b+c}+\frac{b^2}{c+a}+\frac{c^2}{a+b}\geq\frac{a+b+c}{2}+\frac{(a+b-2c)^2}{2(a+b+c)}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.093. Prove that in any triangle ABC the following inequality holds

$$\frac{(b+c)a}{m_a^2}+\frac{(c+a)b}{m_b^2}+\frac{(a+b)c}{m_c^2}\geq 8$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.094. Prove that in any acute triangle ABC the following inequality holds

$$\frac{\cos B\cos C}{\sin A}+\frac{\cos C\cos A}{\sin B}+\frac{\cos A\cos B}{\sin C}\leq\frac{\sqrt{3}}{2}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.095. Let a,b,c be the side lengths of a triangle ABC with inradius r and circumradius R. Prove that

$$(b^4+c^4)\sin^2 A+(c^4+a^4)\sin^2 B+(a^4+b^4)\sin^2 C\leq$$

$$\leq\frac{81}{4}(3R^4-16r^4).$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.096. Let ABC be a triangle and w_a,w_b,w_c are bisectors of ABC. Prove that

$$\frac{1}{aw_a^2}+\frac{1}{bw_b^2}+\frac{1}{cw_c^2}\geq\frac{1}{R\Delta}$$

where R is the circumradius of ABC, Δ is area of ABC.

Proposed by Mehmet Şahin - Ankara - Turkey

SP.097. Let a,b,c be the side lengths of a triangle ABC with incentre I, circumradius R and inradius r. Prove that

$$\frac{\sqrt{AI}}{a}+\frac{\sqrt{BI}}{b}+\frac{\sqrt{CI}}{c}\leq\frac{\sqrt{2}}{2}\cdot\frac{\sqrt{R+r}}{r}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.098. Let ABC be an acute triangle with orthocenter H. Prove that

$$AH\cdot BH+BH\cdot CH+CH\cdot AH\leq 6Rr,$$

where R and r are the circumradius and inradius respectively of triangle ABC.

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.099. Let a,b,c be non-negative such that $a+b+c=3$. Prove that

$$\Big|(a-b)(b-c)(c-a)\Big|\leq\frac{3\sqrt{3}}{2}$$

Equality occurs when?

Proposed by Nguyen Ngoc Tu - Ha Giang – Vietnam

SP.100. Let a, b, c be the lengths of the sides of a triangle with perimeter 3 and inradius r. Prove that

$$288r^2 \leq \frac{(a+b)^4}{a^2+b^2} + \frac{(b+c)^4}{b^2+c^2} + \frac{(c+a)^4}{c^2+a^2} \leq \frac{2}{r^2}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.101. Let a, b and c be the side lengths of a triangle with inradius r. Prove that

$$\sqrt[4]{\frac{1}{a^4+2b^2c^2} + \frac{1}{b^4+2c^2a^2} + \frac{1}{c^4+2a^2b^2}} \leq \frac{\sqrt{3}}{6r}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.102. Let ABC be a triangle with circumradius R and inradius r. Prove that

$$4 \leq \sec^2\frac{A}{2} + \sec^2\frac{B}{2} + sec^2\frac{C}{2} \leq \frac{2R}{r}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.103. Let m, n be positive real numbers. Prove that

$$\left(\frac{1}{m} + \frac{1}{n}\right)^{-1} \leq \frac{4034 - 2015m}{m+2017} + \frac{4034 - 2015n}{n+2017} + \frac{m+n+2009}{2}$$

Proposed by Iuliana Traşcă - Romania

SP.104. Prove that in any triangle ABC the following relationship holds:

$$r\sum\frac{1}{\sin\frac{A}{2}} + \frac{abc}{2}\sum\frac{1}{\sqrt{abs(s-c)}} \leq 6R$$

s - semiperimeter; r - inradius; R - circumradius

Proposed by Daniel Sitaru - Romania

SP.105. Let G be the centroid in ΔABC. Prove that:

$$\cot(\widehat{GBA}) + \cot(\widehat{GCB}) + \cot(\widehat{GAC}) > \cot A + \cot B + \cot C + 3$$

Proposed by Daniel Sitaru - Romania

UNDERGRADUATE PROBLEMS

UP.091. Let be $a \in \mathbb{R}_+^*$ and the continuous functions $f, g, h : \mathbb{R} \to \mathbb{R}$ where f and g are odd and h is even. Prove that:

$$\int_{-a}^{a} f(x)\cdot\ln(1+e^{g(x)})\cdot\arctan(h(x))dx = \int_0^a f(x)g(x)\arctan(h(x))dx.$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.092. Calculate:

$$\lim_{n\to\infty} \sqrt[3]{n^2}\left(\sqrt[3(n+1)]{(n+1)!} - \sqrt[3n]{n!}\right).$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.093. Let $(a_n)_{n\geq 1}, (b_n)_{n\geq 1}$ be positive real sequences such that there exists $\lim_{n\to\infty} \frac{a_{n+1}}{n\cdot a_n}$ and $\lim_{n\to\infty}(b_n - u\cdot a_n)$. Find

a) $\lim_{n\to\infty}\left(\sqrt[n+1]{b_{n+1}} - \sqrt[n]{b_n}\right)$;

b) $\lim_{n\to\infty}\left(\frac{(n+1)^2}{\sqrt[n+1]{b_{n+1}}} - \frac{n^2}{\sqrt[n]{b_n}}\right)$.

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.094. Let $(s_n)_{n\geq 1}, s_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Calculate:

$$\lim_{n\to\infty}\left(s_n \cdot \sqrt[n+1]{(n+1)!} - \frac{\pi^2}{6}\cdot\sqrt[n]{n!}\right).$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.095. Let $(s_n)_{n\geq 1}, s_n = \sum_{k=1}^{n} \frac{1}{k^2}$ and let $(a_n)_{n\geq 1}$ be a positive real sequence such that $\lim_{n\to\infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*$. Calculate:

$$\lim_{n\to\infty}\left(s_n \cdot \sqrt[n+1]{a_{n+1}} - \frac{\pi^2}{6}\cdot\sqrt[n]{a_n}\right).$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.096. Let $(s_n)_{n\geq 1}, s_n = \sum_{k=1}^{n} \frac{1}{k^2}$. Calculate:

$$\lim_{n\to\infty}\left(s_n \cdot \sqrt[n+1]{(2n+1)!!} - \frac{\pi^2}{6}\cdot\sqrt[n]{(2n-1)!!}\right).$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.097. Evaluate $I_n = \int \frac{x^{n+1}+(n-1)(n+1)x^{n-1}}{(x^n+n^2)^2}dx$, where $n \in \mathbb{N}^*$.

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.098. Let $a, b \in \mathbb{R}, a < b$ and $f, g : \mathbb{R} \to \mathbb{R}$ continuous functions such that

$$f(x)f(a+b-x) = 1, g(x) = g(a+b-x), \forall x \in \mathbb{R}.$$

Show that

$$\int_a^b \frac{g(x)}{1+f(x)}dx = \frac{1}{2} \cdot \int_a^b g(x)dx$$

Proposed by D.M. Bătineţu – Giurgiu, Neculai Stanciu – Romania

UP.099. In an arbitrary triangle ABC denote by l_a, m_a, h_a respectively the lengths of the internal angle-bisector, the median and the altitude corresponding to the side $a = BC$ of the triangle. Prove that:

a) $\frac{l_a^2}{h_a^2} + \frac{l_b^2}{h_b^2} + \frac{l_c^2}{h_c^2} \ge 2\frac{l_a}{h_a} \cdot \frac{l_b}{h_b} \cdot \frac{l_c}{h_c} + 1.$

b) $\frac{m_a^2}{h_a^2} + \frac{m_b^2}{h_b^2} + \frac{m_c^2}{h_c^2} \le 2\frac{m_a}{h_a} \cdot \frac{m_b}{h_b} \cdot \frac{m_c}{h_c} + 1.$

c) explain why each of a) and b) are equivalent to the fundamental inequality of the triangle.

Proposed by Vasile Jiglău - Romania

UP.100. In $\Delta ABC; m_a, m_b, m_c$ - median's length. Prove that:

$$3(a^2+b^2+c^2) < 4(am_c + bm_a + cm_b)$$

Proposed by Daniel Sitaru - Romania

UP.101. Prove that if $a, b, c \in (1, \infty)$ then:

$$3\sqrt{2} + \int_1^a x\sin\frac{\pi}{3x}dx + \int_1^b x\sin\frac{\pi}{3x}dx + \int_1^c x\sin\frac{\pi}{3x}dx > \sqrt{3+a^2+b^2+c^2}$$

Proposed by Daniel Sitaru - Romania

UP.102. Solve for real numbers:

$$n^{n(x_1^2-x_2)} + n^{n(x_2^2-x_3)} + \ldots + n^{n(x_{n-1}^2-x_n)} + n^{n(x_n^2-x_1)} = \frac{n}{\sqrt[4]{n^n}}$$

$$n \in \mathbb{N}; n \ge 2$$

Proposed by Daniel Sitaru - Romania

UP.103. Prove that in any triangle ABC the following relationship holds:

$$|\cos A| + |\cos B| + |\cos C| \le \sum\left(\sqrt{|\cos A \cos B|} + \sqrt{\left|\cos\frac{C}{2}\sin\frac{B-A}{2}\right|}\right)$$

Proposed by Daniel Sitaru - Romania

UP.104. Prove that if $x_i \in (0, \infty); i \in \overline{1, n}; n \in \mathbb{N}; n \geq 3;$ $x_{n+1} = x_1; x_1 x_2 \cdot \ldots \cdot x_n = 1$, then

$$\sum_{i=1}^{n} \frac{\frac{x_i}{x_{i+1}} + \frac{x_{i+1}}{x_i} + 1}{\sqrt{x_i^2 + x_i x_{i+1} + x_{i+1}^2}} \geq n\sqrt{3}$$

Proposed by Daniel Sitaru - Romania

UP.105. In $ABC; a, b, c$ - length sides; s - semiperimeter; A, B, C - angled's measures. Prove that:

$$\left(\frac{A^3}{b} + \frac{B^3}{c} + \frac{C^3}{a}\right)\left(\frac{A^3}{c} + \frac{B^3}{a} + \frac{C^3}{b}\right)\left(\frac{A^3}{a} + \frac{B^3}{b} + \frac{C^3}{c}\right) \geq \frac{\pi^9}{216s^3}$$

Proposed by Daniel Sitaru - Romania

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