R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

If $a,b,c>0$ then:

$$\sum\frac{(a^7+b^7)^3}{(a^4+b^4)(a^5+b^5)(a^6+b^6)}\ge 3a^2b^2c^2$$

Proposed by Daniel Sitaru – Romania

Solution by Lazaros Zachariadis-Thessaloniki-Greece

$$\left(\frac{a^7+b^7}{2}\right)^{\frac{1}{7}}\ge\left(\frac{a^4+b^4}{2}\right)^{\frac{1}{4}}\Rightarrow\frac{(a^7+b^7)^4}{2^4}\ge\frac{(a^4+b^4)^7}{2^7}\Rightarrow$$

$$\Rightarrow a^7+b^7\ge\frac{1}{2^{\frac{3}{4}}}\cdot(a^4+b^4)^{\frac{7}{4}}\Rightarrow\frac{a^7+b^7}{a^4+b^4}\ge\frac{1}{2^{\frac{3}{4}}}\cdot(a^4+b^4)^{\frac{3}{4}}$$

$$\left(\frac{a^7+b^7}{2}\right)^{\frac{1}{7}}\ge\left(\frac{a^5+b^5}{2}\right)^{\frac{1}{5}}\Rightarrow\left(\frac{a^7+b^7}{2}\right)^{5}\ge\frac{\left(a^5+b^5\right)^7}{2^7}\Rightarrow$$

$$\Rightarrow a^7+b^7\ge\frac{1}{2^{\frac{3}{5}}}\left(a^5+b^5\right)^{\frac{7}{5}}\Rightarrow\frac{a^7+b^7}{a^5+b^5}\ge\frac{1}{2^{\frac{2}{5}}}\cdot\left(a^5+b^5\right)^{\frac{2}{5}}$$

$$\left(\frac{a^7+b^7}{2}\right)^{\frac{1}{7}}\ge\left(\frac{a^6+b^6}{2}\right)^{\frac{1}{6}}\Rightarrow\frac{(a^7+b^7)^6}{2^6}\ge\frac{(a^6+b^6)^7}{2^7}\Rightarrow$$

$$\Rightarrow a^7+b^7\ge\frac{1}{2^{\frac{1}{6}}}\cdot(a^6+b^6)^{\frac{7}{6}}\Rightarrow\frac{a^7+b^7}{a^6+b^6}\ge\frac{1}{2^{\frac{1}{6}}}\cdot(a^6+b^6)^{\frac{1}{6}}$$

$$\frac{(a^7+b^7)(a^7+b^7)(a^7+b^7)}{(a^4+b^4)(a^5+b^5)(a^6+b^6)}\ge 2^{-\frac{3}{4}}\cdot 2^{-\frac{2}{5}}\cdot 2^{-\frac{1}{6}}\cdot(a^4+b^4)^{\frac{3}{4}}\cdot\left(a^5+b^5\right)^{\frac{2}{5}}(a^6+b^6)^{\frac{1}{6}}$$

$$\ge 2^{-\frac{3}{4}}\cdot 2^{-\frac{2}{5}}\cdot 2^{-\frac{1}{6}}\left(2\sqrt{(ab)^4}\right)^{\frac{3}{4}}\cdot\left(2\sqrt{(ab)^5}\right)^{\frac{2}{5}}\cdot\left(2\sqrt{(ab)^6}\right)^{\frac{1}{6}}$$

$$=2^{-\frac{3}{4}}\cdot 2^{-\frac{2}{3}}\cdot 2^{-\frac{1}{6}}\cdot 2^{\frac{3}{4}}\cdot 2^{\frac{3}{5}}\cdot 2^{\frac{1}{6}}\cdot((ab)^2)^{\frac{3}{4}}\cdot\left((ab)^{\frac{5}{2}}\right)^{\frac{3}{5}}\cdot((ab)^3)^{\frac{1}{6}}$$

$$=(ab)^{\frac{3}{2}}\cdot(ab)\cdot(ab)^{\frac{1}{2}}=(ab)^{\frac{6}{2}}=(ab)^3$$

Thus $\sum_{cyc}\frac{\left(a^7+b^7\right)^3}{(a^4+b^4)(a^5+b^5)(a^6+b^6)}\ge(ab)^3+(bc)^3+(ca)^3\ge 3\cdot\sqrt[3]{(abc)^6}=3\cdot(abc)^2$