If $a,b,c>0, a^2+b^2+c^2=3$ then:

$$6\left(\frac{b^2}{\sqrt{a^2+3}}+\frac{c^2}{\sqrt{b^2+3}}+\frac{a^2}{\sqrt{c^2+3}}\right)\geq(a+b+c)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Dimitris Kastriotis-Athens-Greece, Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solution 3 by Lazaros Zachariadis-Thessaloniki-Greece

Solution 1 by Dimitris Kastriotis-Athens-Greece

$$(a+b+c)^2\leq 3(a^2+b^2+c^2)\Rightarrow\frac{a+b+c}{3}\leq 1\quad (1)$$

If $a\leq b\leq c$ then $a^2\leq b^2\leq c^2$ and $\frac{1}{\sqrt{a^2+3}}\geq\frac{1}{\sqrt{b^2+3}}\geq\frac{1}{\sqrt{c^2+3}}$

By the Rearrangement Inequality we deduce

$$\frac{b^2}{\sqrt{a^2+3}}+\frac{c^2}{\sqrt{b^2+3}}+\frac{a^2}{\sqrt{c^2+3}}\geq\frac{a^2}{\sqrt{a^2+3}}+\frac{b^2}{\sqrt{b^2+3}}+\frac{c^2}{\sqrt{c^2+3}}\quad (2)$$

Let $f(x)=\frac{x^2}{\sqrt{x^2+1}}, x\in(0,\sqrt{3}), f'(x)=\frac{(x^2+6)x}{(x^2+3)^{\frac{3}{2}}}$

$f''(x)=-\frac{3(x^2-6)}{(x^2+3)^{\frac{5}{2}}}>0, \forall x\in(0,\sqrt{3})\Rightarrow f$: convexe on the interval $(0,\sqrt{3})$

By Jensen's inequality we deduce

$$\frac{a^2}{\sqrt{a^2+3}}+\frac{b^2}{\sqrt{b^2+3}}+\frac{c^2}{\sqrt{c^2+3}}\geq 3f\left(\frac{a+b+c}{3}\right)=3\frac{\left(\frac{a+b+c}{3}\right)^2}{\sqrt{\left(\frac{a+b+c}{3}\right)^2+3}}\overset{(1)}{\geq}\frac{(a+b+c)^2}{6}\quad (3)$$

$$\left.\begin{matrix}(2)\\(3)\end{matrix}\right\}\Rightarrow\frac{b^2}{\sqrt{a^2+3}}+\frac{c^2}{\sqrt{b^2+3}}+\frac{a^2}{\sqrt{c^2+3}}\geq\frac{(a+b+c)^2}{6}$$

$$\Rightarrow 6\left(\frac{b^2}{\sqrt{a^2+3}}+\frac{c^2}{\sqrt{b^2+3}}+\frac{a^2}{\sqrt{c^2+3}}\right)\geq(a+b+c)^2$$

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Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaidian

If $a, b, c > 0$ ***and*** $\sum a^2 = 3$

On it $\left(\sum \sqrt{a^2+3}\right)^2 \le 3(a^2+3+b^2+3+c^2+3) = 36 \Rightarrow$

$$\Rightarrow \sum \sqrt{a^2+3} \le 6 \quad \textbf{(1)}$$

Prove that:

$$6\left(\sum \frac{a^2}{\sqrt{c^2+3}}\right) \ge \left(\sum a\right)^2$$

$$LHS = 6\left(\sum \frac{a^2}{\sqrt{c^2+3}}\right) \overset{Bergstrom}{\ge} 6 \cdot \frac{(a+b+c)^2}{\sum \sqrt{a^2+3}} \overset{(1)}{\ge} (a+b+c)^2$$

Solution 3 by Lazaros Zachariadis-Thessaloniki-Greece

$$\sqrt{a^2+3} = \sqrt{\frac{(a^2+3)\cdot 4}{4}} = \frac{\sqrt{(a^2+3)\cdot 4}}{2} \le \frac{1}{2} \cdot \frac{a^2+3+4}{2}$$

$$\Rightarrow \sqrt{a^2+3} \le \frac{a^2+7}{4} \Rightarrow \frac{1}{\sqrt{a^2+3}} \ge \frac{4}{a^2+7}$$

Likevise
$$\left.\begin{array}{l} \overset{b^2>0}{\Rightarrow} \dfrac{b^2}{\sqrt{a^2+3}} \ge \dfrac{4b^2}{a^2+7} \\ \dfrac{c^2}{\sqrt{b^2+3}} \ge \dfrac{4c^2}{b^2+7} \\ \dfrac{a^2}{\sqrt{c^2+3}} \ge \dfrac{4a^2}{c^2+7} \end{array}\right\} \overset{(+)}{\Rightarrow} \sum_{cyc} \frac{4a^2}{c^2+7} \overset{BERGSTROM}{\ge} \frac{(2a+2b+2c)^2}{a^2+b^2+c^2+3\cdot 7} = \frac{4(a+b+c)^2}{3+21}$$

$$\overset{6}{\Rightarrow} LHS \ge \frac{24(a+b+c)^2}{24} = (a+b+c)^2 = RHS$$

$$\text{“=”}\ a = b = c = 1$$