If $a, b, c > 0$ then:

$$\left(6\sqrt[3]{\frac{a}{b}}-\frac{a^2}{b^2}\right)+\left(6\sqrt[3]{\frac{b}{c}}-\frac{b^2}{c^2}\right)+\left(6\sqrt[3]{\frac{c}{a}}-\frac{c^2}{a^2}\right)\le 15$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Athanasios Mplegiannis-Greece, Solution 2 by Bibashwan Ghosh-India, Solution 3 by Boris Colakovic-Belgrade-Serbia, Solution 4 by Ravi Prakash-New Delhi-India, Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand, Solution 6 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solution 7 by Soumava Chakraborty-Kolkata-India

Solution 1 by Athanasios Mplegiannis-Greece

The function $f(x)=x^6, x\in[0,+\infty)$, is a concave up function because

$f'(x)=6x^5$ and $f''(x)=30x^4>0, \forall x>0$

The tangent line of the function's graph at point $(1,1)$ is

$$y-f(1)=f'(1)(x-1)\Rightarrow y=6x-5$$

So, $f(x)\ge 6x-5 \Leftrightarrow x^6\ge 6x-5, \forall x\ge 0$, and

$$\begin{Bmatrix} f\left(\sqrt[3]{\frac{a}{b}}\right)\ge 6\cdot\sqrt[3]{\frac{a}{b}}-5 \\ f\left(\sqrt[3]{\frac{b}{c}}\right)\ge 6\cdot\sqrt[3]{\frac{b}{c}}-5 \\ f\left(\sqrt[3]{\frac{c}{a}}\right)\ge 6\cdot\sqrt[3]{\frac{c}{a}}-5 \end{Bmatrix}\Rightarrow\begin{Bmatrix} \frac{a^2}{b^2}\ge 6\cdot\sqrt[3]{\frac{a}{b}}-5 \\ \frac{b^2}{c^2}\ge 6\cdot\sqrt[3]{\frac{b}{c}}-5 \\ \frac{c^2}{a^2}\ge 6\cdot\sqrt[3]{\frac{c}{a}}-5 \end{Bmatrix}\overset{(+)}{\Rightarrow}$$

$$\Rightarrow\left(6\cdot\sqrt[3]{\frac{a}{b}}-\frac{a^2}{b^2}\right)+\left(6\cdot\sqrt[3]{\frac{b}{c}}-\frac{b^2}{c^2}\right)+\left(6\cdot\sqrt[3]{\frac{c}{a}}-\frac{c^2}{a^2}\right)\le 15$$

Equality holds for $\frac{a}{b}=\frac{b}{c}=\frac{c}{a}=1\Rightarrow a=b=c$.

Solution 2 by Bibashwan Ghosh-India

If $sqrt\left(\frac{a}{b}\right)=x$, then,

$f(x)=6x-(x^6)$ ***attains a global maximum at*** $x=1$ ***which means***

$$f(x)\le 5$$

Thus, $f\left(\frac{a}{b}\right)+f\left(\frac{b}{c}\right)+f\left(\frac{c}{a}\right)\le 15$ ***with equality at*** $a=b=c$.

Solution 3 by Boris Colakovic-Belgrade-Serbia

$$\frac{a}{b}=x^3>0,\frac{b}{c}=y^3>0,\frac{c}{a}=z^3>0$$

$$6(x+y+z)-(x^6+y^6+z^6)\le 15\Leftrightarrow$$

$$\Leftrightarrow x^6+y^6+z^6-6(x+y+z)+15\ge 0\Leftrightarrow$$

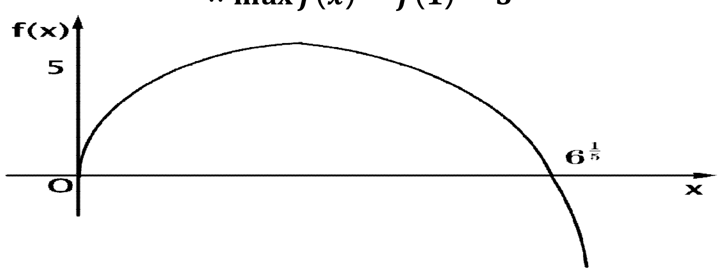
$$\Leftrightarrow \frac{1}{243}(x+y+z)^6-6(x+y+z)+15\ge 0$$

$$x+y+z=t>0$$

$$\frac{1}{243}t^6-6t+15\ge 0\Leftrightarrow \frac{1}{243}(t-3)^2\underbrace{(t^4+6t^3+27t^2+108t+405)}_{>0}\ge 0$$

Solution 4 by Ravi Prakash-New Delhi-India

Let $f(x)=6x-x^6, x>0, f'(x)=6(1-x^5)$
$=6(1-x)(1+x+x^2+x^3+x^4+x^5)$
$f'(x)>0$ ***if*** $0<x<1$, <0 ***if*** $x>1$, $=0$ ***if*** $x=1$
$\therefore \max f(x)=f(1)=5$



$\therefore f(x)\le 5, \forall x>0$

Thus,

$$\left(6\left(\frac{a}{b}\right)^{\frac{1}{3}}-\frac{a^2}{b^2}\right)+\left(6\left(\frac{b}{c}\right)^{\frac{1}{3}}-\frac{b^2}{c^2}\right)+\left(6\left(\frac{c}{a}\right)^{\frac{1}{3}}-\frac{c^2}{a^2}\right)$$

$$=f\left(\left(\frac{a}{b}\right)^{\frac{1}{3}}\right)+f\left(\left(\frac{b}{c}\right)^{\frac{1}{3}}\right)+f\left(\left(\frac{c}{a}\right)^{\frac{1}{3}}\right)\le 15$$

Solution 5 by Sanong Huayrerai-Nakon Pathom-Thailand

Give $\frac{a}{b}=\frac{x^3}{y^3};\frac{b}{c}=\frac{y^3}{z^3},\frac{c}{a}=\frac{z^3}{x^3}$

Hence $\left(6\sqrt[3]{\frac{a}{b}}-\frac{a^2}{b^2}\right)+\left(6\sqrt[3]{\frac{b}{c}}-\frac{b^2}{c^2}\right)+\left(6\sqrt[3]{\frac{c}{a}}-\frac{c^2}{a^2}\right)$

$=\left(6\frac{x}{y}-\frac{x^6}{y^6}\right)+\left(6\frac{y}{z}-\frac{y}{z^6}\right)+\left(6\frac{z}{x}-\frac{z^6}{x^6}\right)\leq 15$ *is to be true*

Because $\forall\, m,n>0$ *we will get that*

$$\frac{m^6}{n^6}+5=\frac{m^6}{n^6}+1+1+(1+1+1)\geq 3\frac{m^2}{n^2}+1+1+1$$

$$=\left(\frac{m^2}{n^2}+1\right)+\left(\frac{m^2}{n^2}+1\right)+\left(\frac{m^2}{n^2}+1\right)\geq\frac{2m}{n}+\frac{2m}{n}+\frac{2m}{n}=6\frac{m}{n}$$

Hence $6\frac{m}{n}-\frac{m^b}{n^b}\leq 5$

Solution 6 by Seyran Ibrahimov-Maasilli-Azerbaidian

If $a,b,c>0,\sum\left(6\sqrt[3]{\frac{a}{b}}-\frac{a^2}{b^2}\right)\leq 15\Rightarrow$

$$\Rightarrow\begin{cases}\frac{a^2}{b^2}+5=\frac{a^2}{b^2}+1+1+1+1+1\geq 6\sqrt[3]{\frac{a}{b}}\\ \frac{b^2}{c^2}+5=\frac{b^2}{c^2}+1+1+1+1+1\geq 6\sqrt[3]{\frac{b}{c}}\\ \frac{c^2}{a^2}+5=\frac{c^2}{a^2}+1+1+1+1+1\geq 6\sqrt[3]{\frac{c}{a}}\end{cases}\overset{\oplus}{\Rightarrow}$$

$$\Rightarrow 15\geq\sum\left(6\sqrt[3]{\frac{a}{b}}-\frac{a^2}{b^2}\right)\Rightarrow\ \textit{(proved)}$$

Solution 7 by Soumava Chakraborty-Kolkata-India

$$\sqrt[3]{\frac{a}{b}} = \sqrt[3]{\frac{a}{b}\cdot 1\cdot 1} \overset{GM-AM}{\overbrace{\leq}} \left(\frac{a}{b}+1+1\right)\cdot\frac{1}{3}$$

$$\rightarrow \begin{cases} 6\sqrt[3]{\frac{a}{b}} \leq 2\left(\frac{a}{b}+2\right) = 4+\frac{2a}{b} \\ 6\sqrt[3]{\frac{b}{c}} \leq 2\left(\frac{b}{c}+2\right) = 4+\frac{2b}{c} \\ 6\sqrt[3]{\frac{c}{a}} \leq 2\left(\frac{c}{a}+2\right) = 4+\frac{2c}{a} \end{cases} \rightarrow LHS \leq 12+2\sum\frac{a}{b}$$

$$12+2\sum\frac{a}{b} \leq 15+\sum\frac{a^2}{b^2} \leftrightarrow \sum\left(\frac{a^2}{b^2}-\frac{2a}{b}+1\right) \geq 0 \leftrightarrow$$

$$\leftrightarrow \sum\left(\frac{a}{b}-1\right)^2 \geq 0$$

$$Equality\ holds\ for\ a=b=c$$