Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro

SOLUTION
PROBLEM JP104 WINTER 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

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1) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_c}+\frac{r_c^2}{h_a m_b}\geq\frac{54r^2}{p^2-r^2-4Rr}$$

Proposed by D.M. Bătineţu-Giurgiu - Romania, Martin Lukarevski - Skopje

Proof.

We prove the following lemma:

Lemma 1.
*2) **In** ΔABC*

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{4(4R+r)^2}{5p^2-3r(4R+r)}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the fact that $h_a\leq m_a$ and Bergström inequality we obtain:

$$\sum\frac{r_a^2}{h_b m_c}\geq\sum\frac{r_a^2}{m_b m_c}\geq\frac{(\sum r_a)^2}{\sum m_b m_c}\geq\frac{(4R+r)^2}{\frac{1}{4}\sum(2a^2+bc)}=\frac{4(4R+r)^2}{2\sum a^2+\sum bc}=$$
$$=\frac{4(4R+r)^2}{2\cdot 2(p^2-r^2-4Rr)+p^2+r^2+4Rr}=\frac{4(4R+r)^2}{5p^2-3r(4R+r)}$$

Equality holds if and only if the triangle is equilateral.

□

Let's pass to solving the inequality from the enunciation.

*Using **Lemma 1** it's enough to prove that* $\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\geq\frac{54r^2}{p^2-r^2-4Rr}$.

This inequality can be transformed equivalently:

$$2(4R+r)^2(p^2-r^2-4Rr)\geq 27r^2(5p^2-3r^2-12Rr)\Leftrightarrow$$
$$\Leftrightarrow p^2(32R^2+16Rr-133r^2)\geq 2r(4R+r)^3-81r^3(4R+r)$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$
and from the observation that $32R^2+16Rr-133r^2>0$ (see Euler's inequality $R\geq 2r$).

It remains to prove that:

$$(16Rr-5r^2)(32R^2+16Rr-133r^2)\geq 2r(4R+r)^3-81r^3(4R+r)\Leftrightarrow$$
$$\Leftrightarrow 32R^3-159Rr^2+62r^3\geq 0\Leftrightarrow(R-2r)(32R^2+64Rr-31r^2)\geq 0$$

obviously from Euler's inequality.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1)** can be rewritten:*

1) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\ge\frac{108r^2}{a^2+b^2+c^2}$$

Proof.

*Using **Lemma 1** and the identity $ab+bc+ca=p^2+r^2+4Rr$ it suffices to prove that*

$$\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\ge\frac{108r^2}{p^2+r^2+4Rr}$$

This inequality transformed equivalently:

$$(4R+r)^2(p^2+r^2+4Rr)\ge 27r^2(5p^2-3r^2-12Rr)\Leftrightarrow$$

$$\Leftrightarrow p^2(16R^2+8Rr+r^2-135r^2)+r(4R+r)^3+81r^3(4R+r)\ge 0\Leftrightarrow$$

$$\Leftrightarrow p^2(8R^2+4Rr-67r^2)+32R^3r+24R^2r^2+168Rr^3+41r^4\ge 0$$

We distinguish the following cases:

Case 1). If $8R^2+4Rr-67r^2\ge 0$, the inequality is obvious.

Case 2). If $8R^2+4Rr-67r^2<0$, the inequality can be rewritten:

$$32R^3r+24R^2r^2+168Rr^3+41r^4\ge p^2(67r^2-4Rr-8r^2)$$

which follows from Gerretsen's inequality $p^2\le 4R^2+4Rr+3r^2$. It remains to prove that:

$$32R^3r+24R^2r^2+168Rr^3+41r^4\ge(4R^2+4Rr+3r^2)(67r^2-4Rr-8R^2)\Leftrightarrow$$

$$\Leftrightarrow 8R^4+20R^3r-51R^2r^2-22Rr^3-40r^4\ge 0\Leftrightarrow(R-2r)(8R^3+36R^2r+21Rr^2+20r^3)\ge 0,$$

obviously from Euler's inequality $R\ge 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

5. In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\ge\frac{108r^2}{ab+bc+ca}\ge\frac{108r^2}{a^2+b^2+c^2}.$$

Proof.

*We use inequality **4)** and inequality $a^2+b^2+c^2\ge ab+bc+ca$.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **4)** can also be strengthened:*

6) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{9r\sqrt{3}}{p}$$

Proof.

*Using **Lemma 1** it suffices to prove that* $\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\geq\frac{9r\sqrt{3}}{p}$.

This inequality can be transformed equivalently:

$4p(4R+r)^2\geq 9r\sqrt{3}(5p^2-3r^2-12Rr)$, *which follows from Mitrinović's inequality* $p\geq 3r\sqrt{3}$.

It suffices to prove that

$4\cdot 3r\sqrt{3}(4R+r)^2\geq 9r\sqrt{3}(5p^2-3r^2-12Rr)\Leftrightarrow 4(4R+r)^2\geq 15p^2-9r(4R+r)\Leftrightarrow$

$\Leftrightarrow 4(4R+r)^2+9r(4R+r)\geq 15p^2$, *true from Gerretsen's inequality*

$p^2\leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$4(4R+r)^2+9r(4R+r)\geq 15(4R^2+4Rr+3r^2)\Leftrightarrow R^2+2Rr-8r^2\geq 0\Leftrightarrow(R-2r)(R+4r)\geq 0$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **6)** is stronger than inequality **4)**.*

7) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{9r\sqrt{3}}{p}\geq\frac{108r^2}{ab+bc+ca}$$

Proof.

*We use inequality **6)** and the known inequality in triangle* $ab+bc+ca\geq 4\sqrt{3}S$

□

Remark.

*Inequality **6)** can also be strengthened:*

8) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\geq\frac{2p\sqrt{3}}{3R}$$

Proof.

*Using **Lemma 1** it suffices to prove that* $\dfrac{4(4R+r)^2}{5p^2-3r(4R+r)} \geq \dfrac{2p\sqrt{3}}{3R}$.

This inequality can be transformed equivalently:

$6R(4R+r)^2 \geq p\sqrt{3}(5p^2-3r^2-12Rr)$, *which follows from Doucet's inequality* $4R+r \geq p\sqrt{3}$. *It remains to prove that*

$6R(4R+r)^2 \geq (4R+r)(5p^2-3r^2-12Rr) \Leftrightarrow 6R(4R+r) \geq 5p^2-3r^2-12Rr$

true from Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$. *It remains to prove that:*

$6R(4R+r) \geq 5(4R^2+4Rr+3r^2)-3r^2-12Rr \Leftrightarrow 2R^2-Rr-6r^2 \geq 0 \Leftrightarrow (R-r)(2R+3r) \geq 0$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **8)** is stronger than inequality **6)**.*

9) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b} \geq \frac{2p\sqrt{3}}{3R} \geq \frac{9r\sqrt{3}}{p}.$$

Remark.

*We use inequality **8)** and the known inequality in triangle* $2p^2 \geq 27Rr$

(true from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ *and Euler's inequality* $R \geq 2r$*).*

Remark.

We can write the following inequalities:

10) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b} \geq \frac{4(4R+r)^2}{5p^2-3r(4R+r)} \geq \frac{9r\sqrt{3}}{p} \geq \frac{108r^2}{p^2+r^2+4Rr} \geq \frac{54r^2}{p^2-r^2-4Rr}$$

Proof.

*We use **Lemma 1** and the above inequalities.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an apposite sense:

11) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b} \leq \left(\frac{R}{r}\right)^2-\frac{3}{4}\cdot\frac{R}{r}+\frac{1}{2}.$$

Proof.

Let's prove the following lemma:

Lemma 2.
12) In ΔABC

$$\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\le\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the fact that $h_a\le m_a$ we obtain:

$$\sum\frac{r_a^2}{h_b m_c}\le\sum\frac{r_a^2}{h_b h_c}=\sum\frac{\frac{S^2}{(p-a)^2}}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{4}\sum\frac{bc}{(p-a)^2}=\frac{1}{4}\det\frac{p^2(r-8R)+(4R+r)^3}{rp^2}=$$
$$=\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}$$

The equality holds if and only if the triangle is equilateral.

□

*Let's pass to solving inequality **11)**.*

*Using **Lemma 2** it suffices to prove that* $\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}\le\left(\frac{R}{r}\right)^2-\frac{3}{4}\cdot\frac{R}{r}+\frac{1}{2}$

This inequality can be transformed equivalently:

$$p^2(r-8R)+(4R+r)^3\le p^2(4R^2-3Rr+2r^2)\Leftrightarrow p^2(4R^2+5Rr+r^2)\ge r(4R+r)^3$$

which follows from inequality $p^2\ge\frac{r(4R+r)^2}{R+r}$

(true from Gerretsen's inequality $p^2\ge 16Rr-5r^2$ and Euler's inequality $R\ge 2r$).

The equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

1) In ΔABC

$$\frac{4(4R+r)^2}{5p^2-3r(4R+r)}\le\frac{r_a^2}{h_b m_c}+\frac{r_b^2}{h_c m_a}+\frac{r_c^2}{h_a m_b}\le\frac{p^2(r-8R)+(4R+r)^3}{4rp^2}$$

Proof.

*See **Lemma 1 and Lemma 2***

The equality holds if and only if the triangle is equilateral.

□

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