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TRIANGLE INEQUALITY - 528
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1) In ΔABC

$$\frac{\cot A}{p-a}+\frac{\cot B}{p-b}+\frac{\cot C}{p-c}\leq\frac{1}{r}-\frac{R-2r}{2Rr}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

Let's prove the following lemma:

Lemma 1.
2) In ΔABC

$$\frac{\cot A}{p-a}+\frac{\cot B}{p-b}+\frac{\cot C}{p-c}=\frac{5p^2-(4R+r)^2}{2rp^2}.$$

Proof.

We have:

$$\sum\frac{\cot A}{p-a}=\sum\frac{\frac{\cos A}{\sin A}}{p-a}=\sum\frac{\frac{b^2+c^2-a^2}{2bc}\cdot\frac{2R}{a}}{p-a}=\frac{R}{abc}\sum\frac{b^2+c^2-a^2}{p-a}=$$
$$=\frac{R}{4Rrp}\cdot\frac{10p^2-2(4R+r)^2}{p}=\frac{5p^2-(4R+r)^2}{2rp^2}.$$

□

*Let's pass to solving inequality **1)**.*

*Using **Lemma 1** the inequality can be written* $\frac{5p^2-(4R+r)^2}{2rp^2}\leq\frac{1}{r}-\frac{R-2r}{2Rr}\Leftrightarrow$

$\Leftrightarrow p^2\leq\frac{R(4R+r)^2}{2(2R-r)}$, *which is Blundon's-Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having an opposite sense:

3) In ΔABC

$$\frac{\cot A}{p-a}+\frac{\cot B}{p-b}+\frac{\cot C}{p-c}\geq\frac{4r-R}{2r^2}.$$

Proof.

*Using **Lemma 1** the inequality can be written:*

$$\frac{5p^2-(4R+r)^2}{2rp^2} \geq \frac{4r-R}{2r^2} \Leftrightarrow p^2 \geq \frac{r(4R+r)^2}{R+r}$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC

$$\frac{4r-R}{2r^2} \leq \frac{\cot A}{p-a}+\frac{\cot B}{p-b}+\frac{\cot C}{p-c} \leq \frac{R+2r}{2Rr}.$$

Proof.

*See inequalities **1)** and **3)**.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way we can propose:

5) In ΔABC

$$\frac{1}{p}\left(15-\frac{5r}{R}-\frac{4R}{r}\right) \leq \frac{\cos A}{p-a}+\frac{\cos B}{p-b}+\frac{\cos C}{p-c} \leq \frac{1}{p}\left(1+\frac{r}{R}\right).$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2.
6) In ΔABC

$$\frac{\cos A}{p-a}+\frac{\cos B}{p-b}+\frac{\cos C}{p-c} = \frac{p^2-Rr-4R^2}{Rrp}$$

Proof.

We have $\sum \frac{\cos A}{p-a} = \sum \frac{\frac{b^2+c^2-a^2}{2bc}}{p-a} = \sum \frac{b^2+c^2-a^2}{2(p-a)bc} = \frac{p^2-Rr-4R^2}{Rrp}$

□

Let's pass to solve the double inequality ***5).***

Using ***Lemma 2*** *the double inequality* ***5)*** *can be written*

$$\frac{1}{p}\left(15-\frac{5r}{R}-\frac{4R}{r}\right)\le\frac{p^2-Rr-4r^2}{Rrp}\le\frac{1}{p}\left(1+\frac{r}{R}\right),$$

which follows from Gerretsen's inequality $16Rr-5r^2\le p^2\le 4R^2+4Rr+3r^2$.

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC

$$\frac{5}{2r}-\frac{1}{R}\le\frac{\csc A}{p-a}+\frac{\csc B}{p-b}+\frac{\csc C}{p-c}\le\frac{1}{2r}\left(2+\frac{R}{r}\right).$$

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Proof.

We prove the following lemma:

Lemma 3.
8) In ΔABC

$$\frac{\csc A}{p-a}+\frac{\csc B}{p-b}+\frac{\csc C}{p-c}=\frac{1}{2r}\left[1+\left(\frac{4R+r}{p}\right)^2\right].$$

Proof.

We have:

$$\sum\frac{\csc A}{p-a}=\sum\frac{\frac{1}{\sin A}}{p-a}=\sum\frac{\frac{2R}{a}}{p-a}=2R\sum\frac{1}{a(p-a)}=2R\cdot\frac{p^2+(4R+r)^2}{4Rrp^2}=$$
$$=\frac{1}{2r}\left[1+\left(\frac{4R+r}{p}\right)^2\right]$$

□

Let's pass to solve the double inequality ***7)****.*

Using ***Lemma 3*** *the double inequality* ***7)*** *can be written*

$$\frac{5}{2r}-\frac{1}{R}\le\frac{1}{2r}\left[1+\left(\frac{4R+r}{p}\right)^2\right]\le\frac{1}{2r}\left(2+\frac{R}{r}\right)$$

which follows from Blundon's Gerretsen's inequality $\dfrac{r(4R+r)^2}{R+r}\le p^2\le\dfrac{R(4R+r)^2}{2(2R-r)}$.

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC

$$\frac{12}{p}\le\frac{\csc^2 A}{p-a}+\frac{\csc^2 B}{p-b}+\frac{\csc^2 C}{p-c}\le\frac{1}{p}\left(\frac{2R^2}{r^2}+\frac{5R}{4r}+\frac{3}{2}\right).$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 4
10) In ΔABC

$$\frac{\csc^2 A}{p-a}+\frac{\csc^2 B}{p-b}+\frac{\csc^2 C}{p-c}=\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}.$$

Proof.

We have

$$\sum\frac{\csc^2 A}{p-a}=\sum\frac{\frac{1}{\sin^2 A}}{p-a}=\sum\frac{\frac{4R^2}{a^2}}{p-a}=4R^2\sum\frac{1}{a^2(p-a)}=$$
$$=4R^2\cdot\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{16R^2r^2p^3}=\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}.$$

□

*Let's pass to solve the double inequality **9)**.*

*Using **Lemma 4** the double inequality **7)** can be written*

$$\frac{12}{p}\le\frac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}\le\frac{1}{p}\left(\frac{2R^2}{r^2}+\frac{5R}{4r}+\frac{3}{2}\right).$$

The left inequality is equivalent with:

$$p^4+p^2(2r^2-4Rr)+r(4R+r)^3\ge 48r^2p^2\Leftrightarrow p^2(p^2-46r^2-4Rr)+r(4R+r)^3\ge 0.$$

We distinguish the following cases:

Case 1). If $p^2-46r^2-4Rr\ge 0$, the inequality becomes obviously.

Case 2). If $p^2-46r^2-4Rr<0$, the inequality can be rewritten:

$p^2(46r^2+4Rr-p^2)\le r(4R+r)^3$ *it follows from Blundon-Gerretsen's inequality*

$16Rr-5r^2\le p^2\le\dfrac{R(4R+r)^2}{2(2R-r)}$. *It remains to prove that:*

$$\frac{R(4R+r)^2}{2(2R-r)}\cdot(46r^2+4Rr-16Rr+5r^2)\le r(4R+r)^3\Leftrightarrow 28R^2-55Rr-2r^2\ge 0\Leftrightarrow$$

$\Leftrightarrow(R-2r)(28R+r)\ge 0$, *obviously from Euler's inequality $R\ge 2r$.*

Equality holds if and only if the triangle is equilateral.

Let's solve the inequality from the right:

We have $\dfrac{p^4+p^2(2r^2-4Rr)+r(4R+r)^3}{4r^2p^3}=\dfrac{1}{4r^2p}\left[p^2+2r^2-4Rr+\dfrac{r(4R+r)^3}{p^2}\right]\le$

$$\le\frac{1}{4r^2p}\left[4R^2+4Rr+3r^2+2r^2-4Rr+\frac{r(4R+r)^3}{\frac{r(4R+r)}{R+r}}\right]=\frac{1}{4r^2p}[4R^2+5r^2+(4R+r)(R+r)]=$$
$$=\frac{8R^2+5Rr+6r^2}{4r^2p}=\frac{1}{p}\left(\frac{2R^2}{r^2}+\frac{5R}{4r}+\frac{3}{2}\right).$$

In the above inequality we've used $p^2\le 4R^2+4Rr+3r^2$ and $\dfrac{r(4R+r)^2}{R+r}\le p^2$

it follows from Gerretsen's inequality $16Rr-5r^2\le p^2$.

Equality holds if and only if the triangle is equilateral.

□

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