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TRIANGLE INEQUALITY - 524
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1) In ΔABC

$$\frac{bc}{r_b r_c}+\frac{ca}{r_c r_a}+\frac{ab}{r_a r_b}\geq 5-\frac{2r}{R}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We prove the following lemma:

Lemma 1.
2) In ΔABC

$$\frac{bc}{r_b r_c}+\frac{ca}{r_c r_a}+\frac{ab}{r_a r_b}=1+\left(\frac{4R+r}{p}\right)^2.$$

Proof.

Using the formula $r_a=\dfrac{S}{p-a}$ *we obtain:*

$$\sum\frac{bc}{r_b r_c}=\sum\frac{bc}{\frac{S}{p-b}\cdot\frac{S}{p-c}}=\frac{1}{S^2}\sum bc(p-b)(p-c)=\frac{1}{r^2p^2}\cdot r^2[p^2+(4R+r)^2]=1+\left(\frac{4R+r}{p}\right)^2$$

□

*Let's prove inequality **1)**.*

*Using **Lemma 1** inequality **1)** can be written:*

$1+\left(\dfrac{4R+r}{p}\right)^2\geq 5-\dfrac{2r}{R}\Leftrightarrow p^2\leq\dfrac{R(4R+r)^2}{2(2R-r)}$, *which is Blundon-Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

Remark.

Let's find an inequality having on opposite sense:

3) In ΔABC

$$\frac{bc}{r_b r_c}+\frac{ca}{r_c r_a}+\frac{ab}{r_a r_b}\leq 2+\frac{R}{r}$$

Proposed by Marin Chirciu - Romania

Proof.

*Using Lemma 1 inequality **3)** can be written:*

$$1+\left(\frac{4R+r}{p}\right)^2 \leq 2+\frac{R}{r} \Rightarrow p^2 \geq \frac{r(4R+r)^2}{R+r}$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr - 5r^2$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4) In ΔABC

$$\mathbf{5-\frac{2r}{R} \leq \frac{bc}{r_br_c}+\frac{ca}{r_cr_a}+\frac{ab}{r_ar_b} \leq 2+\frac{R}{r}.}$$

Proof.

*See inequalities **1)** and 3).*

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way we can propose:

5) In ΔABC

$$\mathbf{4 \leq \frac{bc}{h_bh_c}+\frac{ca}{h_ch_a}+\frac{ab}{h_ah_b} \leq 4\left(\frac{R}{r}\right)^2-\frac{3}{4}\cdot\frac{R}{r}+\frac{3}{2}}$$

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Proof.

We prove the following lemma:

Lemma 2.

6) In ΔABC

$$\mathbf{\frac{bc}{h_bh_c}+\frac{ca}{h_ch_a}+\frac{ab}{h_ah_b}=\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{4r^2p^2}}$$

Proof.

Using the formula $h_a=\dfrac{2S}{a}$ *we obtain:*

$$\sum\frac{bc}{h_bh_c}=\sum\frac{bc}{\frac{2S}{b}\cdot\frac{2S}{c}}=\frac{1}{4S^2}\sum b^2c^2=\frac{1}{4r^2p^2}[p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2]=$$

$$=\frac{p^4+p^2(2r^2-8Rr)+r^2(4R+r)^2}{4r^2p^2}.$$

□

*Let's prove the double inequality **5)**.*

*Using **Lemma 2** the left inequality from **5)** can be written:*

$$4 \le \frac{p^4 + p^2(2r^2 - 8Rr) + r^2(4R + r)^2}{4r^2p^2} \Leftrightarrow p^4 + p^2(2r^2 - 8Rr) + r^2(4R+r)^2 \ge 16r^2p^2 \Leftrightarrow$$

$$p^4 - p^2(14r^2 + 8Rr) + r^2(4R + r)^2 \ge 0 \Leftrightarrow p^2(p^2 - 14r^2 - 8Rr) + r^2(4R + r)^2 \ge 0.$$

We distinguish the following cases:

Case 1). If $p^2 - 14r^2 - 8Rr \ge 0$*, the inequality is obvious.*

Case 2). If $p^2 - 14r^2 - 8Rr < 0$*, inequality can be rewritten:*

$p^2(8Rr + 14r^2 - p^2) \le r^2(4R + r)^2$*, which follows from Gerretsen's inequality* $16Rr - 5r^2 \le p^2 \le 4R^2 + 4Rr + 3r^2$*. It remains to prove that:*

$$(4R^2 + 4Rr + 3r^2)(8Rr + 14r^2 - 16Rr + 5r^2) \le r^2(4R + r)^2 \Leftrightarrow$$

$$\Leftrightarrow (4R^2 + 4Rr + 3r^2)(19r - 8R) \le r(4R + r)^2 \Leftrightarrow 8R^3 - 7R^2r - 11Rr^2 - 14r^3 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (R - 2r)(8R^2 + 9Rr + 7r^2) \ge 0,$$ *obviously from Euler's inequality* $R \ge 2r$.

Equality holds if and only if the triangle is equilateral.

*Let's prove the right inequality from **5)**:*

We have

$$\frac{bc}{h_bh_c} + \frac{ca}{h_ch_a} + \frac{ab}{h_ah_b} = \frac{p^4 + p^2(2r^2 - 8Rr) + r^2(4R + r)^2}{4r^2p^2} =$$

$$= \frac{1}{4r^2}\left[p^2 + 2r^2 - 8Rr + \frac{r^2(4R + r)^2}{p^2}\right] \le \frac{1}{4r^2}\left[4R^2 + 4Rr + 3r^2 + 2r^2 - 8Rr + \frac{r^2(4R + r)^2}{\frac{r(4R+r)^2}{R+r}}\right] =$$

$$= \frac{1}{4r^2}(4R^2 - 4Rr + 5r^2 + r(R + r)) = \frac{4R^2 - 3Rr + 6r^2}{4r^2} = 4\left(\frac{R}{r}\right)^2 - \frac{3}{4} \cdot \frac{R}{r} + \frac{3}{2}.$$

In the above inequality we've used $p^2 \le 4R^2 + 4Rr + 3r^2$ *and* $p^2 \ge \frac{r(4R + r)^2}{R + r}$ *which follows from Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC

$$2 + \left(\frac{r}{R}\right)^2 \le \frac{h_bh_c}{bc} + \frac{h_ch_a}{ca} + \frac{h_ah_b}{ab} \le \frac{3r}{R}\left(2 - \frac{r}{R}\right).$$

Proposed by Marin Chirciu - Romania

Proof.

Let's prove the following lemma:

Lemma 3.
8) In ΔABC

$$\frac{h_bh_c}{bc} + \frac{h_ch_a}{ca} + \frac{h_ah_b}{ab} = \frac{p^2 - r^2 - 4Rr}{2R^2}.$$

Proof.

Using the formula $h_a = \dfrac{2S}{a}$ *we obtain:*

$$\sum \frac{h_b h_c}{bc} = \sum \frac{\frac{2S}{b} \cdot \frac{2S}{c}}{bc} = 4S^2 \sum \frac{1}{b^2c^2} = 4r^2p^2 \cdot \frac{\sum a^2}{a^2b^2c^2} =$$
$$= 4r^2p^2 \cdot \frac{2(p^2 - r^2 - 4Rr)}{16R^2r^2p^2} = \frac{p^2 - r^2 - 4Rr}{2R^2}$$

□

Let's prove the double inequality ***7)****.*

Using Lemma 3 the double inequality ***7)*** *can be written:*

$$2+\left(\frac{r}{R}\right)^2 \leq \frac{p^2 - r^2 - 4Rr}{2R^2} \leq \frac{3r}{R}\left(2 - \frac{r}{R}\right)$$

which follows from Gerretsen's inequality $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$.

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC

$$\boldsymbol{\frac{9r}{2R} \leq \frac{r_b r_c}{bc} + \frac{r_c r_a}{ca} + \frac{r_a r_b}{ab} \leq \frac{9}{4}}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 4.

10) In ΔABC

$$\boldsymbol{\frac{r_b r_c}{bc} + \frac{r_c r_a}{ca} + \frac{r_a r_b}{ab} = 2 + \frac{r}{2R}}$$

Proof.

Using the formula $r_a = \dfrac{S}{p-a}$ *we obtain:*

$$\sum \frac{r_b r_c}{bc} = \sum \frac{\frac{S}{p-b} \cdot \frac{S}{p-c}}{bc} = S^2 \sum \frac{1}{bc(p-b)(p-c)} = r^2p^2 \cdot \frac{4R+r}{2Rr^2p^2} = \frac{4R+r}{2R}.$$

□

Let's prove the double inequality ***9)****.*

Using ***Lemma 4*** *the double inequality* ***9)*** *can be written:*

$$\frac{9r}{2R} \leq 2 + \frac{r}{2R} \leq \frac{9}{4} \Leftrightarrow 2r \leq R \text{ (Euler's inequality).}$$

Equality holds if and only if the triangle is equilateral.

□

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