Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
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TRIANGLE INEQUALITY - 532
ROMANIAN MATHEMATICAL MAGAZINE 2017

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1) In ΔABC

$$\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}\geq\frac{1}{2Rr}$$

Proposed by Adil Abdullayev - Baku - Azerbaidian

Proof.

We prove the following lemma:

Lemma 1.
2) In ΔABC

$$\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}=\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2.$$

Proof.

Using the following formulas $\cos^2\frac{A}{2}=\frac{p(p-a)}{bc}$ *and* $r_a=\frac{S}{p-a}$ *we obtain:*

$$\sum\frac{\cos^2\frac{A}{2}}{r_a^2}=\sum\frac{\frac{p(p-a)}{bc}}{\frac{S^2}{(p-a)^2}}=\frac{p}{S^2}\sum\frac{(p-a)^3}{bc}=\frac{p}{r^2p^2}\cdot\frac{\sum a(p-a)^3}{abc}=$$

$$=\frac{1}{r^2p}\cdot\frac{4Rrp^2-2r^2(4R+r)^2}{4Rrp}=\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2.$$

Let's prove inequality ***1).***

Using ***Lemma 1*** *inequality* ***1)*** *becomes:*

$\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2\geq\frac{1}{2Rr}\Leftrightarrow p^2(2R-r)\geq r(4R+r)^2$, *which is true from*

Gerretsen's inequality $p^2\geq 16Rr-5r^2$. *It remains to prove that*

$(16Rr-5r^2)(2R-r)\geq r(4R+r)^2\Leftrightarrow 8R^2-17Rr+2r^2\geq 0\Leftrightarrow(R-2r)(8R-r)\geq 0$

obviously from Euler's inequality $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

□

Remark.

Let's find an inequality having an opposite sense:

3) In ΔABC

$$\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}\le\left(\frac{1}{R}-\frac{1}{r}\right)^2$$

Proposed by Marin Chirciu - Romania

Proof.

*Using **Lemma 1** inequality **3)** can be written:*

$$\frac{1}{r^2}-\frac{1}{2Rr}\left(\frac{4R+r}{p}\right)^2\le\left(\frac{1}{R}-\frac{1}{r}\right)^2\Leftrightarrow p^2\le\frac{R(4R+r)^2}{2(2R-r)}$$

(Blundon - Gerretsen's inequality)

Equality holds if and only if the triangle is equilateral.

□

Remark.

The double inequality can be written:

4. In ΔABC

$$\frac{1}{2Rr}\le\frac{\cos^2\frac{A}{2}}{r_a^2}+\frac{\cos^2\frac{B}{2}}{r_b^2}+\frac{\cos^2\frac{C}{2}}{r_c^2}\le\left(\frac{1}{R}-\frac{1}{r}\right)^2.$$

Proof.

*See inequalities **1)** and **3)**.*

□

Remark.

In the same way we can propose:

5) In ΔABC

$$\frac{1}{R^2p}\le\frac{\sin^2\frac{A}{2}}{r_a^2}+\frac{\sin^2\frac{B}{2}}{r_b^2}+\frac{\sin^2\frac{C}{2}}{r_c^2}\le\frac{1}{4r^2p}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2.
6) In ΔABC

$$\frac{\sin^2\frac{A}{2}}{r_a^2}+\frac{\sin^2\frac{B}{2}}{r_b^2}+\frac{\sin^2\frac{C}{2}}{r_c^2}=\frac{1}{2Rrp}$$

Proof.

Using the following formulas $\sin^2\frac{A}{2}=\frac{(p-b)(p-c)}{bc}$ *and* $r_a=\frac{S}{p-a}$ *we obtain:*

$$\sum\frac{\sin^2\frac{A}{2}}{r_a^2}=\sum\frac{\frac{(p-b)(p-c)}{bc}}{\frac{S^2}{(p-a)^2}}=\frac{\prod(p-a)}{S^2}\sum\frac{1}{bc}=\frac{r^2p}{r^2p^2}\cdot\frac{\sum a}{abc}=\frac{1}{p}\cdot\frac{2p}{4Rrp}=\frac{1}{2Rrp}.$$

□

Let's prove the double inequality ***5)****.*

Using **Lemma 2** *double inequality* ***5)*** *can be written:*

$$\frac{1}{R^2p}\le\frac{1}{2Rrp}\le\frac{1}{4r^2p}\Leftrightarrow 4r^2\le 2Rr\le R^2\Leftrightarrow 2r\le R\ \textit{(Euler's inequality)}.$$

Equality holds if and only if the triangle is equilateral.

□

7) In ΔABC

$$\frac{4}{9R^2}\le\frac{\tan^2\frac{A}{2}}{r_a^2}+\frac{\tan^2\frac{B}{2}}{r_b^2}+\frac{\tan^2\frac{C}{2}}{r_c^2}\le\frac{1}{9r^2}$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 3.
8) In ΔABC

$$\frac{\tan^2\frac{A}{2}}{r_a^2}+\frac{\tan^2\frac{B}{2}}{r_b^2}+\frac{\tan^2\frac{C}{2}}{r_c^2}=\frac{3}{p^2}$$

Proof.

Using the following formulas $\tan^2\frac{A}{2}=\frac{(p-b)(p-c)}{p(p-a)}$ *and* $r_a=\frac{S}{p-a}$ *we obtain:*

$$\sum\frac{\tan^2\frac{A}{2}}{r_a^2}=\sum\frac{\frac{(p-b)(p-c)}{p(p-a)}}{\frac{S^2}{(p-a)^2}}=\frac{\prod(p-a)}{S^2p}\sum 1=\frac{r^2p}{r^2p^3}\cdot 3=\frac{3}{p^2}.$$

□

Let's prove the double inequality ***7)****.*

Using Lemma 3 the double inequality ***7)*** *can be written:*

$$\frac{4}{9R^2}\le\frac{3}{p^2}\le\frac{1}{9r^2}\Leftrightarrow 27r^2\le p^2\le\frac{27R^2}{4}\ \textit{(Mitrinović's inequality)}.$$

Equality holds if and only if the triangle is equilateral.

□

9) In ΔABC

$$\frac{1}{r^2} \le \frac{\cot^2 \frac{A}{2}}{r_a^2} + \frac{\cot^2 \frac{B}{2}}{r_b^2} + \frac{\cot^2 \frac{C}{2}}{r_c^2} \le \frac{4R^2 - 10Rr + 5r^2}{r^4}$$

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Proof.

We prove the following Lemma:

10) In ΔABC

$$\frac{\cot^2 \frac{A}{2}}{r_a^2} + \frac{\cot^2 \frac{B}{2}}{r_b^2} + \frac{\cot^2 \frac{C}{2}}{r_c^2} = \frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2}$$

Proof.

Using the following formulas $\cot^2 \frac{A}{2} = \frac{p(p-a)}{(p-b)(p-c)}$ *and* $r_a = \frac{S}{p-a}$ *we obtain:*

$$\sum \frac{\cot^2 \frac{A}{2}}{r_a^2} = \sum \frac{\frac{p(p-a)}{(p-b)(p-c)}}{\frac{S^2}{(p-a)^2}} = \frac{p}{S^2} \sum \frac{(p-a)^3}{(p-b)(p-c)} = \frac{p}{r^2p^2} \cdot \frac{\sum (p-a)^4}{(p-a)(p-b)(p-c)} =$$

$$\frac{1}{r^2p} \cdot \frac{\sum (p-a)^4}{\prod (p-a)} = \frac{1}{r^2p} \cdot \frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^2p} = \frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2}.$$

□

*Let's prove the double inequality **9**.*

*Using **Lemma 3** the double inequality **9)** can be written:*

$$\frac{1}{r^2} \le \frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2} \le \frac{4R^2 - 10Rr + 5r^2}{r^4}.$$

The first inequality can be transformed equivalently:

$$\frac{1}{r^2} \le \frac{p^4 - 16rp^2 + 2r^2(4R+r)^2}{r^4p^2} \Leftrightarrow p^4 - 16Rrp^2 + 2r^2(4R+r)^2 \ge r^2p^2 \Leftrightarrow$$

$$\Leftrightarrow p^2(p^2 - 16Rr - r^2) + 2r^2(4R+r)^2 \ge 0.$$

We distinguish the following cases:

Case 1). If $p^2 - 16Rr - r^2 \ge 0$*, the inequality is equivalent.*

Case 2). If $p^2 - 16Rr - r^2 < 0$*, the inequality can be rewritten:*

$p^2(16Rr + r^2 - p^2) \le 2r^2(4R+r)^2$*, which follows from Gerretsen's inequality:*

$16Rr - 5r^2 \le p^2 \le 4R^2 + 4Rr + 3r^2$*. It remains to prove that:*

$$(4R^2 + 4Rr + 3r^2)(16Rr + r^2 - 16Rr + 5r^2) \le 2r^2(4R+r)^2 \Leftrightarrow R^2 - Rr - 2r^2 \ge 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(R+r) \ge 0$*, obviously from Euler's inequality* $R \ge 2r$*.*

Equality holds if and only if the triangle is equilateral.

Let's prove the second inequality.

We have $\frac{p^4 - 16Rrp^2 + 2r^2(4R+r)^2}{r^4p^2} = \frac{1}{r^4}\left[p^2 - 16Rr + \frac{2r^2(4R+r)^2}{p^2}\right] \le$

$$\le \frac{1}{r^4}\left[4R^2 + 4Rr + 3r^2 - 16Rr + \frac{2r^2(4R+r)^2}{\frac{r(4R+r)^2}{R+r}}\right] = \frac{1}{r^4}[4R^2 - 12Rr + 3r^2 + 2r(R+r)] =$$

$= \dfrac{4R^2 - 10Rr + 5r^2}{r^4}$, *where, above were used inequalities* $p^2 \leq 4R^2 + 4Rr + 4r^2$ *and*

$p^2 \geq \dfrac{r(4R+r)^2}{R+r}$, *true from Gerretsen's inequality.*

Equality holds if and only if the triangle is equilateral.

□

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