R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

If $a,b,c,d\in\mathbb{N}^*, 1\le a\le b\le c\le d$ then:

$$4\log_{a+1}a\le\log_{a+1}a+\log_{b+1}b+\log_{c+1}c+\log_{d+1}d\le 4\log_{d+1}d$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Le Van-Ho Chi Minh-Vietnam, Solution 2 by Priyanka Banerjee-India, Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

Solution 1 by Le Van-Ho Chi Minh-Vietnam

Put $f(x)=\frac{\ln x}{\ln(x+1)}, x\ge 1$

Then $f'(x)\cdot[\ln(x+1)]^2=\frac{\ln(x+1)}{x}-\frac{\ln x}{x+1}=$

$$=\frac{[(x+1)\ln(x+1)-x\ln(x)]}{x(x+1)}>0$$

Then $f(x)$ *is a positive function, which gives us*

$$4f(a)\le f(a)+f(b)+f(c)+f(d)\le 4f(d)$$

$\rightarrow$ *Q.E.D. Equality holds when* $a=b=c=d=1$.

Solution 2 by Priyanka Banerjee-India

Let $f(x)=\log_{x+1}x$

$$f(x)=\frac{\log_e x}{\log_e(x+1)}\Rightarrow f(x)=\frac{\frac{1}{x}\log_e(x+1)-\frac{1}{x+1}\log_e 2}{(\log_e(x+1))^2}$$

Now, $(x+1)\log_e(x+1)>x\log_e x$

Hence $f(x)>0$

As $a\le b\le c\le d$

$$\log_{a+1}a\le\log_{b+1}b\le\log_{c+1}c\le\log_{d+1}d$$

Hence $4\log_{a+1}a\le\log_{a+1}a+\log_{b+1}b+\log_{c+1}c+\log_{d+1}d\le 4\log_{d+1}d$

Solution 3 by Sanong Huayrerai-Nakon Pathom-Thailand

Because $1 \le a \le b \le c \le d, a, b, c, d \in N$***, we get***

(1) $\log_a b \ge \log_{a+1}(b+1), \log_a c \ge \log_{(a+1)}(c+1), \log_a d \ge \log_{(a+1)}(d+1)$

and (2) $\log_d a \le \log_{(d+1)}(a+1), \log_d b \le \log_{(d+1)}(b+1), \log_d c \le \log_{(d+1)}(c+1)$

from (1), we obtain

$$(\log_a b)(\log_a e)(\log_a d) \ge \left(\log_{(a+1)}(b+1)\right)\left(\log_{(a+1)}(c+1)\right)\left(\log_{(a+1)}(b+1)\right)$$

$$\Rightarrow \frac{(\log_a b)(\log_a c)(\log_a d)}{\left(\log_{(a+1)}(b+1)\right)\left(\log_{(a+1)}(c+1)\right)\left(\log_{(a+1)}(d+1)\right)} \ge 1$$

$$\Rightarrow \frac{\left(\log_{(a+1)} b\right)\left(\log_{(a+1)} e\right)\left(\log_{(a+1)} d\right)}{\left(\log_{(a+1)}(b+1)\right)\left(\log_{(a+1)}(c+1)\right)\left(\log_{(a+1)}(d+1)\right)} \ge (\log_{a+1} a)^3$$

$$\Rightarrow \sqrt[3]{\frac{(\log_{a+1} b)(\log_{a+1} e)(\log_{a+1} d)}{(\log_{a+1} b+1)(\log_{a+1} c+1)(\log_{a+1} d+1)}} \ge \log_{a+1} a$$

$$\Rightarrow 3\sqrt{\frac{(\log_{a+1} b)(\log_{a+1} c)(\log_{a+1} d)}{(\log_{a+1} b+1)(\log_{a+1} c+1)(\log_{a+1} d+1)}} \ge 3\log_{a+1} a$$

$$\to \frac{\log_{a+1} b}{\log_{a+1} b+1} + \frac{\log_{a+1} c}{\log_{a+1} c+1} + \frac{\log_{a+1} d}{\log_{a+1} d+1} \ge 3\log_{a+1} a$$

$$\to \log_{(b+1)} b + \log_{(c+1)} c + \log_{(d+1)} d \ge 3\log_{a+1} a$$

$$\to \log_{a+1} a + \log_{b+1} b + \log_{c+1} c + \log_{d+1} d \ge 4\log_{a+1} a$$

and (2) can show it such like this. Therefore it is to be true.