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If $a, b, c \geq 1$,

$$\Omega(a,b) = \int_a^{2a}\left(\int_b^{2b}\left(\left(\ln(x+y)-\ln x\right)\left(\ln(x+y)-\ln y\right)\right)dy\right)dx$$

then:

$$\frac{1}{\ln 2}\left(\Omega(a,b)+\Omega(b,c)+\Omega(c,b)\right) < \ln 2^{a^2+b^2+c^2}$$

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Solution by Dimitris Kastrotis-Athens-Greece

Let $f(t) = \ln(t+1) - t\ln 2,\ f'(t) = \frac{1}{t+1} - \ln 2,\ f''(t) = -\frac{1}{(t+1)^2} < 0,\ t \geq 1$

$$f' - decreasing,\ f'(t) \leq f'(1) = \frac{1}{2} - \ln 2 < 0 \to f - decreasing$$

$$\underbrace{max}_{t\in[0,\infty)} f(t) = f(1) = 0 \to \ln(t+1) - t\ln 2 \leq 0 \to \ln(t+1) \leq t\ln 2$$

Equality holds for $t = 1$

$$[\ln(x+y) - \ln x]\cdot[\ln(x+y) - \ln y] = \ln\left(\frac{x+y}{x}\right)\cdot \ln\left(\frac{x+y}{y}\right) =$$

$$= \ln\left(1+\frac{y}{x}\right)\ln\left(1+\frac{x}{y}\right) \leq \frac{x}{y}\cdot \ln 2\cdot \frac{y}{x}\cdot \ln 2 = \ln^2 2$$

$$\Omega(a,b) = \int_a^{2a}\left(\int_b^{2b}\left(\left(\ln(x+y)-\ln x\right)\left(\ln(x+y)-\ln y\right)\right)dy\right)dx \leq$$

$$\leq \int_a^{2a}\left(\int_b^{2b}(\ln^2 2)dy\right)dx = \ln^2 2\cdot ab$$

$$\frac{1}{\ln 2}\left(\Omega(a,b)+\Omega(b,c)+\Omega(c,b)\right) < \ln 2\cdot(ab+bc+ca) \leq$$

$$\leq (a^2+b^2+c^2)\ln 2 = \ln 2^{a^2+b^2+c^2}$$