R M M

ROMANIAN MATHEMATICAL MAGAZINE
www.ssmrmh.ro

If $a,b,c\in\mathbb{R}$, $abc\ge 0$, $a+b+c=3$ then:

$$a^2+b^2+c^2+abc\ge ab+bc+ca+1$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by proposer

Let $x=a(b-1)(c-1), y=b(c-1)(a-1), z=c(a-1)(b-1)$ we have

$xyz=abc(a-1)^2(b-1)^2(c-1)^2\ge 0$ at least one of x, y, z is nonnegative; assume that

$x\ge 0\Leftrightarrow a(b-1)(c-1)\ge 0\Leftrightarrow abc\ge ab+ac-a$.

* If $a\le 0$: since $abc\ge 0\Rightarrow bc\le 0$ and since $a+b+c=3\Rightarrow b+c\ge 3$. Hence

$ab+bc+ca+1\le ab+ac+1=a(b+c)+1\le 1<a^2+b^2+c^2+abc$

(because $a\le 0$ and $b+c\ge 3$)

* If $a>0$: Since $a(b-1)(c-1)\ge 0$

$\Rightarrow(b-1)(c-1)\ge 0\Leftrightarrow bc\ge b+c-1\Rightarrow a+bc\ge a+b+c-1=2$ (*)

We have $a^2+b^2+c^2+abc\ge a^2+b^2+c^2+ab+ac-a$

$a^2+b^2+c^2+ab+ac-a\ge ab+bc+ca+1\Leftrightarrow(a^2-2a+1)+(b^2-2bc+c^2)+(a+bc-2)\ge 0$

$\Leftrightarrow(a-1)^2+(b-c)^2+(a+bc-2)\ge 0$

It's true by $(a-1)^2\ge 0,(b-c)^2\ge 0$ and (*). □