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If $a, b, c > 0$ then:

$$\frac{(b+c-a)^2}{(b+c)^2+a^2}+\frac{(c+a-b)^2}{(c+a)^2+b^2}+\frac{(a+b-c)^2}{(a+b)^2+c^2}\geq\frac{3}{5}$$

JAPAN TST

Solution 1 by Soumava Chakraborty-Kolkata-India, Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\sum\frac{(b+c-a)^2}{(b+c)^2+a^2}=\sum\left(1-\frac{2a(b+c)}{(b+c)^2+a^2}\right)=3-2\sum\frac{2a(b+c)}{(b+c)^2+a^2}\geq\frac{3}{5}\leftrightarrow$$

$$\leftrightarrow\sum\frac{2a(b+c)}{(b+c)^2+a^2}\leq\frac{6}{5}\leftrightarrow$$

$$\leftrightarrow 6\prod[(b+c)^2+a^2]-5\sum a(b+c)[(c+a)^2+b^2][(a+b)^2+c^2]\geq 0\leftrightarrow$$

$$\leftrightarrow 3\sum a^6+\sum\left(a^5b+ab^5\right)+3abc\sum a^3+2\sum a^3b^3+12a^2b^2c^2\geq$$

$$\geq\sum(a^4b^2+a^2b^4)+6abc\sum(a^2b+ab^2)\ (to\ prove)(i)$$

$$\sum\left(a^5b+ab^5\right)\geq\sum ab(a^4+b^4)=\frac{1}{2}\sum ab(a^2+b^2)^2\geq\sum a^2b^2(a^2+b^2)=$$

$$=\sum(a^4b^2+a^2b^4)\ (1)$$

$$3abc\sum a^3+9a^2b^2c^2\overset{SCHUR}{\geq}3abc\sum(a^2b+ab^2)$$

$$abc\sum a^3+3a^2b^2c^2\overset{SCHUR}{\geq}abc\sum(a^2b+ab^2)\ (2)$$

$$3\sum a^6+\sum a^3b^3\geq 4\sum(a^2b+ab^2)\ (a)$$

By adding: $\sum x^3+3xyz\geq\sum(x^2y+xy^2)$, $\sum x^3\geq 3xyz\rightarrow 2\sum x^3\geq\sum(x^2y+xy^2)$

$$4\sum a^3b^3\geq 2abc\sum(a^2b+ab^2)\rightarrow 3\sum a^6+\sum a^3b^3\geq 2abc\sum(a^2b+ab^2)(3)$$

By adding $(1),(2),(3)\rightarrow(i)$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a+b+c=p>0$

$$\sum\frac{(b+c-a)^2}{(b+c)^2+a^2}\geq\frac{3}{5}\leftrightarrow\sum\frac{(p-2a)^2}{(p-2a)^2+a^2}\geq\frac{3}{5}$$

By Tangent-Line Method:

$$\frac{(p-2x)^2}{(p-2x)^2+a^2}\geq\frac{23}{25}-\frac{54x}{25p}\leftrightarrow(p-3x)^2(p+6x)\geq 0$$

$$\sum\frac{(b+c-a)^2}{(b+c)^2+a^2}\geq\frac{69}{25}-\frac{54}{25p}(a+b+c)=\frac{3}{5}$$