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In $\Delta\, ABC$:

$$\prod\left(\frac{b^2}{\sin^2\frac{B}{2}}+\frac{c^2}{\sin^2\frac{C}{2}}-\frac{a^2}{\sin^2\frac{A}{2}}\right)\le\prod\left(\frac{a}{\sin\frac{A}{2}}\right)^2$$

*Proposed by Daniel Sitaru – Romania*

*Solution by Soumava Chakraborty-Kolkata-India*

$$\frac{b}{\sin\frac{B}{2}}=\frac{2R\cdot 2\sin\frac{B}{2}\cos\frac{B}{2}}{\sin\frac{B}{2}}\overset{(1)}{=}4\,R\cos\frac{B}{2}$$

*Similarly,* $\frac{c}{\sin\frac{C}{2}}\overset{(2)}{=}4R\cos\frac{C}{2}$ *and* $\frac{a}{\sin\frac{A}{2}}\overset{(3)}{=}4R\cos\frac{A}{2}$

*(1), (2), (3)* $\Rightarrow \frac{b^2}{\sin^2\frac{B}{2}}+\frac{c^2}{\sin^2\frac{C}{2}}-\frac{a^2}{\sin^2\frac{A}{2}}$

$$=8R^2\left(2\cos^2\frac{B}{2}+2\cos^2\frac{C}{2}-2\cos^2\frac{A}{2}\right)$$

$$=8R^2(1+\cos B+1+\cos C-1-\cos A)$$

$$=8R^2\left(1+2\cos\frac{B+C}{2}\cos\frac{B-C}{2}-1+2\sin^2\frac{A}{2}\right)$$

$$=8R^2\cdot 2\sin\frac{A}{2}\left(\cos\frac{B+C}{2}+\cos\frac{B-C}{2}\right)$$

$$\overset{(a)}{=}32R^2\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$$

*Similarly,* $\frac{c^2}{\sin^2\frac{C}{2}}+\frac{a^2}{\sin^2\frac{A}{2}}-\frac{b^2}{\sin^2\frac{B}{2}}\overset{(b)}{=}32R^2\sin\frac{B}{2}\cos\frac{C}{2}\cos\frac{A}{2}$

*and,* $\frac{a^2}{\sin^2\frac{A}{2}}+\frac{b^2}{\sin^2\frac{B}{2}}-\frac{c^2}{\sin^2\frac{C}{2}}\overset{(c)}{=}32R^2\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{B}{2}$

*(a), (b), (c)* $\Rightarrow LHS=32^3R^6\left(\prod\sin\frac{A}{2}\right)\left(\prod\cos\frac{A}{2}\right)^2$ *(i)*

$$RHS = \prod \left(4R\cos\frac{A}{2}\right)^2 \overset{(ii)}{=} 16^3R^6\left(\pi\cos\frac{A}{2}\right)^2$$

$\therefore$ ***given inequality becomes:***

$$8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} \leq 1 \quad \textit{(using (i), (ii))}$$

$$\Leftrightarrow 2\frac{r}{R} \leq 1 \Leftrightarrow R \geq 2r \rightarrow \textit{true (Euler)}$$

***(Proved)***