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MARIN CHIRCIU

Problem PP 26038 Octogon Mathematical Magazine
1) In ΔABC

$$\sum (m_a + m_b)(m_b + m_c) \leq \frac{1}{2}(11p^2 - 9r^2 - 36Rr).$$

Proposed by Mihály Bencze - Romania

Proof.

Using $\sum m_a^2 = \frac{3}{4}\sum a^2, 4m_b m_c \leq 2a^2 + bc, \sum a^2 = 2(p^2 - r^2 - 4Rr)$ *and*

$\sum bc = p^2 + r^2 + 4Rr$ *we obtain:*

$$\sum (m_a+m_b)(m_b+m_c) = \sum m_a^2 + 3\sum m_b m_c \leq \frac{3}{4}\sum a^2 + \frac{3}{4}\sum (2a^2+bc) = \frac{3}{4}(3\sum a^2 + \sum bc) =$$

$$= \frac{3}{4}\Big[6(p^2 - r^2 - 4Rr) + p^2 + r^2 + 4Rr\Big] = \frac{3}{4}(7p^2 - 5r^2 - 20Rr).$$

The inequality we have to prove can be written:

$$\frac{3}{4}(7p^2 - 5r^2 - 20Rr) \leq \frac{1}{2}(11p^2 - 9r^2 - 36Rr) \Leftrightarrow p^2 \geq 3r(4R + r)$$

which follows from Gerretsen's inequality: $p^2 \geq 16Rr - 5r^2$

It remains to prove that: $16Rr - 5r^2 \geq 3r(4R + r) \Leftrightarrow R \geq 2r$,

obviously from Euler's inequality.

Equality holds if and only if the triangle is equilateral.

□

Remark 1.

*Inequality **1)** can be developed:*

2) In ΔABC

$$\sum (m_a + \lambda m_b)(m_b + \lambda m_c) \leq \frac{\lambda + 1}{4}\Big[(5\lambda + 6)p^2 - (\lambda + 2) \cdot 3r(4R + r)\Big]$$

where $\lambda \in \mathbb{R}$.

Proposed by Marin Chirciu - Romania

Proof.

$$\sum(m_a+\lambda m_b)(m_b+\lambda m_c)=\lambda\sum m_a^2+(\lambda^2+\lambda+1)\sum m_bm_c\leq$$
$$\leq\lambda\cdot\frac{3}{4}\sum a^2+(\lambda^2+\lambda+1)\cdot\frac{1}{4}\sum(2a^2+bc)=$$
$$=\frac{1}{4}\Big[(2\lambda^2+5\lambda+2)\sum a^2+(\lambda^2+\lambda+1)\sum bc\Big]=$$
$$=\frac{1}{4}\Big[(2\lambda^2+5\lambda+2)\cdot 2(p^2-r^2-4Rr)+(\lambda^2+\lambda+1)(p^2+r^2+4Rr)\Big]=$$
$$=\frac{1}{4}\Big[(5\lambda^2+11\lambda+5)p^2-(\lambda^2+3\lambda+1)\cdot 3r(4R+r)\Big].$$

The inequality we have to prove can be written:

$$\frac{1}{4}\Big[(5\lambda^2+11\lambda+5)p^2-(\lambda^2+3\lambda+1)\cdot 3r(4R+r)\Big]\leq$$
$$\leq\frac{\lambda+1}{4}\Big[(5\lambda+6)p^2-(\lambda+2)\cdot 3r(4R+r)\Big]\Leftrightarrow$$

$\Leftrightarrow p^2\geq 3r(4R+r)$, *which follows from* $p^2\geq 16Rr-5r^2$ *(Gerretsen) and* $R\geq 2r$ *(Euler).*

Equality holds if and only if the triangle is equilateral.

□

Note

For $\lambda=1$ *we obtain inequality* ***1)****.*

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3) In ΔABC

$$\sum\frac{1}{(m_a+m_b)^2}\geq\frac{18}{11p^2-9r^2-36Rr}.$$

Proposed by Mihály Bencze - Romania

Proof.

We use the inequality $x^2+y^2+z^2\geq xy+yz+zx$*, for* $x=\frac{1}{m_a+m_b}$, $y=\frac{1}{m_b+m_c}$,

$z=\frac{1}{m_c+m_a}$ *and inequality* ***1)****. We obtain:*

$$\sum\frac{1}{(m_a+m_b)^2}\geq\sum\frac{1}{(m_a+m_b)(m_b+m_c)}\geq\frac{9}{\sum(m_a+m_b)(m_b+m_c)}\geq$$
$$\geq\frac{9}{\frac{1}{2}(11p^2-9r^2-36Rr)}=\frac{18}{11p^2-9r^2-36Rr}.$$

Equality holds if and only if the triangle is equilateral.

□

Remark 2.

Inequality ***3)*** *can be developed:*

4) In ΔABC

$$\sum\frac{1}{(m_a+\lambda m_b)^2}\geq\frac{36}{(\lambda+1)[(5\lambda+6)p^2-(\lambda+2)\cdot 3r(4R+r)]},\text{ where }\lambda\geq 0.$$

Proposed by Marin Chirciu - Romania

Proof.

We use the inequality $x^2 + y^2 + z^2 \geq xy + yz + zx$,

for $x = \dfrac{1}{m_a + \lambda m_b}, y = \dfrac{1}{m_b + \lambda m_c}, z = \dfrac{1}{m_c + \lambda m_a}$ *and inequality **2)**. We obtain:*

$$\sum \frac{1}{(m_a + \lambda m_b)^2} \geq \sum \frac{1}{(m_a + \lambda m_b)(m_b + \lambda m_c)} \geq \frac{9}{\sum (m_a + \lambda m_b)(m_b + \lambda m_c)} \geq$$

$$\geq \frac{9}{\frac{\lambda+1}{4}[(5\lambda + 6)p^2 - (\lambda + 2) \cdot 3r(4R + r)]} = \frac{36}{(\lambda + 1)[(5\lambda + 6)p^2 - (\lambda + 2) \cdot 3r(4R + r)]}.$$

Equality holds if and only if the triangle is equilateral.

□

Note

For $\lambda = 1$ *we obtain inequality **3)**.*

Mathematics Department, "Theodor Costescu" National Economic College, Drobeta Turnu - Severin, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com