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1) In ΔABC

$$\sum r_a(r_b - r_c)^2 \geq \frac{2p^2(p^2 - 3r^2 - 12Rr)}{4R + r}$$

Proposed by Mihály Bencze - Romania

Proof.

We prove the following lemma:

Lemma 1.
2) In ΔABC

$$\sum r_a(r_b - r_c)^2 = 4p^2(R - 2r).$$

Proof.

We have

$$\sum r_a(r_b - r_c)^2 = \sum r_a(r_b^2 + r_c^2 - 2r_b r_c) = \sum r_a(r_b^2 + r_c^2) - 6r_a r_b r_c =$$

$$= \sum r_a(r_a^2 + r_b^2 + r_c^2 - r_a^2) - 6r_a r_b r_c =$$

$$= \sum r_a \sum r_a^2 - \sum r_a^3 - 6r_a r_b r_c = (4R+r)\Big[(4R+r)^2 - 2p^2\Big] - \Big[(4R+r)^3 - 12Rrp^2\Big] = 4p^2(R-2r)$$

□

Let's solve the inequality in the statement.

*Using **Lemma 1** the inequality can be written:*

$$4p^2(R - 2r) \geq \frac{2p^2(p^2 - 3r^2 - 12Rr)}{4R + r} \Leftrightarrow p^2 \leq 8R^2 - 2Rr - r^2$$

which follows from Gerretsen's inequality: $p^2 \leq 4R^2 + 4Rr + 3r^2$. *It remains to prove that:*

$$4R^2 + 4Rr + 3r^2 \leq 8R^2 - 2Rr - r^2 \Leftrightarrow 2R^2 - 3Rr - 2r^2 \geq 0 \Leftrightarrow (R - 2r)(2R + r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark 1.

The inequality can be developed:

3) In ΔABC

$$\sum r_a(r_b-r_c)^2 \geq \frac{np^2(p^2-3r^2-12Rr)}{4R+r}, \text{ where } n \leq 4.$$

Proof.

If $n \leq 0$ the inequality is immediate because $p^2-3r^2-12Rr \geq 0$ true from Gerretsen's inequality: $p^2 \geq 16Rr-5r^2$ and Euler's inequality $R \geq 2r$.

Next we consider $n > 0$.

*Using **Lemma 1** we write the inequality:*

$$4p^2(R-2r) \geq \frac{np^2(p^2-3r^2-12Rr)}{4R+r} \Leftrightarrow np^2 \leq 16R^2+(12n-28)Rr+(3n-8)r^2$$

which follows from Gerretsen's inequality: $p^2 \leq 4R^2+4Rr+3r^2$ and the condition $n > 0$.

It remains to prove that:

$$n(4R^2+4Rr+3r^2) \leq 16R^2+(12n-28)Rr+(3n-8)r^2 \Leftrightarrow (4-n)R^2+(2n-7)Rr-2r^2 \geq 0.$$

$\Leftrightarrow (R-2r)\Big[(4-n)R+r\Big] \geq 0$ *obviously from Euler's inequality $R \geq 2r$ and the condition $n \leq 4$.*

Equality holds if and only if the triangle is equilateral.

□

Note

*For $n = 2$ we obtain inequality **1)**.*

Remark 2.

*The best inequality having the form of **3)** it's obtained for $n = 4$:*

4) In ΔABC

$$\sum r_a(r_b-r_c)^2 \geq \frac{4p^2(p^2-3r^2-12Rr)}{4R+r} \geq \frac{np^2(p^2-3r^2-12Rr)}{4R+r}$$

Proof.

*We use inequality **3)** for $n = 4$ and* $\frac{4p^2(p^2-3r^2-12Rr)}{4R+r} \geq \frac{np^2(p^2-3r^2-12Rr)}{4R+r}$, *true from $p^2-3r^2-12Rr \geq 0$ and the condition $n \leq 4$.*

Equality holds if and only if the triangle is equilateral.

□

Remark 3.

*Inequality **3)** can also be developed:*

5) In ΔABC

$$\sum r_a(r_b-r_c)^2 \geq \frac{np^2(p^2+(2\lambda-27)r^2-\lambda Rr)}{4R+r}, \text{ where } n \leq 4 \text{ and } \lambda \geq 11.$$

Proposed by Marin Chirciu - Romania

Proof.

If $n \le 0$ *the inequality is immediate because* $p^2 + (2\lambda - 27)r^2 - \lambda Rr \ge 0$
true from Gerretsen's inequality: $p^2 \ge 16Rr - 5r^2$, *Euler's inequality* $R \ge 2r$
and the condition $\lambda \ge 11$.
Next we consider $n > 0$.
Using ***Lemma 1*** *we write the inequality:*

$$4p^2(R-2r) \ge \frac{np^2(p^2+(2\lambda-27)r^2-\lambda Rr)}{4R+r} \Leftrightarrow$$

$$\Leftrightarrow 4(R-2r)(4R+r) \ge n(p^2+(2\lambda-27)r^2-\lambda Rr)$$

which follows from Gerretsen's inequality: $p^2 \le 4R^2+4Rr+3r^2$ *and the condition* $n > 0$.
It remains to prove that:

$$4(R-2r)(4R+r) \ge n(4R^2+4Rr+3r^2+(2\lambda-27)r^2-\lambda Rr) \Leftrightarrow$$

$$\Leftrightarrow (16-4n)R^2+(\lambda n-4n-28)Rr+(24-2\lambda n-8)r^2 \ge 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)\Big[(16-4n)R+(4+\lambda n-12n)r\Big] \ge 0$$ *obviously from Euler's inequality* $R \ge 2r$
and the conditions $n \le 4, \lambda \ge 11$.
Equality holds if and only if the triangle is equilateral.

□

Note

For $n = 2$ *and* $\lambda = 12$ *we obtain inequality* ***1)****, and for* $\lambda = 16$ *we obtain inequality* ***5).***

Remark 4.

The best inequality having the form of ***5)*** *we obtain for* $n = 4$ *and* $\lambda = 11$*:*

6) In ΔABC

$$\sum r_a(r_b-r_c)^2 \ge \frac{4p^2(p^2-5r^2-11Rr)}{4R+r} \ge \frac{np^2(p^2+(2\lambda-27)r^2-\lambda Rr)}{4R+r}$$

where $n \le 4$ and $\lambda \ge 11$.

Proof.

We use inequality ***5)*** *for* $n = 4$ *and* $\lambda = 11$ *and*

$$\frac{4p^2(p^2-5r^2-11Rr)}{4R+r} \ge \frac{np^2(p^2+(2\lambda-27)r^2-\lambda Rr)}{4R+r}$$ *is true from the condition* $n \le 4$
and $p^2 - 5r^2 - 11Rr \ge p^2 + (2\lambda-27)r^2 - \lambda Rr \Leftrightarrow (\lambda - 11)(R-2r) \ge 0$,
and the condition $\lambda \ge 11$.
Equality holds if and only if the triangle is equilateral.

□

Remark 5.

In the same way we can propose:

7) In ΔABC

$$\sum a(b-c)^2 \ge nS(R-2r), \text{ where } n \le 4.$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the following lemma:

Lemma 2
8) In ΔABC

$$\sum a(b-c)^2 = 2p(p^2+r^2-14Rr).$$

Proof.

We have

$$\sum a(b-c)^2 = \sum a(b^2+c^2-2bc) = \sum a(b^2+c^2)-6abc = \sum a(a^2+b^2+c^2-a^2)-6abc =$$
$$= \sum a \sum a^2 - \sum a^3 - 6abc = 2p\cdot 2(p^2-r^2-4Rr) - 2p(p^2-3r^2-6Rr) - 6\cdot 4Rrp =$$
$$= 2p(p^2+r^2-14Rr).$$

□

Let's solve the proposed inequality.

*Using **Lemma 2** we write the inequality:*

$2p(p^2+r^2-14Rr) \geq nrp(R-2r)$, *which follows from Gerretsen's inequality:* $p^2 \geq 16Rr-5r^2$

It remains to prove that:

$4r(R-2r) \geq nr(R-2r) \Leftrightarrow (4-n)(R-2r) \geq 0$, *obviously from Euler's inequality* $R \geq 2r$ *and the condition* $n \leq 4$.

Equality holds if and only if the triangle is equilateral.

□

Remark 6.

*The best inequality having the form of **7)** it's obtained for* $n = 4$:

9) In ΔABC

$$\sum a(b-c)^2 \geq 4S(R-2r) \geq nS(R-2r), \text{ where } n \leq 4.$$

Proof.

*See inequality **7)** for* $n = 4$, *and* $4S(R-2r) \geq nS(R-2r) \Leftrightarrow (4-n)(R-2r) \geq 0$,

obviously from $n \leq 4$ *and* $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

10) In ΔABC

$$\sum h_a(h_b-h_c)^2 \geq \frac{nS^2(R-2r)}{R^2}, \text{ where } n \leq 2.$$

Proposed by Marin Chirciu - Romania

Proof.

We prove the followin lemma:

Lemma 3.
11) In ΔABC

$$\sum h_a(h_b-h_c)^2=\frac{rp^2(p^2+r^2-14Rr)}{R^2}$$

Proof.

We have:

$$\sum h_a(h_b-h_c)^2=\sum h_a(h_a^2+h_c^2-2h_bh_c)=\sum h_a(h_b^2+h_c^2)-6h_ah_bh_c=$$
$$=\sum h_a(h_a^2+h_b^2+h_c^2-h_a^2)-6abc=$$
$$=\sum h_a\sum h_a^2-\sum h_a^3-6h_ah_bh_c=\frac{rp^2(p^2+r^2-14Rr)}{R^2}, \text{ the last equality follows from:}$$
$$\sum h_a=\frac{p^2+r^2+4Rr}{2R}, \sum h_a^2=\left(\sum h_a\right)^2-2\sum h_bh_c, \sum h_bh_c=\frac{2rp^2}{R}$$
$$\sum h_a^3=\left(\sum h_a\right)^3-3\prod(h_b+h_c), \prod(h_b+h_c)=\frac{rp^2(p^2+r^2+4Rr)}{R^2}$$

□

Let's solve the proposed inequality.

*Using **Lemma 3** we write the inequality:*

$$\frac{rp^2(p^2+r^2-14Rr)}{R^2}\geq\frac{nr^2p^2(R-2r)}{R^2}\Leftrightarrow p^2+r^2-14Rr\geq nr(R-2r)$$

which follows from Gerretsen's inequality: $p^2\geq 16Rr-5r^2$*. It remains to prove that:* $16Rr-5r^2+r^2-14Rr\geq nr(R-2r)\Leftrightarrow 2r(R-2r)\geq nr(R-2r)\Leftrightarrow(2-n)(R-2r)\geq 0$,

obviously from Euler's inequality $R\geq 2r$ *and the condition* $n\leq 2$.

Equality holds if and only if the triangle is equilateral.

□

Remark 7.

*The best inequality having the form of **10)** it's obtained for* $n=2$:

12. In ΔABC

$$\sum h_a(h_b-h_c)^2\geq\frac{2S^2(R-2r)}{R^2}\geq\frac{nS^2(R-2r)}{R^2}, \text{ where } n\leq 4.$$

Proof.

*See inequality **10)** for* $n=2$, *and* $\frac{2S^2(R-2r)}{R^2}\geq\frac{nS^2(R-2r)}{R^2}\Leftrightarrow(2-n)(R-2r)\geq 0$,

obviously from $n\leq 2$ *and* $R\geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

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