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MATH ADVENTURES
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151 - 200

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151. An Inequality with Two Cyclic Sums

If $a, b, c > 0$, and $abc = 1$ then

$$\sum_{cylc}(a + \sqrt[3]{a} + \sqrt[3]{a^2}) \geq 9 \sum_{cycl} \frac{1}{1 + \sqrt[3]{b^2} + \sqrt[3]{c}}.$$

Proposed by Daniel Sitaru

Solution (by Nguyen Tien Lam).

Let $\sqrt[3]{a} = x$, $\sqrt[3]{b} = y$, $\sqrt[3]{c} = z$. Then still $xyz = 1, x, y, z > 0$.

By the Cauchy - Schwarz inequality,

$$(1 + y^2 + z)(x^2 + 1 + z) \geq (x + y + z)^2 \geq 3^2 = 9,$$

implying $\frac{9}{1+y^2+z} \leq x^2 + 1 + z$. Similarly, $\frac{9}{1+z^2+x} \leq y^2 + 1 + x$ and $\frac{9}{1+x^2+y} \leq z^2 + 1 + y$. Adding the three gives,

$$\begin{aligned} 9 \sum_{cycl} \frac{1}{1 + y^2 + x} &\leq 3 + \sum_{cycl} x + \sum_{cycl} x^2 \\ &\leq \sum_{cycl} x^3 + \sum_{cycl} x + \sum_{cycl} x^2 \\ &= \sum_{cycl} a + \sum_{cycl} \sqrt[3]{a} + \sum_{cycl} a^2 \\ &= \sum_{cycl} (a + \sqrt[3]{b^2} + c) \end{aligned}$$

and this is the required inequality. □

Acknowledgment (by Alexander Bogomolny)

This problem from the ***Romanian Mathematical Magazine*** has been kindly posted at ***CutTheKnotMath facebook page*** by Daniel Sitaru. The above solution is by Nguyen Tien Lam.

152. An Inequality with Constraint in Four Variables IV

If $a, b, c, d > 0$ and $a + b + c + d = 3$ then

$$27 + 3(abc + bcd + cda + dab) \geq \sum_{cycl} a^3 + 54\sqrt{abcd}$$

Proposed by Daniel Sitaru

Solution 1 (by Kevin Soto Palacios).

$(x + y)^3 = x^3 + y^3 + 3xy(x + y)$. With $x = a + b$ and $y = c + d$ we have

$$\left(\sum_{cycl} a\right)^3 = (a + b)^3 + (c + d)^3 + 3(a + b)(c + d) \sum_{cycl} a \Leftrightarrow$$

$$27 = \sum_{cycl} a^3 + 3ab(a + b) + 3cd(c + d) + 9(a + b)(c + d) \Leftrightarrow$$

$$27 = \sum_{cycl} a^3 + 3ab(a + b) + 3cd(c + d) + 9(ac + ad + bc + bd) \Leftrightarrow$$

$$27 + 3 \sum_{cycl} abc = \sum_{cycl} a^3 + 3ab \sum_{cycl} a + 3cd \sum_{cycl} a + 9(ac + ad + bc + bd) \Leftrightarrow$$

$$27+3\sum_{cycl} abc=\sum_{cycl} a^3+9\sum_{all} ab$$

Suffice it to show that $9\sum_{cycl} ab \ge 54\sqrt{abcd}$. But this is true by the ***AM-GM inequality***. □

Solution 2 (by Leonard Giugiuc).

Denote $s_3=\sum_{cycl} abc$ and $s_3=\sum_{cycl} a^3$ and homogenise using the constraint:

$$\left(\sum_{cycl} a\right)^3+3s_3 \ge S_3+18\left(\sum_{cycl} a\right)\sqrt{abcd}$$

Now, recollect one of Newton's identities:

$$S_3=\left(\sum_{cycl} a\right)\left(\sum_{cycl} a^2\right)-\left(\sum_{cycl} a\right)\left(\sum_{all} ab\right)+3s_3$$

This leads to an equivalent inequality:

$$\left(\sum_{cycl} a\right)^3+\left(\sum_{cycl} a\right)\left(\sum_{cycl} ab\right)-\left(\sum_{cycl} a\right)\left(\sum_{cycl} a^2\right)\ge 18\left(\sum_{cycl} a\right)\sqrt{abcd}$$

This is equivalent to

$$\left(\sum_{cycl} a\right)\left[3\left(\sum_{all} ab\right)\right]\ge 18\left(\sum_{cycl} a\right)\sqrt{abcd}$$

and, finally, to

$$\sum_{all} ab \ge 6\sqrt{abcd}$$

which is true by the AM − GM inequality.

Note the equality is reached for $a=b=c=d=\frac{3}{4}$ or $(3,0,0,0)$ and permutations. □

Acknowledgment (by Alexander Bogomolny)

This problem from the ***Romanian Mathematical Magazine*** has been kindly posted by Daniel Sitaru at the ***CutTheKnotMath facebook page***. Solution 1 is by Kevin Soto Palacios; Solution 2 is by Leo Giugiuc.

153. Dan Sitaru's Inequality by Induction

Prove that if $a,b,c,d\ge 0$ then

$$\frac{3}{a+1}+\frac{3}{b+1}+\frac{2}{c+1}+\frac{1}{d+1}\le 6+\frac{1}{a+b+1}+\frac{1}{a+b+c+1}+\frac{1}{a+b+c+d+1}$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Let $P(n)$ be the statement:

$$\textit{If } x_k\ge 1, k=1,\ldots,n, \textit{ then}$$

$$x_1+x_2+\ldots+x_n\le n-1+x_1x_2\ldots x_n$$

We prove $P(n)$ by ***induction***.

For $n=1, x_1\ge 1-1+x_1$, which is obviously true.

For $n=2, x_1+x_2\le 2-1+x_1x_2$ is equivalent to $(x_1-1)(x_2-1)\ge 0$,

which is true because $x_1, x_2 \geq 1$.
Now assume that for $n = k \geq 2, P(k)$ is true and let there be numbers $x_1, \ldots, x_k, x_{k+1} \geq 1$. We wish to prove that $P(k+1)$ is true, i.e.,

$$x_1 + x_2 + \ldots + x_k + x_{k+1} \leq k + x_1 x_2 \ldots x_k x_{k+1}$$

holds. By the inductive assumption.

$$x_1 + x_2 + \ldots + x_k \leq k - 1 + x_1 x_2 \ldots x_k$$

and, therefore,

$$x_1 + x_2 + \ldots + x_k + x_{k+1} \geq k - 1 + x_1 x_2 \ldots x_k + x_{k+1}.$$

Suffice it to prove that

$$k - 1 + x_1 x_2 \ldots x_k + x_{k+1} \leq k + x_1 x_2 \ldots x_k x_{k+1}.$$

i.e.,

$$x_1 x_2 \ldots x_k + x_{k+1} \leq 1 + x_1 x_2 \ldots x_k x_{k+1}$$

But this is equivalent to $0 \leq x_1 x_2 \ldots x_k (x_{k+1} - 1) - (x_{k+1} - 1)$, i.e.,

$$(x_1 x_2 \ldots x_k - 1)(x_{k+1} - 1) \geq 0,$$

which is obviously true. This completes the induction.
Now, in particular, for $n = 2, 3, 4$, we have

$$x_1 + x_2 \leq 1 + x_1 x_2$$

$$x_1 + x_2 + x_3 \leq 2 + x_1 x_2 x_3$$

$$x_1 + x_2 + x_3 + x_4 \leq 3 + x_1 x_2 x_3 x_4$$

By adding:

$$3x_1 + 3x_2 + 2x_3 + x_4 \leq 6 + x_1 x_2 + x_1 x_2 x_3 + x_1 x_2 x_3 x_4$$

Setting $x_1 = t^a; x_2 = t^b; x_3 = t^c; x_4 = t^d$ leads to

$$3t^a + 3t^b + 2t^c + t^d \leq 6 + t^{a+b} + t^{a+b+c} + t^{a+b+c+d}.$$

The inequality is preserved after integration:

$$3\int_0^1 t^a dt + 3\int_0^1 t^b dt + 2\int_0^1 t^c dt + \int_0^1 t^d dt$$

$$\leq 6 + \int_0^1 t^{a+b} dt + \int_0^1 t^{a+b+c} dt + \int_0^1 t^{a+b+c+d} dt$$

which is the required inequality

$$\frac{3}{a+1} + \frac{3}{b+1} + \frac{2}{c+1} + \frac{1}{d+1} \leq 6 + \frac{1}{a+b+1} + \frac{1}{a+b+c+1} + \frac{1}{a+b+c+d+1}$$

□

Acknowledgment (by Alexander Bogomolny)

This problem from the ***Romanian Mathematical Magazine*** has been kindly posted at the ***CutTheKnotMath facebook page*** by Daniel Sitaru. He later mailed his solution in a LaTex file.

154. Dan Sitaru's Integral Inequality with Powers of a Function

Assume $f:[0,1]\to[0,\infty)$ is Riemann integrable and bounded. (Then the same holds for its integer powers.) Assume also that

$$\int_0^1 f^3(x)dx=\sqrt[7]{2}$$

Prove that

$$\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)\geq 2$$

Proposed by Daniel Sitaru

Solution 1(by Daniel Sitaru).

Using repeatedly the ***Cauchy – Schwarz inequality***,

$$\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)\left(\int_0^1 f^3(x)dx\right)$$

$$=\int_0^1\left(f^2\sqrt{f(x)}\right)^2dx\cdot\int_0^1\left(f^3\sqrt{f(x)}\right)^2dx\cdot$$

$$\cdot\int_0^1\left(f^4\sqrt{f(x)}\right)^2dx\cdot\int_0^1\left(f\sqrt{f(x)}\right)^2dx$$

$$\geq\left(\int_0^1 f^6(x)dx\right)^2\cdot\left(\int_0^1 f^6(x)dx\right)^2=\left[\left(\int_0^1 f^6(x)dx\right)\cdot\left(\int_0^1 1^2dx\right)\right]^2$$

$$\geq\left(\int_0^1 f^3(x)dx\right)^8=\sqrt[7]{2^8}$$

so that

$$\sqrt[7]{2}\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)\geq\sqrt[7]{2^8}$$

In other words,

$$\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)\geq 2$$

Equality is attained for $f(x)\equiv\sqrt[21]{2}$. □

Solution 2(by Chris Kyriazis).

By *Hölder's inequality*,

$$\left(\int_0^1 f^n(x)dx\right)^{\frac{3}{n}}\left(\int_0^1 dx\right)^{\frac{n-3}{2n}}\left(\int_0^1 dx\right)^{\frac{n-3}{2n}}\geq\int_0^1 f^3(x)dx=\sqrt[7]{2},$$

implying, in particular,

$$\int_0^1 f^5(x)dx\geq\sqrt[21]{2^5},$$

$$\int_0^1 f^7(x)dx\geq\sqrt[21]{2^7},$$

$$\int_0^1 f^9(x)dx\geq\sqrt[21]{2^9}.$$

Multiplying the three we have

$$\left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right) \geq \sqrt[21]{2^{5+7+9}} = 2.$$

□

Solution 3(by Amit Itagi).

Consider the convex function $g(x) = x^k$ with $k > 1$. Applying ***Jensen's inequality*** (noting that powers of f are non-negative and Riemann – integrable),

$$g\left(\int_0^1 f^3(x)dx\right) \leq \int_0^1 g\left(f^3(x)\right)dx$$

$$\Rightarrow \left(\int_0^1 f^3(x)\right)^k \leq \int_0^1 f^{3k}(x)dx$$

Multiplying the inequalities for $k = \frac{5}{3}, \frac{7}{3}, \frac{9}{3}$ together,

$$2 = \left(\int_0^1 f^3(x)dx\right)^7 \leq \left(\int_0^1 f^5(x)dx\right)\left(\int_0^1 f^7(x)dx\right)\left(\int_0^1 f^9(x)dx\right)$$

□

Solution 4 (by N. N. Taleb).

By *Hölder inequality*,

$$\left(\int_0^1 f^{3p}(x)dx\right)^{\frac{1}{p}}\left(\int_0^1 1^q dx\right)^{\frac{1}{q}} \geq \int_0^1 f^3(x)dx$$

Hence, $\int_0^1 f^{3p}(x)dx \geq (\sqrt[7]{2})^p$.

Allora, taking $p = \left\{\frac{5}{3}, \frac{7}{3}, \frac{9}{3}\right\}$ and merging, we get the result.

□

Acknowledgment (by Alexander Bogomolny)

This problem from the ***Romanian Mathematical Magazine*** has been kindly posted at the ***CutTheKnotMath facebook page*** by Daniel Sitaru. He later mailed his solution (Solution 1) in LaTex file, along with a solution (Solution 2) by Chris Kyriazis (Greece). Solution 3 is by Amit Itagi; Solution 4 is by N. N. Taleb.

155. An Inequality in Three (Or Is It Two) Variables

If $x, y > 0; z \in \mathbb{R}$ ***then:***

$$\frac{(x+y)^2}{(x\sin^2 z + y\cos^2 z)(x\cos^2 z + y\sin^2 z)} + \frac{x}{y} + \frac{y}{x} \geq 6.$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

Denote $\sin^2 z = a; \cos^2 z = b$. Then

$$\frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{ax+by} =$$

$$= \frac{x^2}{axy+bxz} + \frac{y^2}{ayz+bxy} + \frac{z^2}{axz+byz}$$

$$\geq \frac{(x+y+z)^2}{(xy+yz+zx)(a+b)}$$

$$\geq \frac{3(xy+xz+yz)}{(xy+yz+zx)(a+b)} = \frac{3}{a+b}$$

so that

$$(1) \qquad \frac{x}{ay+bz} + \frac{y}{az+bx} + \frac{z}{az+by} \geq \frac{3}{a+b}$$

For $z = x$ in (1):

$$\frac{x(ay+by)+x(ay+bx)}{(ay+bx)(ax+by)} + \frac{y}{x(a+b)} \geq \frac{3}{a+b}$$

$$\frac{x(ax+ay+bx+by)}{(ay+bx)(ax+by)} + \frac{y}{x(a+b)} \geq \frac{3}{a+b}$$

$$(2) \qquad \frac{x(a+b)(x+y)}{(ax+by)(bx+ay)} + \frac{1}{a+b}\cdot\frac{y}{x} \geq \frac{3}{a+b}$$

Similarly, with $z = y$ in (1):

$$(3) \qquad \frac{y(a+b)(x+y)}{(ax+by)(bx+ay)} + \frac{1}{a+b}\cdot\frac{x}{y} \geq \frac{3}{a+b}$$

By adding (2) and (3):

$$(4) \qquad \frac{(a+b)(x+y)^2}{(ax+by)(bx+ay)} + \frac{1}{a+b}\left(\frac{x}{y}+\frac{y}{x}\right) \geq \frac{6}{a+b}$$

Now, replace back $a = \sin^2 z; b = \cos^2 z$ in (4):

$$\frac{(\sin^2 z + \cos^2 z)(x+y)^2}{(x\sin^2 z + y\cos^2 z)(x\cos^2 z + y\sin^2 z)} + \frac{x}{y} + \frac{y}{x} \geq 6$$

Equality holds for $x = y$. □

Solution 2(by Alexander Bogomolny).

Let's prove a little more general result:

$$\textit{If } x,y,z,b > 0 \textit{ and } a+b=1:$$

$$\frac{(x+y)^2}{(xa+yb)(xb+ya)} + \frac{x}{y} + \frac{y}{x} \geq 6.$$

By the ***AM-GM inequality*** $ab \leq \frac{1}{4}$ and $a^2+b^2 \leq \frac{1}{2}(a+b)^2 = \frac{1}{2}$. Using that

$$(xa+yb)(xb+ya) = x^2ab + xyb^2 + xya^2 + y^2ab$$

$$\leq \frac{1}{4}(x^2+y^2) + \frac{1}{2}xy = \frac{1}{4}(x+y)^2$$

Thus

$$\frac{(x+y)^2}{(x\sin^2 z + y\cos^2 z)(x\cos^2 z + y\sin^2 z)} + \frac{x}{y} + \frac{y}{x}$$

$$\geq \frac{(x+y)^2}{\frac{(x+y)^2}{4}} + \frac{x}{y} + \frac{y}{x} = 4 + 2 = 6.$$

□

Solution 3(by Ravi Prakash, Kevin Soto Palacios).

Let $a = x\sin^2 z + y\cos^2 z, b = y\sin^2 z + x\cos^2 z$. Then $x + y = a + b$ and the inequality to prove becomes

$$\frac{(a+b)^2}{ab} + \frac{x}{y} + \frac{y}{x} \geq 6.$$

This is the same as

$$\frac{a}{b} + \frac{b}{a} + 2 + \frac{x}{y} + \frac{y}{x} \geq 2 + 2 + 2 = 6.$$

□

Acknowledgment (by Alexander Bogomolny)

This problem from the ***Romanian Mathematical Magazine*** has been kindly posted at the ***CutTheKnotMath facebook page*** by Daniel Sitaru. He later mailed his solution (Solution 1) in a LaTeX file. Solution 3 is by Ravi Prakash and independently by Kevin Soto Palacios.

156. An Inequality in Four Weighted Variables

Let $a, b, c, d > 0$. Then:

$$(a+c)^c(b+d)^d(c+d)^{c+d} \leq c^c d^d (a+b+c+d)^{c+d}$$

Proposed by Daniel Sitaru

Solution 1(by Kevin Soto Palacios).

The inequality is equivalent to

$$\left(\frac{a+b+c+d}{c+d}\right)^{c+d} \geq \left(\frac{a+c}{c}\right)^c \left(\frac{b+d}{d}\right)^d,$$

or,

$$\left(\frac{a+b+c+d}{c+d}\right)^{c+d} \geq \left(\frac{a}{c}+1\right)^c \left(\frac{b}{d}+1\right)^d$$

Now, by the ***weighted AM-GM inequality***,

$$\frac{(\frac{a}{c}+1)c + (\frac{b}{d}+1)d}{c+d} \geq \sqrt[c+d]{\left(\frac{a}{c}+1\right)^c \left(\frac{b}{d}+1\right)^d},$$

i.e.,

$$\left(\frac{a+b+c+d}{c+d}\right)^{c+d} \geq \left(\frac{a}{c}+1\right)^c \left(\frac{b}{d}+1\right)^d$$

QED. □

Solution 2(by Amit Itagi).

Let $p = a + c$ and $q + b + d$. Noting that the log function is concave, **Jensen's inequality** implies

$$\frac{c}{c+d}\log\left(\frac{p}{c}\right) + \frac{d}{c+d}\log\left(\frac{q}{d}\right) \leq \log\left(\frac{p+q}{c+d}\right).$$

The arguments lie in the domain of the log function as $p > c$ and $q > d$. Applying the monotonically increasing exponential function to both sides,

$$\left(\frac{p}{c}\right)^{\frac{c}{c+d}} \left(\frac{q}{d}\right)^{\frac{d}{c+d}} \leq \left(\frac{p+q}{c+d}\right)$$

$$\Rightarrow \left(\frac{p}{c}\right)^c \left(\frac{q}{d}\right)^d \leq \left(\frac{p+q}{c+d}\right)^{c+d}$$

$$\Rightarrow p^c q^d (c+d)^{c+d} \le c^c d^d (p+q)^{c+d}$$
$$\Rightarrow (a+c)^c (b+d)^d (c+d)^{c+d} \le c^c d^d (a+b+c+d)^{c+d}$$

□

Solution 3(by Daniel Sitaru).

Define $f:(0,\infty)\to\mathbb{R}$, with $f(x)=\log(x+1)$. We have, $f'(x)=\frac{1}{x+1}$; $f''(x)=\frac{-1}{(x+1)^2}<0$; so that f is concave. By Jensen's inequality,

$$c\log\left(1+\frac{a}{c}\right)+d\log\left(1+\frac{b}{d}\right)=(c+d)\left(\frac{c}{c+d}\log\left(1+\frac{a}{c}\right)+\frac{d}{c+d}\log\left(1+\frac{b}{d}\right)\right)$$
$$\le (c+d)\log\left(1+\frac{c}{c+d}\cdot\frac{a}{c}+\frac{d}{c+d}\cdot\frac{b}{d}\right)$$
$$=(c+d)\log\left(1+\frac{a}{c+d}+\frac{b}{c+d}\right)$$
$$=\log\left(1+\frac{a+b}{c+d}\right)^{c+d}$$

Further

$$\log\left[\left(1+\frac{a}{c}\right)^c\cdot\left(1+\frac{b}{d}\right)^d\right]\le\log\left(\frac{a+b+c+d}{c+d}\right)^{c+d}$$
$$\left(1+\frac{a}{c}\right)^c\cdot\left(1+\frac{b}{d}\right)^d\le\left(\frac{a+b+c+d}{c+d}\right)^{c+d}$$
$$\left(\frac{c+d}{a+b+c+d}\right)^{c+d}\cdot\left(1+\frac{a}{c}\right)^c\cdot\left(1+\frac{b}{d}\right)^d\le 1$$
$$(a+c)^c\cdot(b+d)^d\cdot(c+d)^{c+d}\le c^c\cdot d^d(a+b+c+d)^{c+d}.$$

□

Acknowledgment (by Alexander Bogomolny)

This problem by Daniel Sitaru from the ***Romanian Mathematical Magazine*** has been kindly posted at the ***CutTheKnotMath facebook page*** by Kevin Soto Palacios (Peru), along with his solution. Solution 2 is by Amit Tagi; Solution 3 is by Daniel Sitaru.

157. An Inequality with Cyclic Sums on Both Sides

Let $a,b,c>0$. Then:

$$\sum_{cycl}\frac{a^9}{b^6c^2}\ge\sum_{cycl}\sqrt[6]{\frac{a^{28}}{b^{17}c^5}}.$$

Proposed by Daniel Sitaru

Solution 1(by Ravi Prakash).

Observe that, by the AM-GM inequality,

$$4\cdot\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\ge\left(\frac{a^{36}}{b^{24}c^8}\cdot\frac{b^9}{c^6a^2}\cdot\frac{c^9}{a^6b^2}\right)^{\frac{1}{6}}=6\left(\frac{a^{28}}{b^{17}c^5}\right)^{\frac{1}{6}},$$
$$\frac{a^9}{b^6c^2}+4\frac{b^9}{c^6a^2}+\frac{c^9}{a^6b^2}\ge 6\left(\frac{b^{28}}{c^{17}a^5}\right)^{\frac{1}{6}},$$
$$\frac{a^9}{b^6c^2}+\frac{b^9}{c^6a^2}+4\frac{c^9}{a^6b^2}\ge 6\left(\frac{c^{28}}{a^{17}b^5}\right)^{\frac{1}{6}}.$$

Adding up and dividing by 6 yields the required inequality. □

Solution 2 (by Kevin Soto Palacios).

The required inequality is equivalent to

$$\sum_{cycl} \frac{a^9}{b^6c^2} \geq \sqrt[6]{abc} \sum_{cycl} \sqrt{\frac{a^9}{b^6c^2}}.$$

By the ***AM-GM inequality***,

$$\text{(A)} \qquad \frac{a^9}{b^6c^2} + \frac{b^9}{c^6a^2} + \frac{c^9}{a^6b^2} \geq 3\sqrt[3]{abc}.$$

By the ***Cauchy - Schwarz inequality***,

$$\text{(B)} \qquad 3\Big(\frac{a^9}{b^6c^2} + \frac{b^9}{c^6a^2} + \frac{c^9}{a^6b^2}\Big) \geq \left(\sqrt{\frac{a^9}{b^6c^2}} + \sqrt{\frac{b^9}{c^6a^2}} + \sqrt{\frac{c^9}{a^6b^2}}\right)^2.$$

Multiplying (A) and (B) we obtain

$$\Big(\frac{a^9}{b^6c^2} + \frac{b^9}{c^6a^2} + \frac{c^9}{a^6b^2}\Big)^2 \geq \sqrt[3]{abc}\left(\sqrt{\frac{a^9}{b^6c^2}} + \sqrt{\frac{b^9}{c^6a^2}} + \sqrt{\frac{c^9}{a^6b^2}}\right)^2$$

which is the same as

$$\sum_{cycl} \frac{a^9}{b^6c^2} \geq \sqrt[6]{abc} \sum_{cycl} \sqrt{\frac{a^9}{b^6c^2}}$$

□

Solution 3(by Amit Itagi).

Let $a = x^6, b = y^6, c = z^6$. The inequality can be written as

$$\sum_{cycl} \frac{x^{54}}{y^{36}z^{12}} \geq \sum_{cycl} \frac{x^{28}}{y^{17}z^5}.$$

This is equivalent to

$$\sum_{cycl} \frac{x^{90}z^{24}}{y^{36}y^{36}z^{36}} \geq \sum_{cycl} \frac{x^{64}y^{19}}{z^{31}} x^{36}y^{36}z^{36}.$$

Or,

$$\sum_{cycl} x^{90}z^{24} \geq \sum_{cycl} x^{64}y^{19},$$

which follows from ***Muirhead's inequality***. □

Solution 4 (by Nassim Nicolas Taleb).

Rewriting

$$LHS = \frac{a^4b^{15} + a^{15} + c^4 + b^4c^{15}}{a^6b^6c^6}, \quad RHS = \frac{a^2b^{\frac{15}{2}} + a^{\frac{15}{2}}c^2 + b^2c^{\frac{15}{2}}}{a^{\frac{17}{6}}b^{\frac{17}{6}}c^{\frac{17}{6}}}.$$

By the power-mean inequality,

$$\frac{a^4b^{15} + a^{15}c^4 + b^4c^{15}}{a^6b^6c^6} \geq \frac{\left(a^2b^{\frac{15}{2}} + a^{\frac{15}{2}}c^2 + b^2c^{\frac{15}{2}}\right)^2}{3a^6b^6c^6}.$$

Thus, suffice it to prove that

$$\frac{\left(a^2b^{\frac{15}{2}} + a^{\frac{15}{2}}c^2 + b^2c^{\frac{15}{2}}\right)^2}{3a^6b^6c^6} \geq \frac{a^2b^{\frac{15}{2}} + a^{\frac{15}{2}}c^2 + b^2c^{\frac{15}{2}}}{a^{\frac{17}{6}}b^{\frac{17}{6}}c^{\frac{17}{6}}},$$

or that

$$\frac{\left(a^2b^{\frac{15}{2}}+a^{\frac{15}{2}}c^2+b^2c^{\frac{15}{2}}\right)\left(a^2b^{\frac{15}{2}}+a^{\frac{15}{2}}c^2+b^2c^{\frac{15}{2}}-3a^{\frac{19}{6}}b^{\frac{19}{6}}c^{\frac{19}{6}}\right)}{3a^6b^6c^6}\geq 0.$$

The latter inequality is equivalent to

$$a^2b^{\frac{15}{2}}+a^{\frac{15}{2}}c^2+b^2c^{\frac{15}{2}}\geq 3a^{\frac{19}{6}}b^{\frac{19}{6}}c^{\frac{19}{6}}$$

which is true, since by the AM-GM inequality,

$$\frac{a^2b^{\frac{15}{2}}+a^{\frac{15}{2}}c^2+b^2c^{\frac{15}{2}}}{3}\geq\sqrt[3]{a^{\frac{19}{2}}b^{\frac{19}{2}}c^{\frac{19}{2}}}=a^{\frac{19}{6}}b^{\frac{19}{6}}c^{\frac{19}{6}}.$$

□

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted a problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Solution 1 is by Ravi Prakash (India); Solution 2 is by Kevin Soto Palacios (Peru); Solution 3 is by Amit Itagi (USA); Solution 4 is by N. N. Taleb.

158. Inequalities with Double And Triple Integrals

Prove that:

(A) $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos\left(\frac{x+y}{2}\right)dxdy\geq\frac{\pi}{2}$$

(B) $$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_0^1\cos(xz+y(1-z))dxdydz\geq\frac{\pi}{2}$$

Proposed by Daniel Sitaru

Solution 1 (by Quang Minh Tran).

For all $z\in[0,1]$ and $x,y\in\left[0,\frac{\pi}{2}\right]$, ***Jensen's inequality*** gives

$$\cos(zx+(1-z)y)\geq z\cos x+(1-z)\cos y$$

We have

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos(zx+(1-z)y)dxdy$$
$$\geq z\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos dxdy+(1-z)\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos ydxdy$$
$$=z\frac{\pi}{2}+(1-z)\frac{\pi}{2}=\frac{\pi}{2}.$$

Taking $z=\frac{1}{2}$ solves (A).
Further, $\int_0^1\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos(xy+y(1-z))dxxy\geq\int_0^1\frac{\pi}{2}=\frac{\pi}{2}$ which solves (B).

□

Solution 2(by Michel Rebeiz).

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos\left(\frac{x+y}{2}\right)dxdy=\int_0^{\frac{\pi}{2}}2\left[\sin\left(\frac{x+y}{2}\right)\right]_0^{\frac{\pi}{2}}dy$$
$$=2\int_0^{\frac{\pi}{2}}\left[\sin\left(\frac{\frac{\pi}{2}+y}{2}\right)-\sin\left(\frac{y}{2}\right)\right]dy$$

$$= 2\left[-2\cos\sin\left(\frac{\frac{\pi}{2}+y}{2}\right)+2\cos\left(\frac{y}{2}\right)\right]_0^{\frac{\pi}{2}}$$

$$= 4\Big[-\sin\frac{\pi}{2}+\sin\frac{\pi}{4}+\cos\frac{\pi}{4}-\cos 0\Big] = 4\left[\frac{\sqrt{2}}{2}+\frac{\sqrt{2}}{2}-1\right] = 4(\sqrt{2}-1) > \frac{\pi}{2}.$$

This solves (A). □

Solution 3(same solution by Rozeta Atanasova, Soumitra Mandal, Nassim Nicolas Taleb).

$$\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\cos\Big(\frac{x+y}{2}\Big)dxdy$$

$$= \int_0^{\frac{\pi}{2}}\cos\frac{x}{2}dx\int_0^{\frac{\pi}{2}}\cos\frac{y}{2}dy - \int_0^{\frac{\pi}{2}}\sin\frac{x}{2}dx\int_0^{\frac{\pi}{2}}\frac{y}{2}dy$$

$$= 4\left(\sin^2\frac{\pi}{2} - \Big(\cos\frac{\pi}{4}-\cos 0\Big)^2\right) = 4(\sqrt{2}-1) > \frac{\pi}{2}$$

This solves (A) □

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted a problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Solution 1 is by Quang Minh Tran; Solution 2 is by Michel Rebeiz; Solution 3 is by Rozeta Atanasova; Diego Alvariz and N. N. Taleb have independently come with Solution 1.

159. Minimum of a Cyclic Sum with Logarithms

Define, for $a, b, c > 1$***,***

$$\Omega(a,b,c) = \sum_{cycl}\frac{\log_a^2 b + \log_a b\cdot\log_b c + \log_b^2 c}{\log_a b + \log_b c}$$

Find $\min\Omega(a,b,c)$***.***

Proposed by Daniel Sitaru

Solution 1 (same solution by Subhajit Chattopadhyay and Geanina Tudose).

Let $\log_a b = x, \log_b c = y, \log_c a = z$. Clearly $x, y, z = 1$. In terms of x, y, z the function becomes, say,

$$\Omega(a,b,c) = \Omega'(x,y,z) = \sum_{cycl}\frac{x^2+xy+y^2}{x+y}$$

$$= \sum_{cycl}\frac{(x+y)^2-xy}{x+y}$$

$$= 2(x+y+z) - \Big(\frac{xy}{x+y}+\frac{yz}{y+z}+\frac{zx}{z+x}\Big)$$

Now, by the ***AM-HM inequality***, $-\frac{xy}{x+y} \ge -\frac{x+y}{2}$, such that

$$\Omega'(x,y,z) = 2(x+y+z) - \Big(\frac{xy}{x+y}+\frac{yz}{y+z}+\frac{zx}{z+x}\Big)$$

$$\ge 2(x+y+z) - \frac{2}{4}(x+y+z) = \frac{3}{2}(x+y+z)$$

$$\geq \frac{9}{2}\sqrt[3]{xyz} = \frac{9}{2}$$

Since equality is attained for $x = y = z = 1, \min \Omega'(x, y, z) = \frac{9}{2}$, implying $\min \Omega(a, b, c) = \frac{9}{2}$, attained for $a = b = c$. □

Solution 2 (by Kevin Soto Palacios).

With the same notation as in Solution 1, and using the AM-GM inequality,

$$\sum_{cycl} \frac{x^2 + xy + y^2}{x + y} \geq \sum_{cycl} \frac{3(x + y)^2}{4(x + y)}$$

$$= \sum_{cycl} \frac{3}{4}(x + y) = \frac{3}{2}(x + y + z)$$

$$\geq \frac{9}{2}\sqrt[3]{xyz} = \frac{9}{2}.$$

Equality occurs for $x = y = z = 1$. □

Solution 3 (by Nassim Nicolas Taleb).

We expand $\Omega(a, b, c) = \sum_{cycl} \frac{\frac{\log^2 a}{\log^2 b} + \frac{\log c}{\log a} + \frac{\log^2 c}{\log^2 b}}{\frac{\log b}{\log a} + \frac{\log c}{\log b}}$

Now let $x = \log a, y = \log b, z = \log c$. The function transforms to

$$\Omega_2(x, y, z) = \sum_{cycl} \frac{\frac{y^2}{x^2} + \frac{x^2}{z^2} + \frac{y}{x}}{\frac{y}{x} + \frac{y}{z}}$$

$$\frac{x^4 + y^2z^2 + x^y z}{x^3 z + xyz^2}$$

By the AM-GM inequality,

$$\sum_{cycl} \frac{x^4 + y^2z^2 + x^y z}{x^3 z + xyz^2} \geq 3\sqrt[3]{\prod_{cycl} \frac{x^4 + x^2yz + y^2z^2}{x^4 + x^2yz}}$$

Now, using $x^4 + x^2yz + y^2z^2 \geq \frac{3}{4}(x^2 + yz)^2$,

$$3\sqrt[3]{\prod_{cycl} \frac{x^4 + x^2yz + y^2z^2}{x^4 + x^2yz}} \geq 3\sqrt[3]{\prod_{cycl} \frac{3}{4} \cdot \frac{(x^2 + yz)^2}{x^4 + x^2yz}}$$

$$= \frac{9}{4} \cdot \frac{(y^2 + xz)^{\frac{1}{3}}(x^2 + zy)^{\frac{1}{3}}(z^2 + xy)^{\frac{1}{3}}}{x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}}.$$

Now, with the AM-GM inequality,

$$\frac{9}{4} \cdot \frac{(y^2 + xz)^{\frac{1}{3}}(x^2 + zy)^{\frac{1}{3}}(z^2 + xy)^{\frac{1}{3}}}{x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}} \geq \frac{9}{4} \cdot \frac{(2\sqrt{y^2xz})^{\frac{1}{3}}(2\sqrt{x^2zy})^{\frac{1}{3}}(2\sqrt{z^2xy})^{\frac{1}{3}}}{x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}}$$

$$= \frac{9}{4} \cdot \frac{2 \cdot x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}}{x^{\frac{2}{3}}y^{\frac{2}{3}}z^{\frac{2}{3}}} = \frac{9}{2}.$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted a problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Solution 1 is by

Subhajit Chattopadhyay and, independently, by Geanina Tudose; Solution 2 is by Kevin Soto Palacios; Solution 3 is by N.N. Taleb.

160. A System of Two Equations Replete with Squares

Solve for real numbers:

$$\begin{cases} \frac{x^2}{25} + \frac{y^2}{16} = 1 \\ x^2 + y^2 = \left(\frac{x^2}{5} + \frac{y^2}{4}\right)^2 \end{cases}$$

Proposed by Daniel Sitaru

Solution (by Seyran Ibrahimov).

Let, for simplicity, $a = x^2, b = y^2$. The system can be written as

$$\begin{cases} 16a + 25b = 400 \\ a + b = \frac{(4a+5b)^2}{400} \end{cases}$$

The second equation transforms into $400(a + b) = (4a + 5b)^2$. Replacing 400 from the first equation gives

$$(16a + 25b)(a + b) = (4a + 5b)^2,$$

i.e.,

$$16a^2 + 25ab + 16ab + 25b^2 = 16a^2 + 40ab + 25b^2,$$

which simplifies to $ab = 0$, same as $xy = 0$. Note that x, y can't vanish simultaneously. Thus two cases: either $x = 0$ or $y = 0$. The first case gives solutions $(0, \pm 4)$, the second $(\pm 5, 0)$. □

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the above problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***, along with the above solution by Seyran Ibrahimov.

161. Dan Sitaru's Cyclic Inequality In Three Variables with Constraints II

Assume $x, y, z > 0$ ***and*** $x^2 + y^2 + z^2 = 12$. ***Prove that***

$$\sum_{cycl} \frac{\frac{x}{y} + 1 + \frac{y}{x}}{\frac{1}{x} + \frac{1}{y}} \le 9.$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

We first prove that

For $x, y > 0$,

$$\frac{x^2 + xy + y^2}{x + y} \le \frac{3}{2}\sqrt{\frac{x^2 + y^2}{2}}.$$

For a proof, we have a sequence of equivalent inequalities:

$$\frac{(x^2 + xy + y^2)^2}{(x + y)^2} \le \frac{9}{8}(x^2 + y^2)$$

$$8(x^2 + xy + y^2) \le 9(x^2 + y^2)(x + y)^2$$

$$x^4+y^4-6x^2y^2+2x^3y+2xy^3\ge 0$$
$$(x^2-y^2)^2+2x^3y-2x^2y^2+2xy^3-2x^2y^2\ge 0$$
$$(x^2-y^2)^2+2x^2y(x-y)-2xy^2(x-y)\ge 0$$
$$(x^2-y^2)^2+2yx(x-y)^2\ge 0$$

which is true since $x,y>0$. Using that and the ***Cauchy - Schwarz inequality***,

$$\sum_{cycl}\frac{\frac{x}{y}+1+\frac{y}{x}}{\frac{1}{x}+\frac{1}{y}}=\sum_{cycl}\frac{x^2+xy+y^2}{x+y}$$
$$\le\frac{3}{2}\sum_{cycl}\sqrt{\frac{x^2+y^2}{2}}\le\frac{3}{2}\sqrt{(1^2+1^2+1^2)\sum_{cycl}\frac{x^2+y^2}{2}}$$
$$=\frac{3}{2}\sqrt{3\sum_{cycl}x^2}=\frac{3}{2}\sqrt{3\cdot 12}=\frac{3}{2}\cdot 6=9.$$

□

Solution 2 (by Leonard Giugiuc).

Started with $(x-y)^2(x^2+4xy+y^2)$ we arrive through a chain of equivalent inequalities to $\frac{x^2+xy+y^2}{x+y}\le\frac{3}{2}\sqrt{\frac{x^2+y^2}{2}}$, proceeding from which we apply ***Jensen's inequality:***

$$\sum_{cycl}\sqrt{\frac{x^2+y^2}{2}}\le\sqrt{3\sum_{cycl}\frac{x^2+y^2}{2}}=6.$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the above problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and later emailed me his solution (Solution 1) of the problem in a LaTex file. Solution 2 is by Leo Giugiuc.

162. An Inequality for Sides and Area

Prove that in any ΔABC

$$\sum_{cycl}\frac{(a^2-ab+b^2)^2}{a^2+4ab+b^2}\ge\frac{2S}{\sqrt{3}}$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

We shall prove that for positive x,y,

$$\frac{(x^2-xy+y^2)^2}{x^2+4xy+y^2}\ge\frac{1}{12}(x^2+y^2).$$

To this end, introduce $s=x+y$ and $p=xy$. Obviously, $\frac{s}{2}\ge\sqrt{p}$, implying $\frac{s^2}{p}\ge 4$.

We have a sequence of equivalent inequalities:

$$\frac{(x^2-xy+y^2)^2}{x^2+4xy+y^2}\ge\frac{1}{12}(x^2+y^2)$$

$$\frac{(s^2-3p)^2}{s^2+2p} \geq \frac{1}{12}(s^2-2p)$$
$$12(s^2-3p)^2 \geq s^4-4p^2$$
$$12s^4-72s^2p+108p^2 \geq s^4-4p^2$$
$$11s^4-72s^2p+112p^2 \geq 0.$$

Define $t = \frac{s^2}{p}$ and note that $t \geq 4$. The last inequality reduces to $11t^2 - 72t + 112 \geq 0$ which is the same as $(t-4)(11t-28) \geq 0$, which is true since $t \geq 4$. Returning to the original inequality,

$$\sum_{cycl} \frac{(a^2-ab+b^2)^2}{a^2+4ab+b^2} \geq \frac{1}{12}\sum_{cycl} a^2+b^2$$
$$= \frac{1}{6}(a^2+b^2+c^2) \geq \frac{1}{6}4\sqrt{3}S = \frac{2S}{\sqrt{3}},$$

by ***Weitzenböck's inequality***. Equality is attained only for equilateral triangles. □

Acknowledgment (by Alexander Bogomolny)

The problem (from the ***Romanian Mathematical Magazine***) has been kindly posted by Daniel Sitaru at the ***CutTheKnotMath facebook page***, Daniel later emailed me his solution in a LaTex file.

163. Dan Sitaru's Amazing, Never Ending Inequality

Assume $a, b, c > 0$. Prove that

$$\sum_{cycl}\left(\frac{a}{b}\right)^2 \cdot \sum_{cycl}\left(\frac{a}{b}\right)^4 \cdot \sum_{cycl}\left(\frac{a}{b}\right)^8 \geq \sum_{cycl}\left(\frac{a}{c}\right) \cdot \sum_{cycl}\left(\frac{b}{a}\right) \cdot \sum_{cycl}\left(\frac{b}{c}\right)$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Below, we shall be using repeatedly ***the inequality***

$$x^2+y^2+z^2 \geq xy+yz+zx$$

For example, with $x = \frac{a}{b}, y = \frac{b}{c}, z = \frac{c}{a}$, we get

$$\sum_{cycl}\left(\frac{a}{b}\right)^2 \geq \sum_{cycl}\left(\frac{a}{b}\cdot\frac{b}{c}\right) = \sum_{cycl}\left(\frac{a}{c}\right)$$

Similarly,

$$\sum_{cycl}\left(\frac{a}{b}\right)^4 \geq \sum_{cycl}\left(\frac{a}{b}\cdot\frac{b}{c}\right)^2 = \sum_{cycl}\left(\frac{a}{c}\right)^2 \geq \sum_{cycl}\left(\frac{a}{c}\cdot\frac{b}{a}\right) = \sum_{cycl}\left(\frac{b}{c}\right),$$

$$\sum_{cycl}\left(\frac{a}{b}\right)^8 \geq \sum_{cycl}\left(\frac{a}{b}\cdot\frac{b}{c}\right)^4 = \sum_{cycl}\left(\frac{a}{c}\right)^4 \geq \sum_{cycl}\left(\frac{a}{c}\cdot\frac{b}{a}\right)^2$$
$$= \sum_{cycl}\left(\frac{b}{c}\right)^2 \geq \sum_{cycl}\left(\frac{b}{c}\cdot\frac{c}{a}\right) = \sum_{cycl}\left(\frac{c}{a}\right).$$

We now only need to multiply the three inequalities. □

Extra (by Alexander Bogomolny)

The problem admits multiple variations. For example, from

$$\sum_{cycl}\left(\frac{a}{b}\right)^4 \geq \sum_{cycl}\left(\frac{a}{b}\cdot\frac{b}{c}\right)^2$$
$$= \sum_{cycl}\left(\frac{a}{c}\right)^2 \geq \sum_{cycl}\left(\frac{a}{c}\cdot\frac{c}{b}\right)$$
$$= \sum_{cycl}\left(\frac{a}{b}\right),$$

we get that, for $k \geq 2, \sum_{cycl}\left(\frac{a}{b}\right)^{2^k} \geq \sum_{cycl}\left(\frac{a}{b}\right)$ and thus

$$\prod_{k=2}^{n}\left[\sum_{cycl}\left(\frac{a}{b}\right)^{2^k}\right] \geq \left[\sum_{cycl}\left(\frac{a}{b}\right)\right]^{n-1}.$$

By the same token, the original inequality could have been written as

$$\sum_{cycl}\left(\frac{a}{b}\right)^2\cdot\sum_{cycl}\left(\frac{a}{b}\right)^4\cdot\sum_{cycl}\left(\frac{a}{b}\right)^8 \geq \sum_{cycl}\left(\frac{a}{c}\right)\cdot\sum_{cycl}\left(\frac{c}{b}\right)\cdot\sum_{cycl}\left(\frac{b}{a}\right)$$

or as

$$\sum_{cycl}\left(\frac{a}{b}\right)^2\cdot\sum_{cycl}\left(\frac{a}{b}\right)^4\cdot\sum_{cycl}\left(\frac{a}{b}\right)^8 \geq \left[\sum_{cycl}\left(\frac{a}{c}\right)\right]^3$$

and in general

$$\prod_{k=1}^{n}\left[\sum_{cycl}\left(\frac{a}{b}\right)^{2^k}\right] \geq \left[\sum_{cycl}\left(\frac{a}{c}\right)\right]^{n}.$$

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the above problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and later emailed me his solution of this amazing prolem in a LaTex file.

164. A Cyclic Inequality in Triangle for Integer Powers

a, b, c are the side lengths of ΔABC; $n \geq 0$, an integer. Prove that

$$\sum_{cycl}\frac{a^{n+1}}{b+c-a} \geq \sum_{cycl} a^n.$$

Proposed by Daniel Sitaru

Solution(by Daniel Sitaru).

$$\sum_{cycl}\frac{a^{n+1}}{b+c-a} - \sum_{cycl} a^n = \sum_{cycl}\left(\frac{a^{n+1}}{b+c-a} - a^n\right)$$
$$= \sum_{cycl}\frac{a^{n+1} - a^n b - a^n c + a^{n+1}}{b+c-a} = \sum_{cycl}\frac{a^n(a-b) + a^n(a-c)}{b+c-a}$$
$$= \sum_{cycl}\frac{a^n(a-b)}{b+c-a} + \sum_{cycl} a^n(a-c)b+c-a$$

$$= \sum_{cycl} \frac{a^n(a-b)}{b+c-a} + \sum_{cycl} b^n(b-a)c + a - b$$

$$= \sum_{cycl}(a-b)\Big(\frac{a^n(c+a-b) - b^n(b+c-a)}{(b+c-a)(c+a-b)}\Big)$$

$$= \sum_{cycl}(a-b)\Big(\frac{a^n c + a^{n+1} - a^n b - b^{n+1} - b^n c + ab^n}{(b+c-a)(c+a-b)}\Big)$$

$$= \sum_{cycl}(a-b)\Big(\frac{a(a^n+b^n) - b(a^n+b^n) + c(a^n+b^n)}{(b+c-a)(c+a-b)}\Big)$$

$$= \sum_{cycl}(a-b)\Big(\frac{(a-b)(a^n+b^n) + c(a^n-b^n)}{(b+c-a)(c+a-b)}\Big).$$

We now consider two cases:

$$n \geq 0$$

$$\sum_{cycl}(a-b)\Big(\frac{(a-b)(a^n+b^n) + c(a^n-b^n)}{(b+c-a)(c+a-b)}\Big)$$

$$= \sum_{cycl}(a-b)^2\Big(\frac{(a^n+b^n) + \sum_{k=0}^{n-1} a^{n-1-k}b^k}{(b+c-a)(c+a-b)}\Big) \geq 0.$$

$$\sum_{cycl}(a-b)\Big(\frac{(a-b)(a^n+b^n) + c(a^n-b^n)}{(b+c-a)(c+a-b)}\Big)$$

$$= \sum_{cycl}(a-b)\Big(\frac{(a-b)\cdot 2}{(b+c-a)(c+a-b)}\Big)$$

$$= \sum_{cycl}(a-b)^2\Big(\frac{2}{(b+c-a)(c+a-b)}\Big) \geq 0.$$

□

Acknowledgment (by Alexander Bogomolny)
The problem (from the ***Romanian Mathematical Magazine***) has been kindly posted by Daniel Sitaru at the ***CutTheKnotMath facebook page***, Daniel later emailed me his solution in a LaTeX file.

165. An Inequality with Cyclic Sums on Both Sides II

Let $a, b, c > 0$. Then:

$$\sum_{cycl} \sqrt[6]{ab^2c^3} \geq \sum_{cycl} \sqrt[30]{a^9b^{10}c^{11}}.$$

Proposed by Daniel Sitaru

Solution 1(same solution by Kevin Soto Palacios and Ravi Prakash).

Introduce new variables, with $x^{90} = ab^2c^3, y^{90} = bc^2a^3, z^{90} = ca^2b^3$, and $x, y, z > 0$. We have $(xyz)^{90} = (abc)^6$, i.e., $(xyz)^{15} = abc$ and $(xyz)^{120} = (abc)^8$. The required inequality is equivalent to

$$x^{15} + y^{15} + z^{15} \geq (x^3 + y^3 + z^3)(xyz)^4$$

By the ***AM-GM inequality***,

$$7x^{15} + 4y^{15} + 4z^{15} \geq 15\sqrt[15]{(x^{15})^7(y^{15})^4(z^{15})4} = 15x^7y^4z^4$$

$$7y^{15} + 4z^{15} + 4x^{15} \geq 15\sqrt[15]{(y^{15})^7(z^{15})^4(x^{15})4} = 15y^7z^4x^4$$

$$7z^{15} + 4x^{15} + 4y^{15} \geq 15\sqrt[15]{(z^{15})^7(x^{15})^4(y^{15})4} = 15z^7x^4y^4$$

Adding up gives

$$15(x^{15} + y^{15} + z^{15}) \geq 15(x^3 + y^3 + z^3)(xyz)^4 \Leftrightarrow$$

$$x^{15} + y^{15} + z^{15} \geq (x^3 + y^3 + z^3)(xyz)^4.$$

□

Solution 2(by Mohamed Jamal).

Since the inequality is homogeneous, we may assume $abc = 1$. Then the inequality becomes

$$\sum_{cycl} \sqrt[6]{\frac{a}{b}} \geq \sqrt[30]{\frac{a}{b}}.$$

By the AM-GM inequality,

$$2\sqrt[6]{\frac{a}{b}} + \sqrt[6]{\frac{b}{c}} + \sqrt[6]{\frac{c}{a}} + 1 \geq 5 \cdot \sqrt[30]{\frac{a}{b}}$$

$$2\sqrt[6]{\frac{b}{c}} + \sqrt[6]{\frac{c}{a}} + \sqrt[6]{\frac{a}{b}} + 1 \geq 5 \cdot \sqrt[30]{\frac{c}{a}}$$

$$2\sqrt[6]{\frac{c}{a}} + \sqrt[6]{\frac{a}{b}} + \sqrt[6]{\frac{b}{c}} + 1 \geq 5 \cdot \sqrt[30]{\frac{c}{a}}$$

Adding up gives

$$4\sum_{cycl} \sqrt[6]{\frac{a}{b}} + 3 \geq 5\sum_{cycl} \sqrt[30]{\frac{a}{b}}$$

Thus, suffice it to prove that

$$\frac{1}{4}\left(5\sum_{cycl} \sqrt[30]{\frac{a}{b}} - 3\right) \geq \sum_{cycl} \sqrt[30]{\frac{a}{b}}$$

i.e., $\sum_{cycl} \sqrt[30]{\frac{a}{b}} \geq 3$, which is obvious. □

Solution 3(by Nguyen Ngoc Tu).

Take $(x, y, z) = (\sqrt[30]{a}, \sqrt[30]{b}, \sqrt[30]{c}).x, y, z > 0$. We have to prove

$$(xyz)^5(x^5y^{10} + y^5z^{10} + z^5x^{10}) \geq x^9y^{10}z^{11} + y^9z^{10}x^{11} + z^9x^{10}y^{11},$$

or, equivalently,

$$x^5y^{10} + y^5z^{10} + z^5x^{10} \geq (xyz)(xy^2 + yz^2 + zx^2)$$

Assume $xyz = 1$. We have to prove that $\sum_{cycl}(xy^2)^5 \ge \sum_{cycl} xy^2$, which is

$$\sum_{cycl} X^5 \ge \sum_{cycl} X, \text{ with } (X,Y,Z) = (xy^2, yz^2, zx^2), X,Y,Z > 0 \text{ and } XYZ = 1.$$

We have

$$(X^5+Y^5+Z^5)(X+Y+Z) \ge (X^3+Y^3+Z^3)^2 \Rightarrow$$
$$X^5+Y^5+Z^5 \ge \frac{(X^3+Y^3+Z^3)^2}{X+Y+Z}$$
$$(X^3+Y^3+Z^3)(X+Y+Z) \ge (X^2+Y^2+Z^2)^2$$
$$\ge \frac{(X+Y+Z)^4}{9} \Rightarrow$$
$$X^3+Y^3+Z^3 \ge \frac{(X+Y+Z)^3}{9} \Rightarrow$$
$$X^5+Y^5+Z^5 \ge \frac{(X^3+Y^3+Z^3)^2}{X+Y+Z}$$
$$\ge \frac{(X+Y+Z)^5}{81} \ge X+Y+Z.$$

□

Solution 4(by Sanong Hauerai).

Set $x = a^5b^{10}c^{15}, y = b^5c^{10}a^{15}, z = c^5a^{10}b^{15}$. Then

$$2\sqrt[30]{x} + 2\sqrt[30]{y} + \sqrt[30]{z} \ge 5\sqrt[5]{\sqrt[30]{xxyyz}} = 5\sqrt[30]{a^{10}b^9c^{11}}$$
$$2\sqrt[30]{y} + 2\sqrt[30]{z} + \sqrt[30]{x} \ge 5\sqrt[30]{b^{10}c^9a^{11}}$$
$$2\sqrt[30]{z} + 2\sqrt[30]{x} + \sqrt[30]{y} \ge 5\sqrt[30]{c^{10}a^9b^{11}}$$

It follows that

$$\sum_{cycl} \sqrt[6]{ab^2c^3} \ge \sum_{cycl} \sqrt[30]{a^9b^{10}c^{11}}.$$

□

Solution 5 (by Nassim Nicholas Taleb).

We can rewrite the inequality as $\sum_{cycl} \sqrt[30]{a^5b^{10}c^{15}} \ge \sum_{cycl} \sqrt[30]{a^9b^{10}c^{11}}$ and then use convexity arguments.

In general, for $q > p_1 > p_2 > 0$ and $n > 0$,

$$\sum_{cycl} \sqrt[n]{a^{q-p_1}b^qc^{q+p_1}} \ge \sum_{cycl} \sqrt[n]{a^{q-p_2}b^qc^{q+p_2}} \ge 3a^{\frac{q}{n}}b^{\frac{q}{n}}c^{\frac{q}{n}}.$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted a problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Solution 1 is by Kevin Soto Palacios (Peru) and, independently, by Ravi Prakash (India); Solution 2 is by Mohamed Jamal (Morocco); Solution 3 is by Nguyen Ngoc Tu (Vietnam); Solution 4 is by Sanong Hauerai (Thailand), Solution 5 is by N.N. Taleb (USA/Lebanon).

166. Another Inequality with Logarithms, But Not Really

Prove that if $\boldsymbol{x, y, z \in (0,1)}$ ***or*** $\boldsymbol{x, y, z \in (1, \infty)}$ ***then:***

$$\sum_{cycl} \frac{\log_y^3 x + \log_z^3 y}{\log_y^2 x + \log_z x + \log_z^2 y} \geq 2.$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Let $a = \log_y x; b = \log_z y; c = \log_x z$. Since $\log_z x = \log_z y \cdot \log_y x = ab$, the inequality reduces to

$$\sum \frac{a^3 + b^3}{a^2 + ab + b^2} \geq 2,$$

with $abc = 1$.
To continue,

$$\begin{aligned} \frac{a^3 + b^3}{a^2 + ab + b^2} &= \frac{(a+b)(a^2 - ab + b^2)}{a^2 + ab + b^2} \\ &\geq \frac{a+b}{3} \end{aligned}$$

because $\frac{a^2-ab+b^2}{a^2+ab+b^2} \geq \frac{1}{3}$, which is equivalent to $2(a-b)^2 \geq 0$.
Thus using the ***AM-GM inequality***,

$$\begin{aligned} \sum \frac{a^3 + b^2}{a^2 + ab + b^2} &\geq \sum \frac{a+b}{3} = \frac{2}{3}(a+b+c) \\ &\overbrace{\geq}^{AM-GM} \frac{2}{3} \cdot 3\sqrt[3]{abc} = 2. \end{aligned}$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly reposted the above problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and later emailed me his solution in a LaTex file.

167. An Inequality in Fractions with Absolute Values

Assume $\boldsymbol{a, b, c \in \mathbb{R}, a \neq b \neq c \neq a}$. ***Define***

$\boldsymbol{\omega = \min\{|a+b|, |b+c|, |c+a|\}}$ ***and***

$\boldsymbol{\Omega = \max\{|a|, |b|, |c|\}}$. ***Prove that***

$$\omega < \frac{1}{3}\left(\sum_{cycl} \frac{a|a| - b|b|}{a - b}\right) < 2\Omega.$$

Proposed by Daniel Sitaru

Solution (same solution by Soumava Chakraborty and Ravi Prakash).

Note that the function $f(x,y) = \dfrac{x|x| - y|y|}{x - y}$ satisfy $f(x,y) = f(-x,-y)$ and so too

$$f(-x,-y) + f(-y,-z) + f(-z,-x) = f(x,y) + f(y,z) + f(z,x).$$

If so, suffice it to consider only two cases: 1) $a, b, c \geq 0$ and 2) $a, b \geq 0$ and $c \leq 0$. In the first case, we can simply remove the absolute values throughout to obtain

$$\min\{(a+b), (b+c), (c+a)\} < \frac{1}{3}\sum_{cycl}\frac{a^2-b^2}{a-b}$$

$$= \frac{1}{3}\sum_{cycl}(a+b) < 2\max\{a, b, c\}.$$

In the second case, let $|c| = -c$. Then

$$f(a,b)+f(b,c)+f(c,a) = \frac{a|a|-b|b|}{a-b} + \frac{b|b|-c|c|}{b-c} + \frac{c|c|-a|a|}{c-a}$$

$$= (a+b) + \frac{b^2+|c|^2}{b+|c|} + \frac{|c|^2+a^2}{|c|+a}$$

$$< (a+b) + (b+|c|) + (|c|+a) < 6\max\{|a|, |b|, |c|\}.$$

On the other hand, for $x, y \geq 0, x^2+y^2 \geq |x^2-y^2| = |x-y| \cdot (x+y)$, with equality only when of x, y is zero. Thus, it follows that $\frac{b^2+c^2}{b+|c|} \geq |b-|c|| = |b+c|$ and, similarly, $\frac{|c|^2+a^2}{|c|+a} \geq |a-|c|| = |c+a|$. Adding up gives

$$\frac{a|a|-b|b|}{a-b} + \frac{b|b|-c|c|}{b-c} + \frac{c|c|-a|a|}{c-a} \geq (a+b) + |b+c| + |c+a|$$

$$> 3\min\{|a+b|, |b+c|, |c+a|\}$$

No two of a, b, c may vanish simultaneously. □

Acknowledgment (by Alexander Bogomolny)

This problem from his book "Algebraic Phenomenon" has been kindly posted at the ***CutTheKnotMath facebook page*** by Daniel Sitaru.

Soumava Chakraborty gave a proof of $|x+y| \leq \frac{x|x|-y|y|}{x-y} \leq |x|+|y|$ by considering four cases and a similar proof has been submitted by Ravi Prakash.

168. A Cyclic Inequality of Degree Four

Prove that if $a, b, c > 0$, then:

$$a^4b + b^4c + c^4a + 2(a+b+c) \geq \sqrt{3}(ab+bc+ca)$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Observe that the inequality is equivalent to

$$\sum_{cycl} b(a^4 - a\sqrt{3} + 2) \geq 0.$$

But

$$a^4 - a\sqrt{3} + 2(a^2-1)^2 + 2a^2 - a\sqrt{3} + 1$$

$$= (a^2-1)^2 + \left(\sqrt{2}a - \sqrt{\frac{3}{8}}\right)^8 + \frac{5}{8} > 0.$$

□

Extension (by Alexander Bogomolny)

Note that the original inequality is weak, for example,

$$a^2 - a\sqrt{5} + 2 = (a^2-1)^2 + 2a^2 - a\sqrt{5} + 1$$

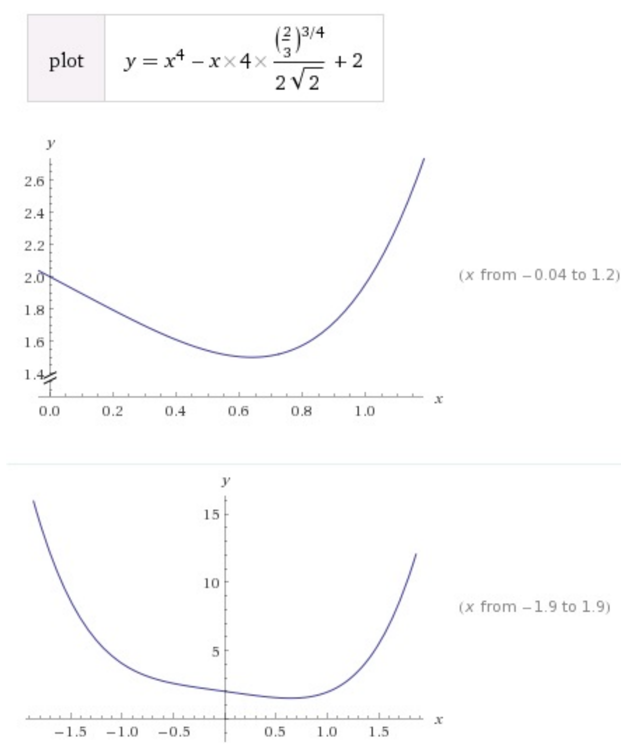
$$= (a^2-1)^2 + \left(\sqrt{2}a - \frac{5}{8}\right)^2 + \frac{3}{8}$$

$$> 0$$

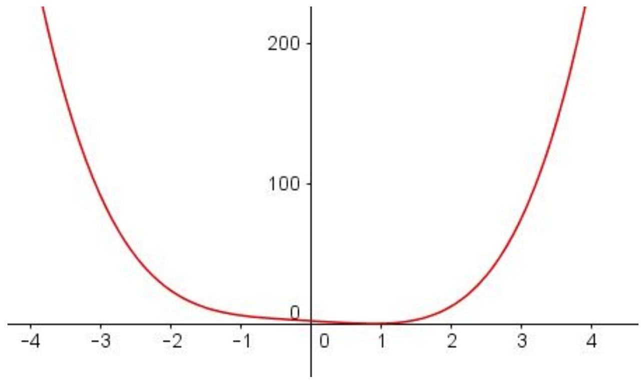
Show that $\sum_{cycl} + 2\sum_{cycl} a \geq \sqrt{5}\sum_{cycl} ab$ is also true which organically leads to the question of finding the maximum k such that

$$a^2b + b^4c + c^4a + 2(a+b+c) \geq k(ab+bc+ca)$$

Simply following the derivation above, the maximal k satisfies, $k \geq k_0 = 2\sqrt{2}$.
But we can do better.
Let $f(x) = x^4 - xk + 2$. Then $f'(x) = 4x^3 - k = 0$ leads to $x = \sqrt[3]{\frac{k}{4}}$. It is easy to see that this is a local minimum because $f''\left(\sqrt[3]{\frac{k}{5}}\right) \geq 0$.
Now, $f\left(\sqrt[3]{\frac{k}{4}}\right)^4 = \left(\sqrt[3]{\frac{k}{4}}\right)^4 - k\sqrt[3]{\frac{k}{4}} + 2 = 2 - 3\left(\sqrt[3]{\frac{k}{4}}\right)^4$. From this, $k_1 = 4\left(\frac{2}{3}\right)^{\frac{3}{4}}$
Now, this is an improvement since $\frac{k_1}{k_0} \approx 1.0434$.
The graph (courtesy wolframalpha) shows that this may not be the last word:



However, the graph generated by GeoGebra tells us a different story:



Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted the above problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and Leo Giugiuc messaged his solution practically right away.
Jeffrey Samuelson gave an answer to the extended question.

169. A Problem with a Magical Solution from Secrets in Inequality

Let x, y, z be positive real numbers satisfying

$$2xyz = 3x^2 + 4y^2 + 5z^2.$$

Find the minimum of the expression

$$P = 3x + 2y + z$$

Proposed by Pham Kim Hung

Solution by author.

Let $a = 3x, b = 2y, c = z$. Then $P = 3x + 2y + z = a + b + c$ and

$$a^2 + 3b^2 + 15c^2 = abc.$$

We'll make a double application of the weighted AM-GM inequality:

$$\frac{\sum w_k x_k}{\sum w_k} \geq \sqrt[\sum w_k]{\prod x_k^{w_k}}.$$

First, with $w_1 = \frac{1}{2}, w_2 = \frac{1}{3}, w_3 = \frac{1}{6} (w_1 + w_2 + w_3 = 1!)$,

(1) $$a + b + c \geq (2a)^{\frac{1}{2}}(3b)^{\frac{1}{3}}(6c)^{\frac{1}{6}}.$$

Then, with $w_1 = \frac{1}{4}, w_2 = \frac{3}{9} = \frac{1}{3}, w_3 = \frac{15}{36} = \frac{5}{12} (w_1 + w_2 + w_3 = 1!)$,

(2) $$\begin{aligned} a^2 + 3b^2 + 15c^2 &\geq (4a^2)^{\frac{1}{4}}(9b^2)^{\frac{3}{9}}(36c^2)^{\frac{15}{36}} \\ &= (4a^2)^{\frac{1}{4}}(9b^2)^{\frac{1}{3}}(36c^2)^{\frac{5}{12}}. \end{aligned}$$

Multiplying (1) and (2),

$$(a + b + c)(a^2 + 3b^2 + 15c^2) \geq 36abc,$$

which implies $a + b + c \geq 36$, the quantity that is attained for $x = y = z = 6$, making it the sought minimum. □

Aknowledgment (by Alexander Bogomolny)

This problem #81 from Phan Kim Hung's *Secrets in Inequalities*, (GIL Publishing House, 2007). I am grateful to Dan Sitaru who mailed me the problem and helped understand its solution.

170. An Equation in Factorials

Solve in natural numbers the following equation

$$\frac{1^2 \cdot 2! + 2^2 \cdot 3! + \ldots + n^2(n+1)! - 2}{(n+1)!} = 108.$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

We'll use the ***induction*** in n to prove

$$1^2 \cdot 2! + 2^2 \cdot 3! + \ldots + n^2(n+1)! - 2 = (n+2)!(n-1).$$

The claim-holds for $n = 1 : 1^2(1+1)! - 2 = (1+1)(1+2)(1-1)$.
Assume $P(k) =: 1^2 \cdot 2! + 2^2 \cdot 3! + \ldots + k^2(k+1)! - 2 = (k+2)!(k-1)$ is true, and let's prove $P(k+1)$, i.e.,

$$1^2 \cdot 2! + 2^2 \cdot 3! + \ldots + k^2(k+1)! + (k+1)^2(k+2)! - 2 = (k+3)!k.$$

We have

$$\begin{gathered} 1^2 \cdot 2! + 2^2 \cdot 3! + \ldots + k^2(k+1)! + (k+1)^2(k+2)! - 2 \\ = (k+2)!(k-1) + (k+1)^2(k+2)! \\ = (k+2)![(k-1) + (k+1)^2] = (k+2)![k^2 + 3k] = (k+3)!k. \end{gathered}$$

as required. Thus we rewrite the original equation:

$$\frac{(n+2)!(n-1)}{(n+1)!} = 108,$$

or, $n^2 + n - 110 = 0$, giving two roots, $n = 10$ that solves the problem and a superfluous one $n = -11$. □

Solution 2 (by Kunihiko Chikaya).

We'll unfold the ***telescoping sum:***

$$\begin{aligned} \sum_{k=1}^{n} k^2(k+1)! &= \sum_{k=1}^{n} [(k+2)^2 - 4(k+1)](k+1)! \\ &= \sum_{k=1}^{n} [(k+2)(k+2)! - 4(k+1)(k+1)!] \\ &= \sum_{k=1}^{n} [(k+3-1)(k+2)! - 4(k+2-1)(k+1)!] \\ &= \sum_{k=1}^{n} [(k+3)! - (k+2)! - 4(k+2)! + 4(k+1)!] \\ &= \sum_{k=1}^{n} [(k+3)! - 5(k+2)! + 4(k+1)!] \\ &= \sum_{k=4}^{n+3} k! - 5\sum_{k=3}^{n+2} k! + 4\sum_{k=2}^{n+1} k! \end{aligned}$$

$$= (n+3)! - 4(n+2)! + 2 = (n+2)!(n-1) + 2.$$

Thus the equation reduces to $(n+2)(n-1) = 108$ from which $n = 10$. □

Solution 3 (by Amit Itagi).

Let

$$S_k = \frac{1^2 2! + 2^2 3! + \ldots + k^2(k+1)! - 2}{(k+1)!}.$$

We can rule out $n = 1, 2$ as solution by observation. Thus, we are guaranteed to have an S_2 and an S_3 and verify that both are integers. In general, for some $k \geq 3$,

$$(k+1)!S_k - k!S_{k-1} = k^2(k+1)!$$

$$\Rightarrow (k+1)S_k - S_{k_1} = k^2(k+1)$$

By observing this equation, we claim that $S_k$is a quadratic polynomial in k. Let $S_k = ak^2 + bk + c$. Pugging this expression back into the recurrence relation and evaluating the undetermined coefficients,

$$(k+1)(ak^2 + bk + c) - [a(k-1)^2 + b(k-1) + c] = k^3 + k^2$$

$$ak^3 + bk^2 + (2a + c)k + (b - a) = k^3 + k^2$$

Thus, $a = 1, b = 1, c = -2$ and $S_k = k^2 + k - 2$.

So, the equation becomes $S_n = n^2 + n - 2 = 108$. The two roots are $n = 10$ and $n = -11$. Thus, the only permissible solution in natural numbers is $n = 10$. □

Solution 4(by Nassim Nicolas Taleb).

Writing in Gamma functions, the LHS is $(n) = \frac{-2+\sum_{k=1}^{n} k^2\Gamma(k+2)}{\Gamma(n+1)}$. We have

$$\sum_{k=1}^{n} k^2\Gamma(k+2) = n\Gamma(n+3) - \Gamma(n+3) + 2,$$

and conclude

$$F(n) = \frac{\Gamma(n+3)}{\Gamma(n+2)}(n-1) = (2+n)(n-1).$$

Solving $(2+n)(n-1) = 108$, we get $n = 10$. □

Ackowledgment(by Alexander Bogomolny)

Daniel Sitaru has kindly posted at ***CutTheKnotMath facebook page*** a problem of his from the ***Romanian Mathematical Magazine*** and later sent me a LaTex file with his solution (Solution 1). Solution 2 is by Kunihiko Chikaya; Solution 3 is by Amit Itagi; Solution 4 is by N. N. Taleb.

171. An Inequality with Powers And Logarithm

Prove, for $a \geq b > 0$, the following inequality

$$\frac{a}{b} + \frac{a^2}{b^2} + \frac{a^3}{b^3} + 12\ln b \geq \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + 12\ln a$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

Consider function $f : [1, \infty) \to \mathbb{R}; f(x) = x - \frac{1}{x} - 2\ln x$.

$$f'(x) = 1 + \frac{1}{x^2} - \frac{2}{x} = \frac{x^2 - 2x + 1}{x^2} = \frac{(x-1)^2}{x^2} \geq 0.$$

It follows that the function is increasing: $f(x) \geq f(1) = 0$, for $x \geq 1$. In other words $x - \frac{1}{x} \geq 2\ln x$.
We'll use this result with $x = \frac{a}{b}, \frac{a^2}{b^2}, \frac{a^3}{b^3}$ to obtain

$$\frac{a}{b} - \frac{b}{a} \geq 2\ln\left(\frac{a}{b}\right)$$
$$\frac{a^2}{b^2} - \frac{b^2}{a^2} \geq 2\ln\left(\frac{a}{b}\right)^2 = 4\ln\left(\frac{a}{b}\right)$$
$$\frac{a^3}{b^3} - \frac{b^3}{a^3} \geq 2\ln\left(\frac{a}{b}\right)^3 = 6\ln\left(\frac{a}{b}\right)$$

Adding up and rearranging we get

$$\frac{a}{b} + \frac{a^2}{b^2} + \frac{a^3}{b^3} \geq \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + 12\ln\left(\frac{a}{b}\right),$$

or,

$$\frac{a}{b} + \frac{a^2}{b^2} + \frac{a^3}{b^3} + 12\ln b \geq \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + 12\ln a.$$

□

Extra (by Alexander Bogomolny)
I originally misread the problem as

Prove, for $a, b > 0$, the following inequality

$$\frac{a}{b} + \frac{a^2}{b^2} + \frac{a^3}{b^3} + 12\ln b \geq \frac{b}{a} + \frac{b^2}{a^2} + \frac{b^3}{a^3} + 12\ln a.$$

Find a simple argument to show that, as stated, it could not be true.

Aknowledgment(by Alexander Bogomolny)
Daniel Sitaru has kindly posted at ***CutTheKnotMath facebook page*** a problem of his from the ***Romanian Mathematical Magazine*** and later sent me a LaTeX file with his solution.

Concerning Extra (by Alexandewr Bogomolny)
If the left-hand side is denoted $f(a, b)$, the right-hand side becomes $f(b, a)$ and the misread problem suggests that $f(a, b) \geq f(b, a)$, for $a, b > 0$. The problem allows one to swap variables and claim $f(b, a) \geq f(a, b)$ which, in combination, leads to $f(a, b) = f(b, a)$ and this is patently not true.

172. A Cyclic Inequality in Three Variables XXV

Prove that, for $a, b, c \geq 0$,

$$\sum_{cycl}(a - \sqrt{ab} + b)^2 \cdot \sum_{cycl}(a^2 - ab + b^2)^2 \geq 9a^2b^2c^2$$

Proposed by Daniel Sitaru

Solution 1 (same solution by Kevin Soto Palacios and Seyran Ibrahimov).

By the ***AM-GM inequality***, $a - \sqrt{ab} + b \geq \sqrt{ab}$ and $a^2 - ab + b^2 \geq ab$. It thus follows that

$$\sum_{cycl}(a - \sqrt{ab} + b)^2 \cdot \sum_{cycl}(a^2 - ab + b^2)^2 \geq \sum_{cycl} ab \cdot \sum_{cycl} a^2b^2.$$

Again, with the AM-GM inequality, $\sum_{cycl} ab \geq 3\sqrt[3]{a^2b^2c^2}$ and $\sum_{cycl} a^2b^2 \geq 3\sqrt[3]{a^4b^4c^4}$ so that

$$\sum_{cycl} ab \cdot \sum_{cycl} a^2b^2 \geq 9\sqrt[3]{a^2b^2c^2} \cdot \sqrt[3]{a^4b^4c^4} = 9a^2b^2c^2.$$

□

Solution 2 (same solution by Sanong Hauerai and Abdur Rahman).

With ***Bergström's inequality***,

$$\sum_{cycl} (a - \sqrt{ab} + b)^2 \geq \frac{(2\sum_{cycl} a - \sqrt{ab})^2}{3}$$

$$\geq \frac{(\sum_{cycl} a)^2}{3}$$

$$\geq \frac{3(\sum_{cycl} ab)}{3} \geq \sum_{cycl} ab$$

and, similarly, $\sum_{cycl}(a^2 - ab + b^2)^2 \geq \sum_{cycl} a^2b^2$. Further,

$$\sum_{cycl} ab \cdot \sum_{cycl} a^2b^2 \geq 3\sum_{cycl} a^3b^2c \geq 9\sqrt[3]{a^6b^6c^6} = 9a^2b^2c^2.$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and later communicated the above proof in a LaTex file. Solution 1 is by Kevin Soto Palacios and, independently, by Seyran Ibrahimov; Solution 2 is by Sanong Hauerai and a similar solution by has been posted by Abdur Rahman.

173. An Identity in Triangle with a 135 Degrees Angle

Prove that in ΔABC

$$\frac{s+r}{R+r} \geq \sqrt{2} \Leftrightarrow \max\{A, B, C\} = 135^\circ$$

where s, R, r are the semiperimeter, the circumradius, the inradius of ΔABC, respectively.

Proposed by Mehmet Şahin

Solution (by Daniel Sitaru).

WLOG, assume $A = 135^\circ$ so that, by ***the Law of Sines***, $R = \frac{a}{a \sin A} = \frac{a\sqrt{2}}{2}$; $[\Delta ABC] = \frac{1}{2}\sin 135^\circ$; and, by the ***Law of Cosines***, $a^2 + b^2 + c^2 + \sqrt{2}bc$.

Now we have a sequence of equivalent identities:

$$\frac{s+r}{R+r} \Leftrightarrow \sqrt{2}R + r(\sqrt{2}-1)r$$

$$\Leftrightarrow s = a + r(\sqrt{2}-1)r$$

$$\Leftrightarrow a + b + c = 2a + 2r(\sqrt{2}-1) \Leftrightarrow b + c - a = (\sqrt{2}-1)\frac{\sqrt{2}bc}{2s}$$

$$\Leftrightarrow (a+b+c)(b+c-a) = (2-\sqrt{2})bc$$
$$\Leftrightarrow (b+c)^2 - a^2 = (2-\sqrt{2})bc \Leftrightarrow (b+c)^2 - b^2 - c^2 - \sqrt{2}bc = (2-\sqrt{2})bc$$
$$\Leftrightarrow (2-\sqrt{2})bc = (2-\sqrt{2})bc \Leftrightarrow bc \Leftrightarrow 0 = 0.$$

□

Acknowledgment (by Alexander Bogomolny)
The problem by Mehmet Sahin has been kindly posted by Daniel Sitaru at the ***CutTheKnotMath facebook page***, along with a solution of his.

174. Dan Sitaru's Cyclic Inequality In Three Variables with Constraints III

Let $x, y, z > 0$ and $\sqrt{xy} + \sqrt{yz} + \sqrt{zx} = 2$ then:

$$12 + \sum_{cycl}\left(\sqrt{\frac{x^3}{y}} + \sqrt{\frac{x^3}{y}}\right) \geq 8(x+y+z).$$

Proposed by Daniel Sitaru

Solution 1(by Ravi Prakash).
Let $x = a^2, b = y^2, c = z^2$, where $a, b, c > 0$. Then $ab + bc + ca = 2$. Now consider,

$$\frac{a^3}{b^3} + \frac{b^3}{a} + 6ab - 4a^2 - 4b^2$$
$$= \frac{1}{2ab}(a^2 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4)$$
$$= \frac{1}{ab}(a-b)^4 \geq 0.$$

By deriving two similar inequalities and summing up we get

$$6\sum_{cycl} ab + \sum_{cycl}\left(\frac{a^3}{b} + \frac{b^3}{a}\right) \geq 8\sum_{cycl} a^2$$

which is, after a face lift, the required inequality. □

Solution 2(by Kevin Soto Palacios).

$$6\sum_{cycl}\sqrt{xy} + \sum_{cycl}\left(\sqrt{\frac{x^3}{y}} + \sqrt{\frac{x^3}{y}}\right) \geq 4\sum_{cycl}(x+y).$$

Suffice it to prove that $\sqrt{\frac{x^3}{y}} + \sqrt{\frac{x^3}{y}} + 6\sqrt{xy} \geq 4(x+y)$ which is equivalent to $x^2 + y^2 + 6xy \geq 4\sqrt{xy}(x+y)$, i.e. $(\sqrt{x} - \sqrt{y})^4 \geq 0$. Summing up yields the required inequality. Equality is achieved for

$$x = y = z = \frac{2}{3}$$

□

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted a problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Solution 1 is by Ravi Prakash (India); Solution 2 is by Kevin Soto Palacios (Peru). Hoang Tung, Aziz, Sanong Hauerai have independently arrived at variants of those.

175. An Inequality with Angles and Integers

Prove that for integers k and l, and for any $\alpha,\beta\in\left(0,\frac{\pi}{2}\right)$ the following inequality holds:

$$k^2\tan\alpha+l^2\tan\beta\geq\frac{2kl}{\sin(\alpha+\beta)-(k^2+l^2)\cot(\alpha+\beta)}$$

Proposed by Daniel Sitaru

Solution 1(by Daniel Sitaru).

By the ***AM-GM inequality***,

$$k^2\cos^2\beta+l^2\cos^2\alpha\geq 2\sqrt{k^2l^2\cos^2\alpha\cos^2\beta}$$
$$=2|kl|\cos\alpha\cos\beta\geq 2kl\cos\alpha\cos\beta.$$

Thus, we have a sequence of equivalent inequalities:

$$k^2\cos^2\beta+l^2\cos^2\alpha-2kl\cos\alpha\cos\beta\geq 0$$
$$\frac{k^2\cos^2\beta}{\sin(\alpha+\beta)\cos\alpha\cos\beta}+\frac{l^2\cos^2\alpha}{\sin(\alpha+\beta)\cos\alpha\cos\beta}-\frac{2kl\cos\alpha\cos\beta}{\sin(\alpha+\beta)\cos\alpha\cos\beta}\geq 0$$
$$\frac{k^2\cos\beta}{\sin(\alpha+\beta)\cos\alpha}+\frac{l^2\cos\alpha}{\sin(\alpha+\beta)\cos\beta}-\frac{2kl}{\sin(\alpha+\beta)}\geq 0$$
$$\frac{k^2\cos(\alpha+\beta-\alpha)}{\sin(\alpha+\beta)\cos\alpha}+\frac{l^2\cos(\alpha+\beta-\beta)}{\sin(\alpha+\beta)\cos\beta}-\frac{2kl}{\sin(\alpha+\beta)}\geq 0$$
$$k^2\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}\right)+l^2\left(\frac{\sin\beta}{\cos\beta}+\frac{\cos(\alpha+\beta)}{\sin(\alpha+\beta)}-\frac{2kl}{\sin(\alpha+\beta)}\right)\geq 0$$
$$k^2(\tan\alpha+\cot(\alpha+\beta))+l^2(\tan\beta+\cot(\alpha+\beta))\geq\frac{2kl}{\sin(\alpha+\beta)}$$
$$k^2\tan\alpha+l^2\tan\beta\geq\frac{2kl}{\sin(\alpha+\beta)}-(k^2+l^2)\cot(\alpha+\beta)$$

□

Solution 2 (by Amit Itagi).

$$k^2\tan\alpha+l^2\tan\beta\geq\frac{2kl}{\sin(\alpha+\beta)}-(k^2+l^2)\cot(\alpha+\beta)\Leftrightarrow$$
$$\left(k^2\frac{\sin\alpha}{\cos\beta}+l^2\frac{\sin\beta}{\cos\beta}\right)\sin(\alpha+\beta)+(k^2+l^2)\cos(\alpha+\beta)\geq 2kl\Leftrightarrow$$
$$\left(k^2\frac{\sin\alpha}{\cos\alpha}+l^2\frac{\sin\beta}{\cos\beta}\right)(\sin\alpha\cos\beta+\cos\alpha\sin\beta)$$
$$+(k^2+l^2)(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\geq 2kl\Leftrightarrow$$
$$k^2\left(\frac{\sin^2\alpha\cos\beta}{\cos\alpha}+\cos\alpha\cos\beta\right)+l^2\left(\frac{\sin^2\beta\cos\alpha}{\cos\beta}+\cos\alpha\cos\beta\right)\geq 2kl\Leftrightarrow$$
$$\left(k^2\frac{\cos\beta}{\cos\alpha}\right)(\sin^2\alpha+\cos^2\alpha)+\left(l^2\frac{\cos\alpha}{\cos\beta}\right)(\sin^2\beta+\cos^2\beta)\geq 2kl\Leftrightarrow$$
$$k^2\frac{\cos\beta}{\cos\alpha}+l^2\frac{\cos\alpha}{\cos\beta}\geq 2kl.$$

Note that $\cos\alpha,\cos\beta$ and $\sin(\alpha+\beta)$ are positive over the domain defined in the problem. Thus, the last inequality follows from AM-GM and the first inequality can be derived from the last inequality be reversing all the steps. □

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly emailed me a LaTeX file with his solution (Solution 1) to the above problem, originally from the ***School Science and Mathematics Association***. The problem is by ARKADY ALT, SAN JOSE, CA. Solution 2 is by Amit Itagi.

176. An Inequality with Cyclic Sums on Both Sides III

Show that, for positive real numbers x, y and z,

$$\frac{x^6z^3 + y^6x^3 + z^6y^3}{x^2y^2z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2}$$

Proposed by Iuliana Traşcă

Solution (by Daniel Sitaru).
By the ***AM-GM inequality***,

$$x^6z^3 + x^6z^3 + y^6x^3 \geq 3\sqrt[3]{x^{15}z^6y^6} = 3x^5y^2z^2.$$

Similarly,

$$y^6x^3 + y^6x^3 + z^6y^3 \geq 3x^2y^5z^2$$
$$z^6y^3 + z^6y^3 + x^6z^3 \geq 3x^2y^2z^5$$

By adding up,

$$3(x^6z^3 + y^6x^3 + z^6y^3) \geq 3x^2y^2z^2(x^3 + y^3 + z^3),$$

i.e.,

$$x^6z^3 + y^6z^3 + z^6y^3 \geq x^2y^2z^2(x^3 + y^3 + z^3)$$

Suffice it to prove that

$$\frac{x^2y^2z^2(x^3 + y^3 + z^3)}{x^2y^2z^2} \geq \frac{x^3 + y^3 + z^3 + 3xyz}{2},$$

or, equivalently,

$$2(x^3 + y^3 + z^3) \geq x^3 + y^3 + z^3 + 3xyz,$$

i.e.,

$$x^3 + y^3 + z^3 \geq 3xyz,$$

which is true by the AM-GM inequality: $x^3 + y^3 + z^3 \geq 3\sqrt[3]{x^3y^3z^3} = 3xyz$. □

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly emailed me a LaTex file with his solution to the above problem, originally from the ***School Science and Mathematics Association***. The problem is by Iuliana Traşcă, Scornicesti, Romania.

177. A Limit with Fractions, Roots, Powers and Series

Find the limit

$$\Omega = \lim_{n\to\infty} \frac{\sqrt[3]{(1 + \frac{1}{\sqrt[5]{2}} + \frac{1}{\sqrt[5]{3}} + \cdots + \frac{1}{\sqrt[5]{n}})^2}}{\sqrt[5]{(1 + \frac{1}{\sqrt[3]{2}} + \frac{1}{\sqrt[3]{3}} + \cdots + \frac{1}{\sqrt[3]{n}})^4}}$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

Applying the *Stolz-Cesaro theorem*,

$$\lim_{n\to\infty}\frac{1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\cdots+\frac{1}{\sqrt[3]{n}}}{n^{\frac{2}{3}}}=\lim_{n\to\infty}\frac{\frac{1}{\sqrt[3]{n+1}}}{(n+1)^{\frac{2}{3}}-n^{\frac{2}{3}}}$$

$$=\lim_{n\to\infty}\frac{1}{(n+1)^{\frac{1}{3}}\cdot n^{\frac{2}{3}}\left[(1+\frac{1}{n})^{\frac{2}{3}}-1\right]}=\lim_{n\to\infty}\frac{1}{(1+\frac{1}{n})^{\frac{1}{3}}\cdot\frac{(1+\frac{1}{n})^{\frac{2}{3}}-1}{\frac{1}{n}}}$$

$$=\frac{1}{\frac{2}{3}}=\frac{3}{2}$$

Similarly,

$$\lim_{n\to\infty}\frac{1+\frac{1}{\sqrt[5]{2}}+\frac{1}{\sqrt[5]{3}}+\cdots+\frac{1}{\sqrt[5]{n}}}{n^{\frac{4}{5}}}=\frac{5}{4}.$$

Thus

$$\Omega=\lim_{n\to\infty}\frac{\sqrt[3]{(1+\frac{1}{\sqrt[5]{2}}+\frac{1}{\sqrt[5]{3}}+\cdots+\frac{1}{\sqrt[5]{n}})^2}}{\sqrt[5]{(1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\cdots+\frac{1}{\sqrt[3]{n}})^4}}$$

$$=\lim_{n\to\infty}\frac{(1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\cdots+\frac{1}{\sqrt[3]{n}})^{-\frac{4}{5}}}{(1+\frac{1}{\sqrt[5]{2}}+\frac{1}{\sqrt[5]{3}}+\cdots+\frac{1}{\sqrt[5]{n}})^{-\frac{2}{3}}}$$

$$=\lim_{n\to\infty}\left(\frac{1+\frac{1}{\sqrt[3]{2}}+\frac{1}{\sqrt[3]{3}}+\cdots+\frac{1}{\sqrt[3]{n}}}{n^{\frac{2}{3}}}\right)^{-\frac{4}{5}}\cdot\left(\frac{n^{\frac{5}{4}}}{1+\frac{1}{\sqrt[5]{2}}+\frac{1}{\sqrt[5]{3}}+\cdots+\frac{1}{\sqrt[5]{n}}}\right)^{-\frac{2}{3}}$$

$$=\frac{(\frac{5}{4})^{-\frac{2}{3}}}{(\frac{3}{2})^{-\frac{4}{5}}}=\frac{(\frac{3}{2})^{\frac{4}{5}}}{(\frac{5}{4})^{\frac{2}{3}}}$$

□

Solution 2 (by Amit Itagi).

For some $k>1$, consider the series

$$S_n=\frac{1}{1+\frac{1}{2^{\frac{1}{k}}}+\frac{1}{3^{\frac{1}{k}}}+\cdots+\frac{1}{n^{\frac{1}{k}}}}.$$

This series converges to 0 as $n\to\infty$. We claim this series approaches 0 with the leading term as $\frac{1}{n^q}$ for some $q<1$. Thus, if we express S_n as

$$S_n=\frac{W_n}{n^q},$$

then the series W_n has a finite non-zero limit (say x) as $n\to\infty$. Thus,

$$\frac{1}{S_n}-\frac{1}{S_{n-1}}=\frac{1}{n^{\frac{1}{k}}}$$

$$\frac{n^q}{W_n}-\frac{(n-1)^q}{W_{n-1}}=\frac{1}{n^{\frac{1}{k}}}$$

In the limit as $n\to\infty$,

$$x=n^{\frac{1}{k}}[n^q-(n-1)^q]$$

$$=n^{\frac{q+1}{k}}\left[1-\left(1-\frac{1}{n}\right)^q\right]$$

$$\sim n^{\frac{q+1}{k}} \frac{q}{n}$$

$$\sim q^{n^{\frac{q+1}{k-1}}}$$

For this to be a constant, the leading power of n has to be 0. Thus, $q = 1 - \frac{1}{k}$ and $x = q$.
Thus up to the leading term,

$$\frac{1}{1 + \frac{1}{2^{\frac{1}{3}}} + \frac{1}{3^{\frac{1}{3}}} + \cdots + \frac{1}{n^{\frac{1}{3}}}} \sim \frac{2}{3n^{\frac{2}{3}}}$$

$$\frac{1}{1 + \frac{1}{2^{\frac{1}{5}}} + \frac{1}{3^{\frac{1}{5}}} + \cdots + \frac{1}{n^{\frac{1}{5}}}} \sim \frac{4}{5n^{\frac{4}{5}}},$$

and the limit for the original problem is

$$L = \frac{(\frac{2}{3})^{\frac{4}{5}}}{(\frac{4}{5})^{\frac{2}{3}}} \sim 0.839.$$

□

Solution 3 (by Leonard Giugiuc).
We first prove the following lemma:

If $\alpha > -1$, then

$$\lim_{n\to\infty} \left(\frac{1^\alpha + 2^\alpha + \cdots + n^\alpha}{n^{\alpha+1}} \right) = \frac{1}{\alpha + 1}$$

Indeed,

$$\lim_{n\to\infty} \left(\frac{1^\alpha + 2^\alpha + \cdots + n^\alpha}{n^{\alpha+1}} \right) = \lim_{n\to\infty} \left(\frac{1}{n} \cdot \sum_{k=1}^{n} \left(\frac{k}{n} \right)^\alpha \right)$$

$$= \int_0^1 x^\alpha dx = \frac{1}{\alpha + 1}.$$

Now, with $\alpha = -\frac{1}{5}$,

$$\lim_{n\to\infty} \frac{1 + 2^{-\frac{1}{5}} + \cdots + n^{-\frac{1}{5}}}{n^{\frac{4}{5}}} = \frac{5}{4}$$

and with $\alpha = -\frac{1}{3}$,

$$\lim_{n\to\infty} \frac{1 + 2^{-\frac{1}{3}} + \cdots + n^{-\frac{1}{3}}}{n^{\frac{2}{3}}} = \frac{3}{2}$$

Interchanging the operations,

$$\lim_{n\to\infty} \frac{\left(\sum_{k=1}^{n} \frac{1}{\sqrt[5]{k}} \right)^{\frac{2}{3}}}{\left(\sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} \right)^{\frac{4}{5}}} = \lim_{n\to\infty} \frac{\left(\frac{\left[\sum_{k=1}^{n} \frac{1}{\sqrt[5]{k}} \right]}{n^{\frac{4}{5}}} \right)^{\frac{2}{3}}}{\left(\frac{\left[\sum_{k=1}^{n} \frac{1}{\sqrt[3]{k}} \right]}{n^{\frac{2}{3}}} \right)^{\frac{4}{5}}} = \frac{\left(\frac{3}{2} \right)^{\frac{4}{5}}}{\left(\frac{5}{4} \right)^{\frac{2}{3}}}.$$

□

Solution 4 (by Nassim Nicolas Taleb).

$$a_0 = \int_0^K \frac{1}{\sqrt[5]{u}} du = \frac{5}{4} K^{\frac{4}{5}}$$

$$a_1 = \int_0^K \frac{1}{\sqrt[3]{u}} du = \frac{3}{2} K^{\frac{2}{3}}$$

$\Omega \to \frac{a_0^{\frac{2}{3}}}{a_1^{\frac{4}{5}}}$ for N large, which is what we are looking for. So, as k comes out from

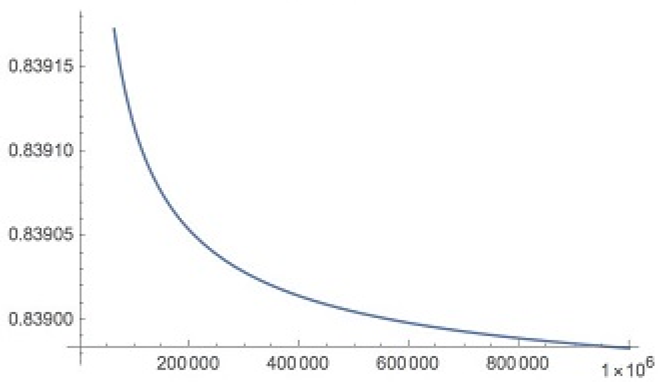
$$\frac{\left(\frac{5K^{\frac{4}{5}}}{4}\right)^{\frac{2}{3}}}{\left(\frac{3K^{\frac{2}{3}}}{2}\right)^{\frac{4}{5}}},$$

$$\frac{a_0^{\frac{2}{3}}}{a_1^{\frac{4}{5}}} = \frac{5^{\frac{2}{3}}}{2^{\frac{8}{15}} 3^{\frac{4}{5}}} \simeq 0.838945.$$

Added a Mathematica double check and intuition builder:

And a Mathematica double check

$$\texttt{Plot}\left[\frac{\texttt{HarmonicNumber[n, k1]}^{\texttt{p}}}{\texttt{HarmonicNumber[n, k2]}^{\texttt{q}}} \texttt{ /. } \left\{\texttt{k1} \to \frac{1}{5}, \texttt{k2} \to \frac{1}{3}, \texttt{p} \to \frac{2}{3}, \texttt{q} \to \frac{4}{5}\right\}, \texttt{\{n, 1, 10^6\}}\right]$$



□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly emailed me a LaTeX file with the above problem and his solution (Solution 1); the problem has been originally published at the ***Romanian Mathematical Magazine***. Solution 2 is by Amit Itagi; Solution 3 is by Leo Giugiuc; Solution 4 is by N. N. Taleb.

178. Four Integrals in One Inequality

If $f : [a, b] \to (0, \infty)$***, where*** $0 < a < b$***, is a continuos increasing function, then***

$$\left(\int_a^b x f(x) dx\right)\left(\int_a^b f^2(x) dx\right)\left(\int_a^b x^3 f(x) dx\right) \geq \frac{a^2 b^2}{b - a}\left(\int_a^b f(x) dx\right)^4$$

Proposed by Daniel Sitaru

Solution (by Leonard Giugiuc).

By *Chebysev's inequality* (this is where we need function $f(x)$ to be increasing)

$$\int_a^b xf(x)dx \geq \frac{1}{b-a}\left(\int_a^b xdx\right)\left(\int_a^b f(x)dx\right) = \frac{a+b}{2}\int_a^b f(x)dx.$$

Similarly,

$$\int_a^b x^3 f(x)dx \geq \frac{a^3+a^2b+ab^2+b^3}{4}\int_a^b f(x)dx$$

By the ***Cauchy - Schwarz inequality***,

$$\int_a^b f^2(x)dx \geq \frac{1}{b-a}\left(\int_a^b f(x)dx\right)^2$$

By the ***AM-GM inequality***, $\frac{a+b}{2} \cdot \frac{a^3+a^2b+ab^2+b^3}{4} \geq a^2b^2$. Multiplying the three inequality yields the required one. □

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Leonard Giugiuc has commented with his solution.

179. A Cyclic Inequality in Three or More Variables

Prove that, for $\boldsymbol{a, b, c > 0}$***, subject to***

$$\sum_{cycl}\frac{1}{a+b} = \frac{1}{9},$$

$$\sum_{cycl}\frac{a+b}{(a-b)^2} + 2\cdot\sum_{cycl}\frac{1}{a} \geq 1.$$

Proposed by Daniel Sitaru

Solution 1 (by Leonard Giugiuc).

We start with proving a lemma:

Prove that, for $a, b > 0$,

$$\frac{a+b}{(a-b)^2} + \frac{1}{a} + \frac{1}{b} \geq \frac{9}{a+b}$$

By cross-multiplying, the above inequality reduces to

$$a^3 - 8a^3b + 18a^2b^2 - 8ab^3 + b^4 = (a^2 - 4ab + b^2)^2 \geq 0.$$

Equality occurs for $a = (2 \pm \sqrt{3})b$.

The given inequality is an immediate consequence of the lemma:

$$\sum_{cycl}\frac{a+b}{(a-b)^2} + 2\cdot\sum_{cycl}\frac{1}{a} = \sum_{cycl}\Big[\frac{a+b}{(a-b)^2} + \frac{1}{a} + \frac{1}{b}\Big]$$

$$\geq \sum_{cycl}\frac{9}{a+b} = 9\cdot\frac{1}{9} = 1.$$

□

A little extra(by Alexander Bogomolny)

The appearance of the above inequality and the proof suggest a little more general result:

Prove that, for an integer $n \geq 3$ and n positive real numbers $a, b, c, \ldots,$ subject to

$$\sum_{cycl} \frac{1}{a+b} = \frac{1}{9},$$

$$\sum_{cycl} \frac{a+b}{(a-b)^2} + 2\sum_{cycl} \frac{1}{a} \geq 1.$$

It is interesting that by just looking at the constraint and the inequality, it is impossible to ascertain the number of variables involved. The situation changes if we replace "cyclic" sums with "symmetric" sums: symmetric in the sense that every variable is paired with any other exactly once. For $n = 3$ there is not difference, but for $n = 4$, the cyclic pairing consists of the four pairs $(a, b), (b, c), (c, d), (d, a)$, whereas the symmetric pairing consists of six pairs: $(a, b), (a, c), (a, d), (b, c), (b, d), (c, d)$. If we write the inequality in the lemma for each of the six pairs and, subsequently, add them all up, we'll get

$$\sum_{sym} \frac{a+b}{(a-b)^2} + 3\Big(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \frac{1}{d}\Big) \geq 1,$$

provided $\sum_{sym} \frac{1}{a+b} = \frac{1}{9}$.
In general,

Prove that, for an integer $n \geq 3$ and n positive real numbers $a, b, c, \ldots,$ subject to

$$\sum_{sym} \frac{1}{a+b} = \frac{1}{9},$$

$$\sum_{sym} \frac{a+b}{(a-b)^2} + (n-1) \cdot \sum_{cycl} \frac{1}{a} \geq 1.$$

It should be noted that the above example, replacing, say, the pair (a, b) with the pair (b, a) would not change any of the expressions we encountered above, which allowed us to disregard half of the pairs. In general, a symmetric sum in four variables of the expressions that depend on only two of them would include twelve terms, for five variables, twenty.

Solution 2 (by Soumitra Mandal).

Using Bergström inequality,

$$\sum_{cycl} \frac{a+b}{(a-b)^2} + 2\sum_{cycl} \frac{1}{a} = \sum_{cycl} (a+b)\Big(\frac{1}{(a-b)^2} + \frac{4}{4ab}\Big)$$

$$\geq \sum_{cycl} (a+b) \cdot \frac{(1+2)^2}{(a-b)^2 + 4ab} = 9\sum_{cycl} \frac{1}{a+b} = 1.$$

Acknowledgment(by Alexander Bogomolny)

Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheMath facebook page***. The post has been commented on by Leonard Giugiuc who supplied the lemma from which the solution is immediate. Solution 2 is by Soumitra Mandal. $\square$

180. Same Integral, Three Intervals

Define

$$I(u,v)=\int_u^v\left(\arctan\left(\frac{u\sin x}{v+u\cos x}\right)+\arctan\left(\frac{v\sin x}{u+v\cos x}\right)\right)dx.$$

Let distinct real numbers a,b,c lie in $\left(0,\frac{\pi}{2}\right)$

Prove that

$$\sum_{cycl}\frac{2}{b-a}I(a,b)\geq\sum_{cycl}\left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right).$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

Let $\alpha=\arctan\left(\frac{u\sin x}{v+u\cos x}\right)+\arctan\left(\frac{v\sin x}{u+v\cos x}\right)$. Then

$$\tan\alpha=\frac{\frac{u\sin x}{v+u\cos x}+\frac{b\sin x}{u+b\cos x}}{1-\frac{uv\sin^2 x}{(v+u\cos x)(u+b\cos x)}}$$

$$\tan\alpha=\frac{u\sin x(u+v\cos x)+v\sin x(v+u\cos x)}{(v+u\cos x)(u+v\cos x)-uv\sin^2 x}$$

$$\tan\alpha=\frac{(u^2+v^2)\sin x+2uv\sin x\cos x}{uv+v^2\cos x+u^2\cos x+uv\cos^2 x-uv\sin^2 x}$$

$$\frac{\sin x(u^2+v^2+2uv\cos x)}{\cos x(u^2+v^2)+uv(1+\cos 2x)}=\tan\alpha$$

$$\frac{\sin x(u^2+v^2+2uv\cos x)}{\cos x(u^2+v^2)+2uv\cos^2 x}=\tan\alpha$$

$$\frac{\sin x}{\cos x}=\tan\alpha\Rightarrow\tan\alpha=\tan x\Rightarrow\alpha=x$$

$$\frac{\sin x}{\cos x}=\tan\alpha\Rightarrow\tan\alpha=\tan x\Rightarrow\alpha=x$$

$$I(u,v)=\int_u^v xdx=\frac{x^2}{2}\Big|_u^v=\frac{v^2-u^2}{2}.$$

Thus we have

$$\frac{2}{b-a}I(a,b)+\frac{2}{c-b}I(b,c)+\frac{2}{a-c}I(a,c)=2(a+b+c).$$

Suffice it to show that

$$2(a+b+c)\geq\sum\left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right).$$

Let $A=\sqrt{\frac{u^2+v^2}{2}}; B=\sqrt{uv}; 2A^2=u^2+v^2; B^2=uv$. Further

$$(u+v)^2=u^2+v^2+2uv=2A^2+2B^2$$

$$u+v\geq A+B\Leftrightarrow(u+v)^2\geq(A+B)^2$$

$$2A^2+2B^2\geq(A+B)^2\Leftrightarrow(A-B)^2\geq 0.$$

It follows that

$$a+b \geq \sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}$$

$$b+c \geq \sqrt{bc}+\sqrt{\frac{b^2+c^2}{2}}$$

$$c+a \geq \sqrt{ac}+\sqrt{\frac{a^2+c^2}{2}}$$

and, finally,

$$2(a+b+c) \geq \sum\left(\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right).$$

□

Solution 2 (by Nassim Nicolas Taleb).

$$I(u,v)=\int_u^v \arctan\left(\frac{u\sin(v)}{u\cos(v)+v}\right)+\arctan\left(\frac{v\sin(v)}{u+v\cos(v)}\right)dx.$$

We have the following property:

$$\arctan(a)+\arctan(b)=\arctan\left(\frac{a+b}{1-ab}\right)+1_{0\leq ab\leq 1}\pi$$

(note the mistake in Abramowicz & Stigum, p 80)

$$\tan(\arctan(a)+\arctan(b))=\frac{a+b}{1-ab},$$

$$a,b\in\left[0,\frac{\pi}{2}\right].$$

Allora

$$\tan\left(\arctan\left(\frac{u\sin(x)}{u\cos(x)+v}\right)+\arctan\left(\frac{v\sin(x)}{u+v\cos(x)}\right)\right)=\tan(x)$$

Since all variables are in $\left(0,\frac{\pi}{2}\right)$, $I(u,v)=\frac{v^2}{2}-\frac{u^2}{2}$, the integrand becomes mysteriously x, so

$$I(a,b)+I(c,a)+I(b,c)=2(a+b+c)$$

We can prove that

$$\frac{\sqrt{a^2+b^2}}{\sqrt{2}}+\frac{\sqrt{a^2+c^2}}{\sqrt{2}}+\frac{\sqrt{b^2+c^2}}{\sqrt{2}}+\sqrt{ab}+\sqrt{ac}+\sqrt{bc}-2(a+b+c)\leq 0$$

for $a,b,c\in\left[0,\frac{\pi}{2}\right]$, with equality for $a=b=c=1$.

□

Sidebar (by Alexander Bogomolny)

In the process found a potential scary error in the literature. People seem to have suspected it on @StackMath

The identity in Abr and Steg, p 80:

```
ArcTan[x] ± ArcTan[y] = ArcTan[(x±y)/(1∓x y)]
```

fails to work: if both x and y are of the same sign, there is a shift by π

```
ArcTan[x] + ArcTan[y] - ArcTan[(x + y)/(1 - x y)] /. {x → 54, y → 54} // FullSimplify
```

π

```
ArcTan[x] + ArcTan[y] - ArcTan[(x + y)/(1 - x y)] /. {x → -55, y → 54} // FullSimplify
```

0

```
ArcTan[x] + ArcTan[y] - ArcTan[(x + y)/(1 - x y)] /. {x → -55, y → -54} // FullSimplify
```

$-\pi$

Riemann Surfaces, sort of. Below is the Abr. & Stig. Now used for 50 years! 4.4.34

$$\arctan z_1 \pm \arctan z_2 = \arctan\left(\frac{z_1 \pm z_2}{1 \mp z_1 z_2}\right)$$

Acknowledgment (by Alexander Bogomolny)
This is a Daniel Sitaru's problem form the ***Romanian Mathematical Magazine***. Daniel has kindly sent me the problem and his solution on a LaTeX file, as did N. N. Taleb (Solution 2). I very much appreciate this kind of thoughtfulness.

181. Dan Sitaru's Inequality with Three Related Integrals and Derivatives

Let be $a > 0; f : [0, a] \to \mathbb{R}; f(a) = f'(a) = 0, f$ is twice continuously differentiable on $[0, a]$, i.e., $f \in C^2[0, a]$. Prove that

$$\left(\int_0^a f(x)dx\right)^4 \leq \frac{a^8}{60}\left(\int_0^a (f'(x))^2 dx\right)\left(\int_0^a (f''(x))^2 dx\right).$$

Proposed by Daniel Sitaru

Solution(same solution by Daniel Sitaru and Amit Itagi).

We shall repeatedly use *integration by parts.*

$$\int_0^a f(x)dx = \int_0^a x' f(x)dx = xf(x)\Big|_0^a - \int_0^a xf'(x)dx$$

$$= -\int_0^a xf'(x)dx$$

$$\left(\int_0^a f(x)dx\right)^2 = \left(\int_0^a xf'(x)dx\right)^2 \leq \int_0^a x^2 dx \cdot \int_0^a (f'(x))^2 dx$$

$$= \frac{x^3}{3}\Big|_0^a \cdot \int_0^a (f'(x))^2 dx = \frac{a^3}{3}\int_0^3 (f'(x))^2 dx$$

$$= 3\left(\int_0^a f(x)dx\right)^2 \le a^3 \int_0^a (f'(x))^2 dx$$

$$\int_0^a f(x)dx = -\int_0^a xf'(x)dx = -\int_0^a \left(\frac{x^2}{2}\right)' f'(x)dx$$

$$= -\left(\frac{x^2}{2} f'(x)\Big|_0^a - \int_0^a \frac{x^2}{2} f''(x)dx\right) = \frac{1}{2}\int_0^a x^2 f''(x)dx$$

$$\le \left(2\int_0^a f(x)dx\right)^2 = \left(\int_0^a x^2 f''(x)dx\right)^2 \le \int_0^a x^4 dx \cdot \left(\int_0^a (f''(x))^2 dx\right)$$

$$= \frac{x^5}{5}\Big|_0^a \left(\int_0^a (f''(x))^2 dx\right) = \frac{a^5}{5}\left(\int_0^a (f''(x))^2 dx\right)$$

$$4\left(\int_0^a f(x)dx\right)^2 \le \frac{a^5}{5}\left(\int_0^a (f''(x))^2 dx\right)$$

$$3\left(\int_0^a f(x)dx\right)^2 \le a^3\left(\int_0^a (f'(x))^2 dx\right)$$

$$12\left(\int_0^a f(x)dx\right)^4 \le \frac{a^8}{5}\left(\int_0^a (f'(x))^2 dx\right)\left(\int_0^a (f''(x))^2 dx\right)$$

$$\left(\int_0^a f(x)dx\right) \le \frac{a^8}{60}\left(\int_0^a (f'(x))^2 dx\right)\left(\int_0^a (f''(x))^2 dx\right)$$

□

Ackonwledgment (by Alexander Bogomolny)
This is a Daniel Sitaru's problem from the ***Romanian Mathematical Magazine***. Daniel has kindly sent me the problem and his solution on a LaTeX file. I very much appreciate this kind of thoughtfulness. Amit Itagi has independently come up with the same solution.

182. Dan Sitaru's Cyclic Inequality in Three Variables

Prove that if $\boldsymbol{a, b, c > 0}$ ***then***

$$\frac{(5a+b)(5b+c)(5c+a)}{27(a+8c)(b+8a)(c+8b)} \ge \frac{8abc}{(5a+4b)(5b+4c)(5c+a)}$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

$$(a-b)^2 \ge 0$$

$$a^2 - 2ab + b^2 \ge 0$$

$$24a^2 + 27ab + 3b^2 \le 25a^2 + 25ab + 4b^2$$

$$3(8a^2 + 8ab + ab + b^2) \le 25a^2 + 5ab + 20ab + 4b^2$$

$$3(8ab + b)(a + b) \le (5a + 4b)(5a + b)$$

$$\frac{5a+b}{3(8a+b)} \ge \frac{a+b}{5a+4b} \overbrace{\ge}^{AM-GM} \frac{2\sqrt{ab}}{5a+4b}.$$

(1) $$\frac{5a+b}{3(8a+b)} \geq \frac{2\sqrt{ab}}{5a+4b}.$$

(2) $$\frac{5b+c}{3(8b+c)} \geq \frac{2\sqrt{bc}}{5b+4c}$$

(3) $$\frac{5c+a}{3(8c+a)} \geq \frac{2\sqrt{ca}}{5c+4a}$$

By multiplying the relationships (1), (2), (3),

$$\frac{(5a+b)(5b+c)(5c+a)}{27(a+8c)(b+8a)(c+8b)} \geq \frac{8abc}{(5a+b)(5b+c)(5c+4a)}$$

□

Solution 2 (by Leonard Giugiuc).

We have $(x-1)^2(25x^2+2x+4) \geq 0, x > 0$, in particular. This is equivalent to

$$(5x^2+4)(5x^2+1) \geq 6x(8x^2+1).$$

Letting $x = \sqrt{\frac{a}{b}}$ translates into

$$(5a+4b)(5a+b) \geq 6\sqrt{ab}(8a+b).$$

This shows that for any number of variables,

$$\prod_{cycl}(5a+4b)(5a+b) \geq 6^n \prod_{cycl} a \prod_{cycl}(8a+b),$$

where n is the number of variables. □

Aknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly shared a problem from the ***Romanian Mathematical Magazine***, with a solution of his mailed on a LaTeX file, which I appreciate greatly. Solution 2 is by Leo Giugiuc.

183. An Inequality in Two or More Variables

Prove that if $a, b, c > 0$ and $abc = 1$ then

$$\frac{a}{1+a} + \frac{b}{(1+a)(1+b)} + \frac{c}{(1+a)(1+b)(1+c)} \geq \frac{7}{8}$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

$$\frac{a}{1+a} + \frac{b}{(1+a)(1+b)} + \frac{c}{(1+a)(1+b)(1+c)}$$
$$= \frac{a(1+b)(1+c)+b(1+c)+c}{(1+a)(1+b)(1+c)}$$
$$= \frac{(1+c)(a+ab+b+1)-1}{(1+a)(1+b)(1+c)}$$
$$= \frac{(1+c)(1+b)(1+c)-1}{(1+a)(1+b)(1+c)}$$
$$= 1 - \frac{1}{(1+a)(1+b)(1+c)} \overbrace{\geq}^{AM-GM} 1 - \frac{1}{2\sqrt{a}\cdot 2\sqrt{b}\cdot 2\sqrt{c}}$$

$$= 1 - \frac{1}{8\sqrt{abc}} = 1 - \frac{1}{8} = \frac{7}{8}$$

□

A little extra(by Alexander Bogomolny)

The above statement and proof extend easily to a number $n \geq 2$ of variables:

$$\textit{Prove that if } a_k > 0, k = 1, 2, \ldots, n \textit{ and } \prod_{k=1}^{n} a_k = 1 \textit{ then}$$

$$\sum_{k=1}^{n} \left(a_k \prod_{i=1}^{k} \frac{1}{1 + a_i} \right) \geq \frac{2^n - 1}{2^n}.$$

The key is the identity derived in the above proof:

$$\sum_{k=1}^{n} \left(a_k \prod_{i=1}^{k} \frac{1}{1 + a_i} \right) = 1 - \prod_{i=1}^{n} \frac{1}{1 + a_i}.$$

The identity can be established by *induction*. Let $s_n = \sum_{k=1}^{n} \left(a_k \prod_{i=1}^{k} \frac{1}{1+a_i} \right)$ and $P_n = 1 - \prod_{i=1}^{n} \frac{1}{1+a_i}$. Then
$S_{n+1} - S_n = a_{n+1} \prod_{i=1}^{n+1} \frac{1}{1+a_i}$, whereas

$$P_{n+1} - P_n = \prod_{i=1}^{n} \frac{1}{1 + a_i} - \prod_{i=1}^{n+1} \frac{1}{1 + a_i} = \prod_{i=1}^{n} \frac{1}{1 + a_i} \left(1 - \frac{1}{1 + a_{n+1}} \right)$$

$$= \prod_{i=1}^{n} \frac{1}{1 + a_i} \left(\frac{a_{n+1}}{1 + a_{n+1}} \right) = a_{n+1} \prod_{i=1}^{n+1} \frac{1}{1 + a_i}.$$

The identity holds for $n = 3$ (and obviously for $n = 1$ and $n = 2$) hence, it holds for any larger n.

Solution 2 (by Amit Itagi).

Multiplying out: $abc + (ab + c) + (bc + a) + (ca + b) \geq 7$, i.e.,
$1 + \left(\frac{1}{c} + c \right) + \left(\frac{1}{a} + a \right) + \left(\frac{1}{b} + b \right) \geq 7$, which follows from the *AM-GM inequality* applied to each pair of parantheses. □

Solution 3 (by Nassim Nicolas Taleb).

$$f = \frac{c}{(a + 1)(b + 1)(c + 1)} + \frac{b}{(a + 1)(b + 1)} + \frac{a}{a + 1}$$

By rearranging the terms,

$$f = 1 - \frac{1}{(a + 1)(b + 1)(c + 1)}$$

$$= 1 - \frac{1}{1 + a + b + c + ab + bc + ca + abc}$$

$$\geq 1 - \frac{27}{(a + b + c + 3)^3}.$$

By the AM-GM inequality,

$$1 + a + b + c + ab + bc + ca + abc \geq 8 * \sqrt[8]{a^4 b^4 c^4} = 8.$$

Can be generalized to n summands. □

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath faceboo page***. He later mailed his solution on a LaTeX file – something I appreciate greatly. Solution 2 is by Amit Itagi; Solution 3 is by N. N. Taleb.
Roland van Gaalen has observed that for a sequence $a_1, a_2, \ldots$ that satisfy $\prod_i a_i = 1$, the sum $\sum_{k=1}^{\infty}\left(a_k \prod_{i=1}^{k} \frac{1}{1+a_i}\right) = 1$.

184. Dan Sitaru's Cyclic Inequality in Three Variables II

Prove that if $a, b, c > 0$ and $a+b+c = 3$ then

$$\sum_{cycl}\sqrt{1+\frac{1}{a^2}+\frac{1}{(a+1)^2}} \ge \frac{9}{12-2(ab+bc+ca)}+3.$$

Proposed by Daniel Sitaru

Solution 1 (same solution by Daniel Sitaru and Ravi Prakash).

We prove that $\sqrt{1+\frac{1}{a^2}+\frac{1}{(a+1)^2}} = 1+\frac{1}{a}-\frac{1}{a+1}$. Indeed, by squaring,

$$1+\frac{1}{a^2}+\frac{1}{(a+1)^2} = 1+\frac{1}{a^2}+\frac{1}{(a+1)^2}+\frac{2}{a}-\frac{2}{a+1}-\frac{2}{a(a+1)}$$

$$0 = 2\left(\frac{1}{a}-\frac{1}{a+1}-\frac{1}{a(a+1)}\right)$$

$$0 = \frac{a+1-a-1}{a(a+1)} \Leftrightarrow 0 = 0.$$

It follows that

$$\sum\sqrt{1+\frac{1}{a^2}+\frac{1}{(a+1)^2}} = \sum\left(1+\frac{1}{a}-\frac{1}{a+1}\right)$$

$$= \sum\left(1+\frac{a+1-a}{a(a+1)}\right)$$

$$3+\sum\frac{1}{a^2+a} \overbrace{\ge}^{Bergstrom} 3+\frac{9}{\sum a^2+\sum a}$$

$$= 3+\frac{9}{(\sum a)^2-2\sum ab+3}$$

$$= 3+\frac{9}{9+3-2\sum ab} = 3+\frac{9}{12-2(ab+bc+ca)}.$$

□

Solution 2 (by Leonard Giugiuc).

We have

$$a^2(a+1)^2+a^2+(a+1)^2 = a^2(a+1)^2+2a(a+1)+1$$
$$= (a^2+a+1)^2.$$

Hence, the required inequality is equivalent to

$$\sum_{cycl}\frac{1}{a(a+1)} \ge \frac{9}{12-2(ab+bc+ca)} \Leftrightarrow$$

$$\sum_{cycl} \frac{1}{a} \geq \frac{9}{12 - 2(ab + bc + ca)} + \sum_{cycl} \frac{1}{a+1}.$$

We'll show that $\frac{3}{2} \geq \frac{9}{12-2(ab+bc+ca)}$ which is equivalent to $3 \geq ab + bc + ca$ and the latter is well known consequence of the constraint $a + b + c = 3$.
Thus, suffice it to prove that $\sum_{cycl} \frac{1}{a} \geq \frac{3}{2} + \sum_{cycl} \frac{1}{a+1}$, which is
$\sum_{cycl} \left(\frac{1-a}{2}\right)\left(\frac{1}{a(a+1)} + \frac{1}{a}\right) \geq 0.$
But the functions $\frac{1-a}{2}$ and $\frac{1}{a(a+1)} + \frac{1}{a}$ are both decreasing, hence, by *Chebyshev's inequality*,

$$\sum_{cycl} \Big(\frac{1-a}{2}\Big)\Big(\frac{1}{a(a+1)} + \frac{1}{a}\Big)$$

$$\geq \frac{1}{3}\Big(\frac{3-a-b-c}{2}\Big)\Big(\frac{1}{a(a+1)} + \frac{1}{a}\Big) = 0.$$

□

Solution 3 (same solution by Nguyen Thanh Nho and Subham Jaiswal).

By *Minkowski's inequality*,

$$\sum_{cycl} \sqrt{1 + \frac{1}{a^2} + \frac{1}{(a+1)^2}} \geq \sqrt{9 + \Big(\sum_{cycl} \frac{1}{a}\Big)^2 + \Big(\sum_{cycl} \frac{1}{a+1}\Big)^2}$$

Now note that

$$\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{1}{a+b+c} = \frac{9}{3} = 3,$$

$$\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1} \geq \frac{1}{a+b+c+3} = \frac{9}{6} = \frac{3}{2}$$

Thus,

$$\sum_{cycl} \sqrt{1 + \frac{1}{a^2} + \frac{1}{(a+1)^2}} \geq \sqrt{9 + 9 + \frac{9}{4}} = \frac{9}{2}.$$

Suffice it to prove that

$$\frac{9}{2} \geq \frac{9}{12 - 2(ab + bc + ca)} + 3,$$

or equivalently,

$$\frac{3}{2} \geq \frac{9}{12 - 2(ab + bc + ca)},$$

i.e.,

$$36 - 6(ab + bc + ca) \geq 12,$$

or

$$ab + bc + ca \leq 3,$$

which is true because

$$ab + bc + ca \leq \frac{1}{3}(a + b + c)^2 = \frac{1}{3}3^2 = 3.$$

□

Solution 4 (by Abdur Rahman).

$y = x^{-2}$ being a convex function,

$$\frac{1+x^{-2}}{2} \geq \left(\frac{1+x}{2}\right)^{-2},$$

i.e.,

$$1 + \frac{1}{x^2} \geq \frac{8}{(1+x)^2},$$

implying

$$1 + \frac{1}{x^2} + \frac{1}{(1+x)^2} \geq \frac{9}{(1+x)^2}.$$

Thus

$$\sum_{cycl} \sqrt{1 + \frac{1}{a^2} + \frac{1}{(1+a)^2}} \geq \sum_{cycl} \sqrt{\frac{9}{(a+1)^2}} = 3\sum_{cycl} \frac{1}{a+1}.$$

By the *AM-HM inequality*, $\sum_{cycl} \frac{1}{1+a} \geq \frac{9}{\sum_{cycl}(a+1)} = \frac{3}{2}$ so that

$$\sum_{cycl} \sqrt{1 + \frac{1}{a^2} + \frac{1}{(1+a)^2}} \geq 3\sum_{cycl} \frac{1}{a+1} \geq \frac{9}{2} \tag{1}$$

Further, $\sum_{cycl} ab \leq \frac{(\sum_{cycl})^2}{3} = 3$, such that $12 - 2\sum_{cycl} ab \geq 6$ and

$$\frac{9}{12 - 2\sum_{cycl} ab} + 3 \leq \frac{9}{6} + 3 = \frac{9}{2}. \tag{2}$$

From (1) and (2),

$$\frac{9}{12 - 2\sum_{cycl} ab} + 3 \leq \frac{9}{2} \leq \sum_{cycl} \sqrt{1 + \frac{1}{a^2} + \frac{1}{(a+1)^2}}$$

□

Solution 5 (by Mike Lawler).

First,

$$1 + \frac{1}{a^2} + \frac{1}{(a+1)^2} = \frac{(a^2+a+1)^2}{a^2(a+1)^2}$$

so that

$$\sqrt{1 + \frac{1}{a^2} + \frac{1}{(a+1)^2}} = \frac{a^2+a+1}{a(a+1)} = 1 + \frac{1}{a(a+1)}.$$

Second,

$$(a+b+c)^2 = 9 = a^2 + b^2 + c^2 + 2(ab+bc+ca)$$

so that

$$12 - 2(ab+bc+ca) = a^2 + b^2 + c^2 + 3.$$

So, the inequality reduces to

$$\sum_{cycl} \left(1 + \frac{1}{a(a+1)}\right) \geq \frac{9}{3 + a^2 + b^2 + c^2} + 3,$$

which simplifies further to

$$\sum_{cycl} \frac{1}{a(a+1)} \geq \frac{9}{a^2 + a + b^2 + b + c^2 + c}.$$

For simplicity, let $x = a(a+1), y = b(b+1), z = c(c+1)$. The required inequality becomes

$$\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \geq \frac{9}{x+y+z}.$$

Multiplying by $(x+y+z)$ we get

$$1 + \frac{y}{x} + \frac{z}{x} + \frac{x}{y} + 1 + \frac{z}{y} + \frac{x}{z} + \frac{y}{z} + 1 \geq 9,$$

or,

$$\left(\frac{y}{x} + \frac{x}{y}\right) + \left(\frac{z}{y} + \frac{y}{z}\right) + \left(\frac{x}{z} + \frac{z}{x}\right) \geq 6$$

which is true because, for $w > 0, w + \frac{1}{w} \geq 2$. □

Solution 6 (by Amit Itagi).

$\sqrt{1 + \frac{1}{a^2} + \frac{1}{(a+1)^2}} = \frac{a^2+a+1}{a(a+1)} \geq \frac{3}{a+1}$ (AM-GM to the numerator), implying

$$\begin{aligned} LHS &\geq 3\left(\frac{1}{a+1} + \frac{1}{b+1} + \frac{1}{c+1}\right) \\ &\geq \frac{27}{(a+1)+(b+1)+(c+1)} \\ &= \frac{9}{2} \text{ (AM-HM)} \end{aligned}$$

From power-mean inequality,

$$a^2 + b^2 + c^2 \geq \frac{(a+b+c)^2}{3} = 3,$$

implying

$$2(ab+bc+ca) = (a+b+c)^2 - (a^2+b^2+c^2) \leq 3^2 - 3 = 6,$$

and,

$$RHS \leq \frac{9}{12-6} + 3 = \frac{9}{2}.$$

Thus,

$$LHS \geq \frac{9}{2} \geq RHS.$$

□

Aknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly shared a problem from his book *Math Accent*, with a solution of his (Solution 1) mailed on a LaTeX file, which I appreciate greatly. He also posted the problem at the ***CutTheKnotMath facebook page*** where it gathered some comments. Solution 2 is by Leo Giugiuc; Solution 3 is by Nguyen Thanh Nho and independently by Subham Jaiswal. Ravi Prakash came up with a solution very close to Solution 1. Solution 4 is by Abdur Rahman; Solution 5 is by Mike Lawler; Solution 6 is by Amit Itagi.

185. An Inequality with Sines But Not in a Triangle

If $a, b, c \in (4, \infty)$ and $abc = 2^{11}$ then

$$\prod_{cycl}\left(a^2 \sin\frac{2\pi}{a} + (a+1)^2 \sin\frac{2\pi}{a+1}\right) > 2^{16}.$$

Proposed by Daniel Sitaru

Solution (same solution by Daniel Sitaru and Nassim Nicolas Taleb).

First off, $x > 4 \Rightarrow x+1 > x > 4 \Rightarrow \frac{1}{x+1} < \frac{1}{x} < \frac{1}{4}$, so that

$$\frac{2}{x+1} < \frac{2}{x} < \frac{1}{2}.$$

And, subsequently,

$$0 < \frac{2\pi}{x+1} < \frac{2\pi}{x} < \frac{\pi}{2}.$$

Now, applying *Jordan's inequality*,

$$\sin\frac{2\pi}{x} \geq \frac{2}{\pi}\cdot\frac{2\pi}{x} = \frac{4}{x},$$

implying

$$a^2 \sin\frac{2\pi}{a} \geq 4a \tag{1}$$

and also

$$(a+1)^2 \sin\frac{2\pi}{a+1} \geq 4(a+1). \tag{2}$$

By adding (1) and (2),

$$a^2\sin\frac{2\pi}{a} + (a+1)^2\sin\frac{2\pi}{a+1} \geq 8a+4$$

$$\overbrace{>}^{AM-GM} \quad 2\sqrt{8a\cdot 4} = 8\sqrt{2a}.$$

$$\prod\left(a^2\sin\frac{2\pi}{a} + (a+1)^2\sin\frac{2\pi}{a+1}\right) > 8^3\cdot 2\sqrt{abc}$$

$$= 2^9\cdot 2\cdot\sqrt{2\cdot 2^{11}} = 2^{10}\cdot 2^6 = 2^{16}.$$

□

Aknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and later mailed me his solution on o LaTeX file which is greatly appreciated. N.N. Taleb has come independently with the same solution.

186. An Inequality with Arbitrary Roots

If $n \in \mathbb{N}, n \geq 2, abc > 1, a+b+c = 3^{n+1}$, ***then***

$$\sum_{cycl}\left(\sqrt[n]{a+\sqrt[n]{a}} + \sqrt[n]{a-\sqrt[n]{a}}\right) < 18$$

Proposed by Daniel Sitaru

Solution 1 (by Khanh Hung Vu).

By the *AM-GM inequality*,

$$\sqrt[n]{a+\sqrt[n]{a}} = \sqrt[n]{a^{\frac{n-1}{n}}(a^{\frac{1}{n}} + a^{-\frac{n-2}{n}})} \leq \frac{(n-1)a^{\frac{1}{n}} + a^{\frac{1}{n}} + a^{-\frac{n-2}{n}}}{n}$$

$$= a^{\frac{1}{n}} + \frac{1}{n}a^{-\frac{n-2}{n}}.$$

Similarly,

$$\sqrt[n]{a-\sqrt[n]{a}} \leq a^{\frac{1}{n}} - \frac{1}{n}a^{-\frac{n-2}{2}}$$

so that

$$\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}} \leq 2\sqrt[n]{a}. \tag{1}$$

On the other hand, again by the AM-GM inequality, $\sqrt[n]{a(3^n)^{n-1}} \leq \frac{a+(n-1)3^n}{n}$, implying

$$\sqrt[n]{a} \leq \frac{a+(n-1)3^n}{n\sqrt[n]{(3^n)^{n-1}}} = \frac{a+(n-1)3^n}{n3^{n-1}}. \tag{2}$$

From (1) and (2),

$$\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}} \leq \frac{2a+(n-1)3^n}{n3^{n-1}}.$$

Thus

$$\sum_{cycl}\Big(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\Big) \leq \sum_{cycl}\frac{2a+(n-1)3^n}{n3^{n-1}}$$
$$= \frac{2\sum_{cycl}a+(n-1)3^{n+1}}{n3^{n-1}} = \frac{2\cdot 3^{n+1}+(n-1)3^{n+1}}{n3^{n-1}}$$
$$= \frac{(n+1)3^{n+1}}{n3^{n-1}} = \frac{9(n+1)}{n} < 18.$$

□

Solution 2 (by Abdur Rahman).

By the mth power theorem, with $m=\frac{1}{n}$, we get

$$\frac{\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}}{2} \leq \Big(\frac{a+\sqrt[n]{a}+a-\sqrt[n]{a}}{2}\Big)^{\frac{1}{n}}$$
$$= \sqrt[n]{a},$$

so that

$$\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}} \leq 2\sqrt[n]{a}.$$

Summing up,

$$\sum_{cycl}\left(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\right) \leq 2\sum_{cycl}\sqrt[n]{a}$$
$$\leq 2\cdot 3\left(\frac{\sum_{cycl}a}{3}\right)^{\frac{1}{n}}$$
$$= 6\Big(\frac{3^{n+1}}{3}\Big)^{\frac{1}{n}} = 6\cdot 3 = 18.$$

□

Solution 3(by Amit Itagi).

From power-mean inequality,

$$2\sqrt[n]{a} > \Big(\sqrt[n]{a+\sqrt[n]{a}}+\sqrt[n]{a-\sqrt[n]{a}}\Big), \text{ (and its cyclic variants)}$$

and

$$6\sqrt[n]{\frac{a+b+c}{3}} = 18 \geq 2(\sqrt[n]{a}+\sqrt[n]{b}+\sqrt[n]{c}).$$

Putting the two together completes the proof. Note that the first inequality has $>$ and not $\geq$ because the two terms in the sum cannot be equal. □

Solution 4 (by Nassim Nicolas Taleb).

By concavity of $(.)^{\frac{1}{n}}$ with $n > 2$, by *Jensen's inequality*,

$$\frac{1}{2}\left(a - a^{\frac{1}{n}}\right)^{\frac{1}{n}} + \frac{1}{2}\left(a + a^{\frac{1}{n}}\right)^{\frac{1}{n}} \leq a^{\frac{1}{n}}$$

Since $\left(\frac{1}{3}\left(a^{\frac{1}{n}} + b^{\frac{1}{n}} + c^{\frac{1}{n}}\right)\right)^n \leq \frac{1}{3}(a+b+c)$, we have

$$lhs \leq 2\left(a^{\frac{1}{n}} + b^{\frac{1}{n}} + c^{\frac{1}{n}}\right) \leq 2(3^{n-1}3^{n+1})^{\frac{1}{n}} = 18$$

□

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Solution 1 is by Khanh Hung Vu; Solution 2 is by Abdur Rahman; Solution 3 is by Amit Itagi; Solution 4 is by N. N. Taleb.

187. An Inequality with Inradius and Excenters

In $\Delta ABC, I$ is the incenter, I_a, I_b, I_c are the excenters, r is the inradius.

Prove that

$$\sum_{cycl} \frac{1}{II_a^2} + \sum_{cycl} \frac{1}{I_aI_b^2} \leq \frac{1}{4r^2}.$$

Proposed by Daniel Sitaru

Solution 1 (by Kevin Soto Palacios).

We know that (with R the circumradius)

$$II_a = 4R\sin\frac{A}{2} \quad II_b = 4R\sin\frac{B}{2} \quad II_c = 4R\sin\frac{C}{2}$$

$$I_aI_b = 4R\cos\frac{C}{2} \quad I_bI_c = 4R\cos\frac{A}{2} \quad I_cI_a = 4R\cos\frac{B}{2}$$

and also $\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{r^2}$. The required inequality is equivalent to

$$\frac{1}{16R^2}\sum_{cycl}\csc^2\frac{A}{2} + \frac{1}{16R^2}\sum_{cycl}\sec^2\frac{A}{2} \leq \frac{1}{4r^2} \Leftrightarrow$$

$$\frac{1}{16R^2}\left(\sum_{cycl}\left(\csc^2\frac{A}{2} + \sec^2\frac{A}{2}\right)\right) \leq \frac{1}{4r^2} \Leftrightarrow$$

$$\frac{1}{16R^2}\left(\sum_{cycl}\left(\tan\frac{A}{2} + \cot\frac{A}{2}\right)^2\right) \leq \frac{1}{4r^2} \Leftrightarrow$$

$$\frac{1}{16R^2}(4\csc^2 A + 4\csc^2 B + 4\csc^2 C) = \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}.$$

□

Solution 2 (by Soumava Chakraborty).

We know that (with R the circumradius)

$$II_a = a\sec\frac{A}{2} = 2R\sin A\sec\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{A}{2}\sec\frac{A}{2}$$
$$= 4R\sin\frac{A}{2},$$
$$I_aI_b = \csc\frac{A}{2} = 2R\sin A\csc\frac{A}{2} = 4R\sin\frac{A}{2}\cos\frac{A}{2}\csc\frac{A}{2}$$
$$= 4R\cos\frac{A}{2}$$

It follows that

$$\sum_{cycl}\frac{1}{II_a^2} + \sum_{cycl}\frac{1}{I_aI_b^2} = \frac{1}{16R^2}\sum_{cycl}\left(\frac{1}{\sin^2\frac{A}{2}} + \frac{1}{\cos^2\frac{A}{2}}\right)$$
$$= \frac{1}{16R^2}\sum_{cycl}\frac{1}{\sin^2\frac{A}{2}\cos^2\frac{A}{2}} = \sum_{cycl}\frac{1}{(4R\sin\frac{A}{2}\cos\frac{A}{2})^2}$$
$$= \sum_{cycl}\frac{1}{(2R\sin A)^2} = \sum_{cycl}\frac{1}{a^2} = \frac{\sum_{cycl}a^2b^2}{a^2b^2c^2}$$
$$\le \frac{4R^2s^2}{a^2b^2c^2} = \frac{4R^2s^2}{16R^2r^2s^2} = \frac{1}{4r^2},$$

where at the penultimate step we used ***Goldstone's inequality***: $\sum_{cycl}a^2b^2 \le 4R^2s^2$, with s the semiperimeter of ΔABC. □

Ackonwledgment (by Alexander Bogomolny)

I am grateful to Daniel Sitaru for kindly posting a problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Solution 1 is by Kevin Soto Palacios, Solution 2 is by Soumava Chakraborty.

188. Dan Sitaru's Cyclic Inequality in Three Variables III

If $a, b, c > 1$ then

$$\sum_{cycl}\frac{a}{(a-1)^2} \ge \sqrt{6}(10 - a - b - c).$$

Proposed by Daniel Sitaru

Solution 1 (by Leonard Giugiuc).

The problem is easily equivalent to

$$4\sqrt{\sum_{cycl}\frac{a+1}{a^2}} \ge \sqrt{6}(7 - a - b - c),$$

for $a, b, c > 0$.

Let's find m, n such that $\frac{\sqrt{x+1}}{x} \ge mx + n$, for $x > 0$. This is the same as $f(x) = mx^2 + nx - \sqrt{x+1} \le 0$. Assuming $f(1) = 0, m + n = \sqrt{2}$. Assuming $f'(1) = 0, 2m + n = \sqrt{12}\sqrt{2}$, solving which $m = -\frac{3}{2\sqrt{2}}$ and $n = \frac{7}{2\sqrt{2}}$.

We shall prove that indeed $g\frac{\sqrt{x+1}}{x} \ge \frac{-3x+7}{2\sqrt{2}}$, for $x > 0$.

If $0 < x \le \frac{7}{3}$, the inequality is equivalent to $(x-1)^2(9x^2 - 24x - 8) \le 0$, which is true because $\frac{4+2\sqrt{6}}{3} \ge \frac{7}{3}$.

If $g > \frac{7}{3}$ then, obviously, $\frac{\sqrt{x+1}}{x} > 0 > \frac{-3x+7}{2\sqrt{2}}$.
Back to the original problem, by *Jensen's inequality*,

$$\sqrt{3\sum_{cycl}\frac{a+1}{a^2}} \geq \sum_{cycl}\frac{\sqrt{a+1}}{a}$$
$$\geq \frac{3(7-a-b-c)}{2\sqrt{2}}.$$

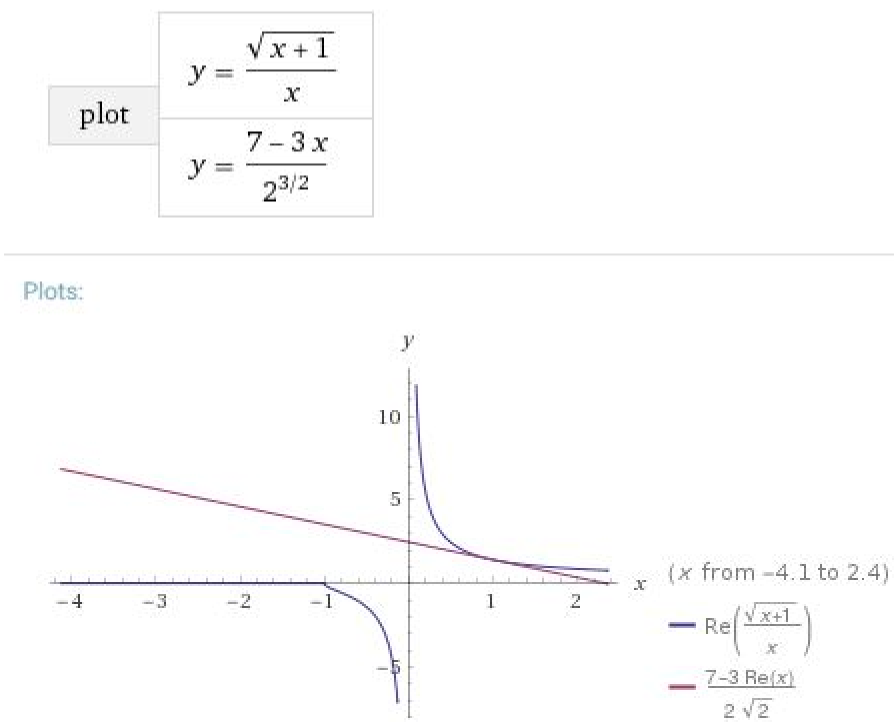
Multiply by 4 to get the required inequality. □

Solution 2 (by Alexander Bogomolny).

This solution is more of an illustration to be first one.
By *Jensen's inequality*,

$$\sqrt{3\sum_{cycl}\frac{a+1}{a^2}} \geq \sum_{cycl}\frac{\sqrt{a+1}}{a}.$$

Thus suffice it to prove that, for > 0, $\frac{\sqrt{x+1}}{x} > \frac{7-3x}{2\sqrt{2}}$. We can see that $\left(\frac{\sqrt{x+1}}{x}\right)(1) = \sqrt{2}$ and $\left(\frac{\sqrt{x+1}}{x}\right)'(1) = -\frac{3}{2\sqrt{2}}$ which makes $\frac{-3x+7}{2\sqrt{2}}$ tangent to $\frac{\sqrt{x+1}}{x}$ at $x = 1$.



The second derivative of $\frac{x+1}{x}$ is easily seen to be negative, making the function convex and insuring the inequality

$$\frac{\sqrt{x+1}}{x} > \frac{7-3x}{2\sqrt{2}}.$$

□

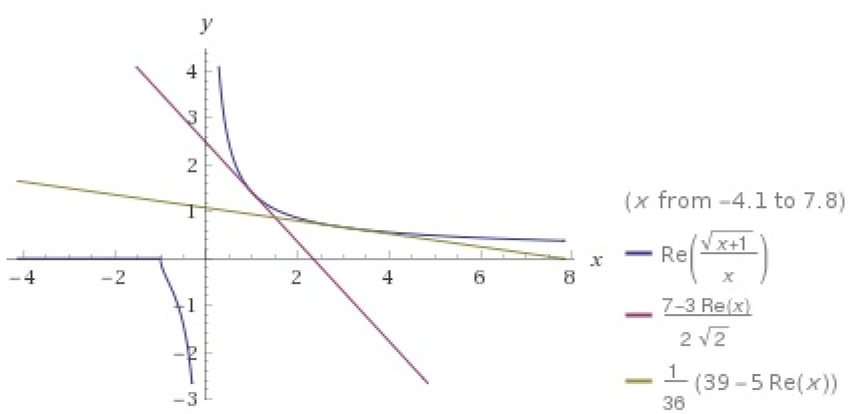
Extra (by Alexander Bogomolny)
Once we surmised the secret behing the problem design, we may try to modify the problem. For example, at $x = 3$, the tangent to function $f(x) = \frac{\sqrt{x+1}}{x}$ is given by $y = -\frac{5}{36} + \frac{13}{12}$, implying

$$\sqrt{\sum_{cycl} \frac{a+1}{a^2}} \geq \sum_{cycl} \frac{\sqrt{a+1}}{a\sqrt{3}}$$

$$\geq \frac{\sqrt{3}}{108}[117 - 5(a+b+c)].$$

	$y = \frac{\sqrt{x+1}}{x}$
plot	$y = \frac{7 - 3x}{2^{3/2}}$
	$y = -5 \times \frac{x}{36} + \frac{13}{12}$

Plots:



Solution 3 (by Amit Itagi).
Let $x = a - 1, y = b - 1$, and $z = c - 1$. Thus, $x, y, z > 0$. Let us rewrite the inequality as

$$4\sqrt{\frac{x+1}{6x^2} + \frac{y+1}{6y^2} + \frac{z+1}{6z^2}} + (x + y + z) \geq 7.$$

$$x + y + z \geq 3(xyz)^{\frac{1}{3}} \text{ (AM-GM)}$$

and

$$4\sqrt{\frac{x+1}{6x^2} + \frac{y+1}{6y^2} + \frac{z+1}{6z^2}}$$

$$= 4\sqrt{\frac{1}{3}\Big[\frac{1}{2}\Big(\frac{1}{x} + \frac{1}{x^2}\Big) + \frac{1}{2}\Big(\frac{1}{y} + \frac{1}{y^2}\Big) + \frac{1}{2}\Big(\frac{1}{z} + \frac{1}{z^2}\Big)\Big]}$$

$$\geq 4\sqrt{\frac{1}{3}\Big(\frac{1}{x^{\frac{3}{2}}}+\frac{1}{y^{\frac{3}{2}}}+\frac{1}{z^{\frac{3}{2}}}\Big)} \text{ (bracket-wise AM-GM)}$$

$$\geq \frac{4}{3}\Big(\frac{1}{x^{\frac{3}{4}}}+\frac{1}{y^{\frac{3}{4}}}+\frac{1}{z^{\frac{3}{4}}}\Big) \text{ (Jensen's/ concavity of the square root)}$$

$$\geq \Big(\frac{1}{xyz}\Big)^{\frac{1}{4}}. \ \textit{(AM-GM)}$$

Let $q = (xyz)^{\frac{1}{7}}$. Thus,

$$LHS \geq 4\Big(\frac{1}{q}\Big)^{\frac{7}{4}} + 3q^{\frac{7}{3}} = 7\left[\frac{\left(\frac{1}{q}\right)^{\frac{7}{4}}}{\frac{7}{4}} + \frac{q^{\frac{7}{3}}}{\frac{7}{3}}\right] \geq 7 \text{ (Young'sinequality).}$$

Let $x = a-1, y = b-1$, and $z = c-1$. Thus, $x, y, z > 0$. Let us rewrite the inequality as

$$4\sqrt{\frac{x+1}{6x^2}+\frac{y+1}{6y^2}+\frac{z+1}{6z^2}} + (x+y+z) \geq 7.$$

$$x+y+z \geq 3(xyz)^{\frac{1}{3}} \ \textit{(AM-GM)}$$

and

$$4\sqrt{\frac{x+1}{6x^2}+\frac{y+1}{6y^2}+\frac{z+1}{6z^2}}$$

$$= 4\sqrt{\frac{1}{6}\Big(\frac{1}{x}+\frac{1}{x^2}+\frac{1}{y}+\frac{1}{y^2}+\frac{1}{z}+\frac{1}{z^2}\Big)} \geq 4\Big(\frac{1}{xyz}\Big)^{\frac{1}{4}}. \ \textit{(AM-GM)}$$

Let $= (xyz)^{\frac{1}{7}}$. Thus,

$$LHS \geq 4\Big(\frac{1}{q}\Big)^{\frac{7}{4}} + 3q^{\frac{7}{3}} = 7\left[\frac{\left(\frac{1}{q}\right)^{\frac{7}{4}}}{\frac{7}{4}} + \frac{q^{\frac{7}{3}}}{\frac{7}{3}}\right] \geq 7 \text{ (Young's inequality).}$$

□

Solution 4 (by Nassim Nicolas Taleb).

Proof by progressive reduction to one single variable.
Let $f = 4\sqrt{\frac{a}{(a-1)^2}+\frac{b}{(b-1)^2}+\frac{c}{(c-1)^2}} - \sqrt{6}(-a-b-c+10)$.
Let $x = \frac{1}{(a-1)}, y = \frac{1}{(b-1)}, z = \frac{1}{(c-1)}$.
Now,

$$f = 4\sqrt{x^2+x+y^2+y+z^2+z} + \sqrt{6}\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}-7\Big)$$

$$\geq 4\sqrt{x^2+x+y^2+y+z^2+z} + 3\sqrt{6}\sqrt[3]{\frac{1}{xyz}} - 7\sqrt{6}$$

$$\geq \sqrt{6}\left(4\sqrt[4]{x}\sqrt[4]{y}\sqrt[4]{z} + \frac{3}{\sqrt[3]{x}\sqrt[3]{y}\sqrt[3]{z}} - 7\right)$$

Now let $X = xyz, X > 0$. We can prove that:

$$\frac{4X^{\frac{7}{12}}+3}{\sqrt[3]{X}} - 7 \geq 0$$

since it is a single function with one variable and its minimum is 0 for $X = x = y = z = 1$.

□

Acknowledgment (by Alexander Bogomolny)
Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. Leo Giugiuc has commented with his solution (Solution 1). Solution 3 (and its concise version) is by Amit Itagi; Solution 4 is by N. N. Taleb.

189. Dan Sitaru's Cyclic Inequality In Three Variables with Constraints IV

Prove that if $x, y, z > 0; 6xyz = \dfrac{1}{x+2y+3z}$ ***then:***

$$\frac{(4x^2y^2+1)(36y^2z^2+1)(9x^2z^2+1)}{230x^2y^2z^2} \geq \frac{1}{(x+2y+3z)^2}$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

First of all, we simplify the problem by replacing the variables:
$a = x, b = 2y, c = 3z$, which reduces the problem to
Prove that if $a, b, c > 0; abc(a+b+c) = 1$ then:

$$\frac{(a^2b^2+1)(b^2c^2+1)(c^2a^2+1)}{a^2b^2c^2} \geq \frac{64}{(a+b+c)^2}.$$

We rewrite the inequality as

$$\left(a^2+\frac{1}{b^2}\right)\left(b^2+\frac{1}{c^2}\right)\left(c^2+\frac{1}{a^2}\right) \geq \frac{64}{(a+b+c)^2}.$$

Note that

$$a^2+\frac{1}{b^2} = a^2+\frac{abc(a+b+c)}{b^2} = a^2+\frac{ac(a+b+c)}{b}$$
$$= \frac{a^2b+a^2c+ac(b+c)}{b} = \frac{a^2(b+c)+ac(b+c)}{b}$$
$$= \frac{a(b+c)(c+a)}{b}.$$

Similarly $b^+\frac{1}{c^2} = \frac{b(c+a)(a+b)}{c}$ and $c^2+\frac{1}{a^2} = \frac{c(a+b)(b+c)}{a}$.
By multiplying the three relationships,

$$\left(a^2+\frac{1}{b^2}\right)\left(b^2+\frac{1}{c^2}\right)\left(c^2+\frac{1}{a^2}\right)$$
$$\geq \frac{a(b+c)(a+c)\cdot b(a+c)(b+a)c\cdot c(a+b)(c+b)}{abc}$$
$$= (a+b)^2(b+c)^2(c+a)^2$$
$$\overbrace{\geq}^{AM-GM} (2\sqrt{ab})^2\cdot(2\sqrt{bc})^2\cdot(2\sqrt{ac})^2$$
$$= 64a^2b^2c^2 = \frac{64}{(a+b+c)^2}$$

The equality holds if $a = b = c = \frac{1}{\sqrt[4]{3}}$, which follows from

$$a\cdot a\cdot a = \frac{1}{a+a+a} \Leftrightarrow a^3 = \frac{1}{3a} \Rightarrow a^4 = \frac{1}{3} \Rightarrow a = \frac{1}{\sqrt[4]{3}}.$$

The equality in the original relationship holds for

$$x = \frac{1}{\sqrt[4]{3}}; y = \frac{1}{2 \cdot \sqrt[4]{3}}; z = \frac{1}{3 \cdot \sqrt[4]{3}}$$

□

Solution 2 (by Amit Itagi).

Using the constraint, the inequality can be written as

$$(4x^2y^2+1)(36y^2z^2+1)(9x^2z^2+1) \geq 64(6xyz)^4$$

Let,

$$x = \sqrt{\frac{ab}{c}}, 2y = \sqrt{\frac{bc}{a}}, 3z = \sqrt{\frac{ca}{b}}.$$

Thus, the inequality and the constraint, respectively, become

$$(a^2+1)(b^2+1)(c^2+1) \geq 64(abc)^2, ab+bc+ca = 1.$$

Let $p = 3(abc)^{\frac{2}{3}}$. AM-GM gives

$$1 = ab+bc+ca \geq 3(abc)^{\frac{2}{3}} = p.$$

The inequality can be simplified to

$$1+(a^2+b^2+c^2)+(a^2b^2+b^2c^2+c^2a^2) - 63(abc)^2 \geq 0.$$

$$\begin{aligned} LHS &\geq 1+3(abc)^{\frac{2}{3}}+3(abc)^{\frac{4}{3}} - 63(abc)^2 \text{ (AM-GM)} \\ &= 1+p+\frac{p^2}{3} - \frac{7}{3}p^3 \\ &= \frac{(1-p)}{3}[7(1-p)^2 - 20(1-p)+16] \\ &= \frac{(1-p)}{3}\left\{\left[(1-p)\sqrt{7} - \frac{10}{\sqrt{7}}\right]^2 + \frac{12}{7}\right\} \geq 0, \end{aligned}$$

from the constraint. □

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the above problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page*** and later emailed me his solution. Solution 2 is by Amit Itagi.

Refinement on Dan Sitaru's Cyclic Inequality In Three Variables

By Nassim Nicolas Taleb

Preliminaries

An *earlier page* dealt with a problem by Dan Sitaru:

Prove that if $x, y, z > 0; 6xyz = \dfrac{1}{x+2y+3z}$ *then:*

$$\frac{(4x^2y^2+1)(36y^2z^2+1)(9x^2z^2+1)}{2304x^2y^2z^2} \geq \frac{1}{(x+2y+3z)^2}$$

While solving that problem, N. N. Taleb has observed (How the problem came about below) the existence of an upper bound on the right-hand side of the inequality and suggested a refinement that is the subject of the present page. An early attempt of solving the new problem (Solution 1) relied on the graphics produced by wolframalpha. Leo Giugiuc devised Solution 2, Amit Itagi Solution 3.

Problem

$$\textit{Prove that if } x, y, z > 0; 6xyz = \frac{1}{x+2y+3z} \textit{ then}$$

$$\frac{(4x^2y^2+1)(36y^2z^2+1)(9x^2z^2+1)}{2304x^2y^2z^2} \geq \frac{1}{3\sqrt{3}}$$

How the problem came about

First of all, we simplify the problem by replacing the variables $:= x, b = 2y, c = 3z$, which reduces the problem to

Prove that if $a, b, c > 0; abc(a+b+c) = 1$ then:

$$\frac{(a^2b^2+1)(b^2c^2+1)(c^2a^2+1)}{a^2b^2c^2} \geq \frac{64}{(a+b+c)^2}.$$

We start with the constraint by applying the AM-GM inequality:

$1 \geq (abc)(a+b+c) \geq abc \cdot 3\sqrt[3]{abc} = 3(abc)^{\frac{4}{3}}$ so that $abc \leq \dfrac{1}{3^{\frac{3}{4}}}$, implying a bound for the RHS of the inequality,

$$\frac{1}{(a+b+c)^2} = (abc)^2 \leq \left(\frac{1}{3^{\frac{3}{4}}}\right)^2 = \frac{1}{3^{\frac{3}{2}}}.$$

Thus, the above problem.

Solution 1.

We start with Amit Itagi's approach for solving the original problem (and copied from Solution 3 below). Let,

$$x = \sqrt{\frac{ab}{c}}, 2y = \sqrt{\frac{bc}{a}}, 3z = \sqrt{\frac{ca}{b}}.$$

Thus, the inequality and the constraint, respectively, become

$$(a^2+1)(b^2+1)(c^2+1) \geq \frac{64abc}{3\sqrt{3}}, ab+bc+ca = 1.$$

Define $p = \sqrt[3]{abc}$. From the constraint, $1 \geq 3\sqrt[3]{a^2b^2c^2} = 3\sqrt[3]{p^2}$, implying $p \in \left[0, \frac{1}{3\sqrt{3}}\right]$. Now, for the left-hand side,

$$(a^2+1)(b^2+1)(c^2+1) = 1 + \sum_{cycl} a^2 + \sum_{cycl} a^2b^2 + a^2b^2c^2$$

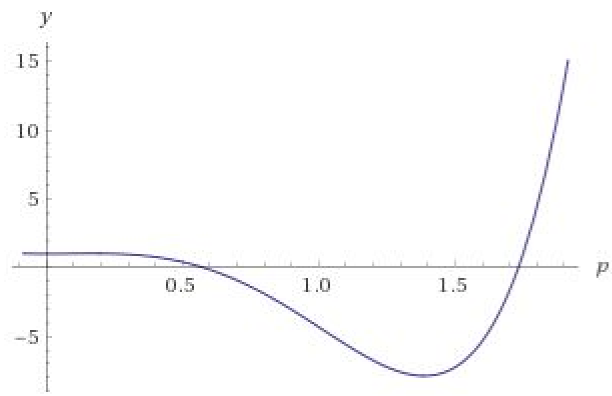
$$\geq 1 + 3\sqrt[3]{a^2b^2c^2} + 3\sqrt[3]{a^4b^4c^4} + a^2b^2c^2$$

$$= 1 + 3p^2 + 3p^4 + p^6$$

We, therefore, define the function

$$f(p) = 1 + 3p^2 + 3p^4 + p^6 - \frac{64}{3\sqrt{3}}p^3.$$

The graph below affirms the inequality $f(p) \geq 0$, for $p \in \left[0, \frac{1}{3\sqrt{3}}\right]$ if we notice that $f\left(\frac{1}{\sqrt{3}}\right) = 0$:



□

Solution 2.

The smartest way is the following. Denote $6x^2yz = \frac{a}{3}, 12xy^2z = \frac{b}{3}, 18xyz^2 = \frac{c}{3}$. Then $a + b + c = 3$ and the required inequality becomes

$$(ab + 3c)(bc + 3a)(ca + 3b) \geq 64(abc)^{\frac{3}{2}}.$$

Let's remark that, since $abc \leq 1, (abc)^{\alpha} \geq (abc)^{\beta}$, for $\alpha \leq \beta$. By the AM-GM inequality,

$$ab + 3c = ab + c + c + c \geq 4(abc^3)^{\frac{1}{4}}$$

Similarly we obtain two additional inequalities, with the product of the three

$$(ab + 3c)(bc + 3a)(ca + 3b) \geq 4^3(abc)^{\frac{5}{4}} \geq 64(abc)^{\frac{3}{2}}$$

because $\frac{5}{4} < \frac{3}{2}$. □

Solution 3.

Using the constraint, the inequality can be written as

$$(4x^2y^2 + 1)(36y^2z^2 + 1)(9x^2z^2 + 1) \geq \frac{64(6xyz)^2}{3\sqrt{3}}.$$

Let,

$$x = \sqrt{\frac{ab}{c}}, 2y = \sqrt{\frac{bc}{a}}, 3z = \sqrt{\frac{ca}{b}}.$$

Thus, the inequality and the constraint, respectively, become

$$(a^2 + 1)(b^2 + 1)(c^2 + 1) \geq \frac{64abc}{3\sqrt{3}}, ab + bc + ca = 1.$$

Let $= \sqrt{3}(abc)^{\frac{1}{3}}$. AM-GM gives

$$1 = ab + bc + ca \geq 3(abc)^{\frac{2}{3}} = p^2 \text{ or } 1 \geq p.$$

The inequality can be simplified to

$$1 + (a^2 + b^2 + c^2) + (a^2b^2 + b^2c^2 + c^2a^2) + (abc)^2 - \frac{64abc}{3\sqrt{3}} \geq 0.$$

$$LHS \geq 1 + 3(abc)^{\frac{3}{2}} + 3(abc)^{\frac{4}{3}} + (abc)^2 - \frac{64abc}{3\sqrt{3}} \text{ (AM-GM)}$$
$$= 1 + p^2 + \frac{p^4}{3} + \frac{p^6}{27} - \frac{64p^3}{27}$$
$$= \frac{(p-1)(p-3)(p^4+4p^3+22p^2+12p+9)}{27} \geq 0,$$

because $(p-1)(p-3) \geq 0$ due to the constraint $0 \leq p \leq 1$. □

190. Small Triangle from Small Triangle

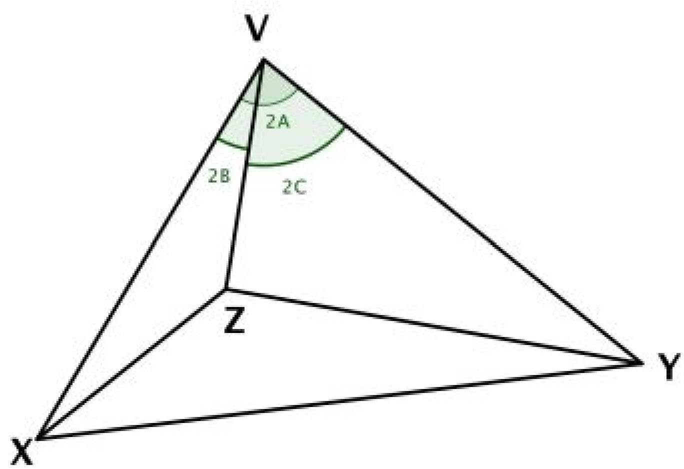
If $0 < A < B + C, 0 < B < C + A, 0 < C < A + B, A + B + C = \pi$ *then*

$$\prod_{cycl} (\sin A + \sin B - \sin C) > 0.$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

Let $VXYZ$ be a tetrahedron as depicted below



$$\angle XVY = 2A; \angle ZVY = 2C; \angle XVY = 2A, VX = VY = VZ = 1.$$
$$XY^2 = VX^2 + VY^2 - 2VX \cdot VY \cdot \cos(\angle XVY) = 1^2 + 1^2 - 2 \cdot 1 \cdot 1 \cdot 1 \cos 2A$$
$$= 2(1 - \cos 2A) = 2(1 - 1 + 2\sin^2 A) = 4\sin^2 A.$$

Thus $XY = 2\sin A$. Similarly, $XZ = 2\sin B, YZ = 2\sin C$.

In $\Delta XYZ, XY + XZ > YZ$ so that $2\sin A + 2\sin B > 2\sin C$, i.e.,

$\sin A + \sin B - \sin C > 0$. Similarly,

$\sin B + \sin C - \sin A > 0$ and $\sin C + \sin A - \sin B > 0$. Hence,

$$\prod_{cycl} (\sin A + \sin B - \sin C) > 0.$$

□

Solution 2 (by Amit Itagi).

WLOG, let $A \geq B \geq C.\pi > A+B+C \geq 2A$. Thus, A is acute and so are B and C. From the ordering of A, B, C and monotonicity of sin in the first quarter,

$$\sin A > \sin B > \sin C \text{ and, therefore, } \sin A + \sin B > \sin C.$$

Additionally,

$$\begin{aligned}\sin A - \sin B &= 2\sin\left(\frac{A-B}{2}\right)\cos\left(\frac{A+B}{2}\right)\\ &< 2\sin\frac{C}{2}\cos\frac{C}{2} = \sin C.\end{aligned}$$

All the angles in the expressions are acute, $C > A - B$ and sin increases monotonically in the first quarter, $A + B > C$ and cos decreases monotoically in the first quarter.

$$\begin{aligned}LHS &= (\sin A + \sin B - \sin C)(\sin B + \sin C - \sin A)(\sin C + \sin A - \sin B)\\ &= (\sin A + \sin B - \sin C)[\sin^2 C - (\sin A - \sin B)^2] > 0,\end{aligned}$$

from the two inequalities already proven. □

Solution 3 (by Leonard Giugiuc).

The numbers $A, B, \pi - A - B$ are angles of a triangle. Hence, the triangle inequalities, combined with the Law of Sines, give

$$\sin A + \sin B > \sin(\pi - A - B) = \sin(A + B).$$

Suffice it to show that $\sin(A + B) \geq \sin C$ which is equivalent to $\sin\left(\frac{A+B-C}{2}\right)\cos\left(\frac{A+B+C}{2}\right) \geq 0$, which is true since $0 < \frac{A+B-C}{2} < \frac{\pi}{2}$ and also

$$0 < \frac{A+B+C}{2} < \frac{\pi}{2}$$

□

Remark (by Alexander Bogomolny)

The three triangle inequalities in the problem inform us that the quantities A, B, C may be looked as the side lengths of a triangle. The condition $A + B + C < \pi$ tells us that the triangle is not big.

The conclusion of the problem is equivalent to saying that $\sin A, \sin B$, and $\sin C$ also $\sin C$ also form a triangle, whose perimeter does not exceed 3 hence the caption.

Furthermore, the given triangle is necessarily acute: $0 < A, B, C < \frac{\pi}{2}$. This is because, say, $A > \frac{\pi}{2}$ would lead to $B + C < \frac{\pi}{2}$, in contradiction with $A < B + C$. We may also claim that the inverse is also true: the angles of an acute triangle satisfy the three triangles inequalities.

Indeed, from $A + B + C = \pi$ and, say $A < \frac{\pi}{2}$, it follows that $B + C > \frac{\pi}{2} > A$.

Thus, a *triangle is acute* iff its angles can be used as the side lengths of a triangle.

Akcnowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly sent me this problem along with the above solution of his. Solution 2 is by Amit Itagi; Soution 3 is by Leonard Giugiuc.

191. An Inequality in Two or More Variables II

Prove that if $\boldsymbol{a, b, c \geq 0}$ ***then***

$$(a+1)^{a+1} \cdot (b+1)^{b+1} \cdot (c+1)^{c+1} \leq e^{a+b+c} \cdot \sqrt{e^{a^2+b^2+c^2}}$$

Proposed by Daniel Sitaru

Solution 1 (by Daniel Sitaru).

Consider function $f : [0, \infty) \to \mathbb{R}$, defined by

$$f(x) = (x+1)\ln(x+1) - x - \frac{x^2}{2}.$$

$$f'(x) = \ln(x+1) + 1 - 1 - x = \ln(x+1) - x$$

$$f''(x) = \frac{1}{1+x} - 1 = -\frac{x}{1+x} < 0$$

$$f''(x) \leq f''(0) \Rightarrow f'(x) \leq f'(0) = 0 \Rightarrow f(x) \leq 0, (\forall)x \geq 0$$

$$(x+1)\ln(x+1) - x - \frac{x^2}{2} \leq 0$$

$$\ln(x+1)^{x+1} \leq x + \frac{x^2}{2}$$

$$(x+1)^{x+1} \leq e^{x+\frac{x^2}{2}}; (\forall)x \geq 0$$

(1) $$(a+1)^{a+1} \leq e^a \cdot \sqrt{e^{a^2}}$$

(2) $$(b+1)^{b+1} \leq e^b \cdot \sqrt{e^{b^2}}$$

(3) $$(c+1)^{c+1} \leq e^c \cdot \sqrt{e^{c^2}}$$

Multiply (1)-(3) to get

$$(a+1)^{a+1} \cdot (b+1)^{b+1} \cdot (c+1)^{c+1} \leq e^{a+b+c} \cdot \sqrt{e^{a^2+b^2+c^2}}$$

□

Solution 2 (by Nassim Nicolas Taleb).

Let $x = a+1$, etc. The inequality becomes:

$$x^x y^y z^z \leq e^{x+y+z-3}\sqrt{e^{(x-1)^2+(y-1)^2+(z-1)^2}}, x, y, z > 1$$

Taking logs on both sides:

$$x\log(x) + y\log(y) + z\log(z) \leq \frac{1}{2}(x^2+y^2+z^2-3)$$

Rewritting. We need to minimize

$$f(x,y,z) = x^2 - 2x\log(x) + y^2 - 2y\log(y) + z^2 - 2z\log(z) - 3$$

which is additively separable into $f(x,y,z) = f_1(x) + f_2(y) + f_3(z)$, with $f_1(x) = x^2 - 2x\log(x) - 1$, etc. The minimum for $f_1(x)$ is for $x = 1$, and so on, hence $f(x,y,z) = 0$ for $x = y = z = 1$, which corresponds to

$$a = b = c = 0$$

□

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the problem of his from the ***Romanian Mathematical Magazine*** at the ***CutTheKnotMath facebook page***. He later mailed

his solution on a LaTex file – something I appreciate greatly. Solution 2 is by N.N. Taleb. As the solution show, the inequality can be extende to any number of variable.

192. Inequality 101 from the Cyclic Inequalities Marathon

Let a, b, c be positive real numbers, subject to $a+b+c=1$Prove that

$$\sum_{cycl} \frac{c^5}{(a+1)(b+1)} \geq \frac{1}{144}$$

Proposed by George Apostolopoulos

Solution 1 (by Soumava Chakraborty).

WLOG, $a \geq b \geq c$, implying $\frac{1}{(b+1)(c+1)} \geq \frac{1}{(c+1)(a+1)} \geq \frac{1}{(b+1)(c+1)}$. It follows by Chebyshev's inequality that

$$LHS \geq \frac{1}{3}(\sum_{cycl} a^5)\Big(\sum_{cycl} \frac{1}{(b+1)(c+1)}\Big) \tag{1}$$

By Chebyshev's inequality,

$$\sum_{cycl} a^5 \geq \frac{1}{3^4(\sum_{cycl} a)^5} = \frac{1}{81}. \tag{2}$$

Further, by Bergström's inqequality,

$$\sum_{cycl} \frac{1}{(b+1)(c+1)} \geq \frac{9}{\sum_{cycl} ab + 2\sum_{cycl} a + 3} = \frac{9}{\sum_{cycl} ab + 5} \tag{3}$$

$$\geq \frac{9}{\frac{16}{3}} = \frac{27}{16}$$

because $\sum_{cycl} ab \leq \frac{1}{3}(\sum_{cycl} a)^2$. With (1)-(3),

$$LHS \geq \frac{1}{3} \cdot \frac{1}{81} \cdot \frac{27}{16} = \frac{1}{144}$$

□

Solution 2 (by Soumitra Mandal).

By *Hölder's inequality,*

$$1 = (a+b+c)^5$$

$$\leq \left(\sum_{cycl} \frac{a^5}{(b+1)(c+1)}\right)\left(\sum_{cycl}(b+1)(c+1)\right)(1+1+1)^3.$$

It follows that

$$\left(\sum_{cycl} \frac{a^5}{(b+1)(c+1)}\right)(ab+bc+ca+5)3^3 \geq 1.$$

The latter implies

$$\Big(\frac{(a+b+c)^2}{3} + 5\Big)\left(\sum_{cycl} \frac{a^5}{(b+1)(c+1)}\right)3^3 \geq 1.$$

And, finally,

$$\sum_{cycl} \frac{a^5}{(b+1)(c+1)} \geq \frac{1}{3^3} \cdot \frac{3}{16} = \frac{1}{144}.$$

Equality is attained at $a = b = c = \frac{1}{3}$. $\square$

Remark (by Alexander Bogomolny)

The latter solution appears to equally well tackle another inequality:

$$\sum_{cycl} \frac{a^5}{(a+1)(b+1)} \geq \frac{1}{144}.$$

Solution 3 (by Nassim Nicolas Taleb).

We use the inequality variant

$$(1) \qquad \left(\frac{1}{3}(a^p + b^p + c^p)\right)^{\frac{1}{p}} \geq \left(\frac{1}{3}(a^q + b^q + c^q)\right)^{\frac{1}{q}}, p \geq q$$

Let $p = 5, q = 1$:

$$\frac{\sqrt[5]{a^5 + b^5 + c^5}}{\sqrt[3]{3}} \geq \frac{1}{3}(a + b + c),$$

so

$$a^5 + b^5 + c^5 \geq \frac{1}{81}$$

On the other hand $\frac{1}{27}(a + b + c + 3)^3 = \frac{64}{27} \geq (a + 1)(b + 1)(c + 1)$

Expanding the lhs:

$$lhs = \frac{a^6 + a^5 + b^6 + b^5 + c^6 + c^5}{(a + 1)(b + 1)(c + 1)} \geq \frac{27}{64}\left(\frac{1}{81} + a^6 + b^6 + c^6\right)$$

Let $p = 6, q = 1$ in (1),

$$a^6 + b^6 + c^6 \geq \left(\frac{\sqrt[6]{3}}{3}\right)^6 = \frac{1}{243}$$

So

$$lhs \geq \frac{27}{64}\left(\frac{1}{81} + \frac{1}{243}\right) = \frac{1}{144}$$

$\square$

Aknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly emailed me a copy of the collection ***RMM CYCLIC INEQUALITIES MARATHON 101-200***. The problem is due to George Apostolopoulos. I copy two proofs: Solution 1 by Soumava Chakraborty, Solution 2 by Soumitra Mandal. Solution 3 is by N. N. Taleb.

193. Adil Abdulayev's Inequality With Angles, Medians, Inradius and Circumradius

In any ΔABC,

$$\frac{A}{m_a} + \frac{B}{m_b} + \frac{C}{m_c} \leq \frac{3\pi}{4R + r}.$$

Proposed by Adil Abdullayev

Solution (by Daniel Sitaru).

By *Chebyshev's inequality,*

$$\sum_{cycl} \frac{A}{m_a} \geq \frac{1}{3} \sum_{cycl} A \cdot \sum_{cycl} \frac{1}{m_a} \cdot \frac{\pi}{3} \sum_{cycl} \frac{1}{m_a}.$$

By *Bergström inequality*,

$$\frac{\pi}{3}\sum_{cycl}\frac{1}{m_a} \geq \frac{\pi}{3}\frac{9}{\sum_{cycl} m_a} \geq \frac{\pi}{3}\cdot\frac{9}{4R+r}$$

$$= \frac{3\pi}{4R+r},$$

due to *Leuenberger's Inequality* $m_a + m_b + m_c \leq 4R + r$. □

Aknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted this problem by Adil Abdulayev at the ***CutTheKnotMath facebook page***, along with his solution. The problem was originally posted to the ***Romanian Mathematical Magazine***.

194. An Inequality with Sides, Cosines, and Semiperimet

In any ΔABC,

$$\sum_{cycl} a^2(b\cos B + c\cos C) \leq \frac{8s^3}{9}$$

Proposed by Daniel Sitaru

Solution 1 (same solution by Kevin Soto Palacios and Amit Itagi).

The problem is the same as

$$\sum_{cycl} a^2(b\cos B + c\cos C) \leq \frac{(a+b+c)^3}{9}.$$

Due to the ***AM-GM inequality***, suffice it to prove that

$$\sum_{cycl} a^2(b\cos B + c\cos C) \leq 3abc$$

We shall show that in fact

$$\sum_{cycl} a^2(b\cos B + c\cos C) = 3abc.$$

We'll prove that identity in the form

$$\sum_{cycl} a^2(b^2+c^2)(2bc\cos A) = 6a^2b^2c^2. \text{ Indeed,}$$

$$\sum_{cycl} a^2(b^2+c^2)(2bc\cos A) = \sum_{cycl} a^2(b^2+c^2)(b^2+c^2-a^2)$$

$$= \sum_{cycl}\left[a^2(b^2+c^2)^2 - \sum_{cycl} a^4(b^2+c^2)\right]$$

$$= \sum_{cycl}[a^2b^4 + a^2c^4 + 2a^2b^2c^2] - \sum_{cycl}[a^4b^2 + a^4c^2] = 6a^2b^2c^2.$$

□

Solution 2 (by Kevin Soto Palacios).

As in Solution 1, we aim to prove

$$\sum_{cycl} a^2(b\cos B + c\cos C) = 3abc.$$

This is equivalent to

$$2\sum_{cycl} \sin^2 A(\sin 2B + \sin 2C) = 12\sin A\sin B\sin C.$$

And further,

$$2\sum_{cycl} \sin^2 A(\sin 2B + \sin 2C) = \sum_{cycl}(1 - \sin 2A)(\sin 2B + \sin 2C)$$

$$= 2\sum_{cycl} \sin 2A - \sum_{cycl} \sin(2B + 2C)$$

$$= 2\sum_{cycl} \sin 2A + \sum_{cycl} \sin 2A = 3\sum_{cycl} \sin 2A = 12\sin A\sin B\sin C,$$

as *is well known.* □

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the problem at the ***CutTheKnotMath facebook page***. This problem of his was originally published at the ***Romanian Mathematical Magazine***. Solutions 1 and 2 are by Kevin Soto Palacios. Amit Itagi independently came up with Solution 1.

195. Seyran Ibrahimov's Inequality

In any ΔABC,

$$\sqrt{3}s \cdot \sum_{cycl} m_a \le 20R^2 + r^2.$$

Proposed by Seyran Ibrahimov

Solution (same solution by Daniel Sitaru and George Apostolopoulos).

We'll emply *Leuenberger's inequality* $\sum_{cycl} m_a \le 4R + r$ *and Mitrinovič's inequality* $\sum_{cycl} a \le 3\sqrt{3}R$:

$$\sqrt{3}s \cdot \sum_{cycl} m_a \le \sqrt{3} \cdot (4R + r) \cdot \frac{1}{2}3\sqrt{3}R$$

$$= \frac{36R^2 + 9Rr}{2}.$$

Suffice it to prove that $\frac{36R^2+9Rr}{2} \le 20R^2 + r^2$. The latter is equivalent to $4R^2 - 9Rr = 2r^2 \ge 0$, i.e., $(R - 2r)(4R - r) \ge 0$ which is true due to *Euler's inequality* $R \ge 2r$. □

Acknowledgment (by Alexander Bogomolny)

Daniel Sitaru has kindly posted the problem and his solution at the ***CutTheKnotMath facebook page*** and helped me out when I lost the link. I am deeply in Dan's debt. The problem is by Seyran Ibrahimov. George Apostolopoulos gave

the same solution. This problem of his was originally published at the ***Romanian Mathematical Magazine***.

196. An Inequality with Two Pairs of Triplets

If $\boldsymbol{a, b, c, x, y, z \in \mathbb{R}, xyz \neq 0}$***, then***

$$(a^2+b^2+c^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{2(ab+bc+ca)(x+y+z)}{xyz} \geq 0$$

Proposed by Daniel Sitaru

Solution 1 (by Ravi Prakash). Pure algebra

$$\begin{aligned}
&(a^2+b^2+c^2)\left(\frac{1}{x^2}+\frac{1}{y^2}+\frac{1}{z^2}\right)+\frac{2(ab+bc+ca)(x+y+z)}{xyz}\\
&=\frac{a^2}{x^2}+\frac{b^2}{y^2}+\frac{c^2}{z^2}+\frac{2ab}{xy}+\frac{2bc}{yz}+\frac{2ca}{zx}\\
&+\frac{a^2}{y^2}+\frac{b^2}{z^2}+\frac{c^2}{x^2}+\frac{2ab}{yz}+\frac{2bc}{zx}+\frac{2ca}{xy}\\
&+\frac{a^2}{z^2}+\frac{b^2}{x^2}+\frac{c^2}{y^2}+\frac{2ab}{zx}+\frac{2bc}{xy}+\frac{2ca}{yz}\\
&=\left(\frac{a}{x}+\frac{b}{y}+\frac{c}{z}\right)^2+\left(\frac{a}{y}+\frac{b}{z}+\frac{c}{x}\right)^2+\left(\frac{a}{z}+\frac{b}{x}+\frac{c}{y}\right)^2 \geq 0.
\end{aligned}$$

□

Solution 2 (by Amit Itagi).

$xyz \neq 0$ implies the finiteness of $p = \frac{1}{x}, q = \frac{1}{y}$, and $r = \frac{1}{z}$. Thus, the inequality can be written as

$$(a^2+b^2+c^2)(p^2+q^2+r^2)+2(ab+bc+ca)(pq+qr+rp) \geq 0.$$

The inequality is trivially satisfied if $a^2+b^2+c^2=0$ or $p^2+q^2+r^2=0$. Let us consider the case when neither is zero. The inequality is separately homogeneous in $\{a, b, c\}$ and $\{p, q, r\}$. Thus, WLOG, we can assume $a^2+b^2+c^2=1$ and $p^2+q^2+r^2=1$.

Let us find the extrema of $ab+bc+ca$ under the constraint $a^2+b^2+c^2=1$ using Lagrange multipliers. The three resulting equations obtained in addition to the constraint are

$$\begin{aligned}
b+c-2\lambda a&=0\\
c+a-2\lambda b&=0\\
a+b-2\lambda c&=0,
\end{aligned}$$

where λ is the Lagrange multiplier. Adding the three equations, we have

$$(a+b+c)(1-\lambda)=0.$$

Thus, either $a+b+c=0$ or $\lambda=1$.

$\lambda=1$ results in $a=b=c$. The constraint implies $a=b=c=\pm\frac{1}{\sqrt{3}}$. Thus, $ab+bc+ca=1$ for this case. If $a+b+c=0$,

$$ab+bc+ca=\frac{(a+b+c)^2-(a^2+b^2+c^2)}{2}=-\frac{1}{2}.$$

The exact same analysis applies to $\{p, q, r\}$.

Thus, the LHS can be written as $1+2uv$ where $u \in \left[-\frac{1}{2}, 1\right]$ and $v \in \left[-\frac{1}{2}, 1\right]$.

This expression will take minimum value when one of $\{u, v\}$ is most negative (takes value $-\frac{1}{2}$) and the other is most positive (takes value $+1$). Thus the minimum value of the LHS is $1 + 2\Big(-\frac{1}{2}\Big)(1) = 0$. □

Acknowledgment (by Alexander Bogomolny)
The problem above was kindly posted to the ***CutTheKnotMath faceboo page*** by Daniel Sitaru, with a solution by Ravi Prakash. Originally, the problem was published by Daniel at the ***Romanian Mathematical Magazine***. Solution 2 is by Amit Itagi.

197. An Inequality in Triangle, with Sides and Medians III

In any ΔABC, with the side lengths a, b, c and then medians m_a, m_b, m_c,

$$\sum_{cycl} \frac{(m_b + m_c - m_a)^3}{m_a} \geq \frac{3}{4}(a^2 + b^2 + c^2).$$

Proposed by Daniel Sitaru

Solution 1 (by Soumitra Mandal).

$$\sum_{cycl}(m_b + m_c - m_a)^2 = 3\sum_{cycl} m_a^2 - 2\sum_{cycl} m_a m_b$$

$$= \frac{9}{4}\sum_{cycl} a^2 - 2\sum_{cycl} m_a m_b \geq \frac{9}{4}\sum_{cycl} a^2 - \frac{1}{2}\sum_{cycl}(2a^2 + bc)$$

$$\geq \frac{9}{4}\sum_{cycl} a^2 - \frac{3}{2}\sum_{cycl} a^2 = \frac{3}{4}\sum_{cycl} a^2.$$

Again,

$$\sum_{cycl} m_a(m_b + m_b - m_a) = 2\sum_{cycl} m_a m_b - \sum_{cycl} m_a^2$$

$$\leq \frac{1}{2}\sum_{cycl}(2a^2 + bc) - \frac{3}{4}\sum_{cycl} a^2 \leq \frac{3}{2}\sum_{cycl} a^2 = \frac{3}{4}\sum_{cycl} a^2.$$

It follows, by *Bergström's inequality*, that

$$\sum_{cycl} \frac{(m_b + m_c - m_a)^3}{m_a} = \sum_{cycl} \frac{(m_b + m_c - m_a)^4}{m_a(m_b + m_c - m_a)}$$

$$\geq \frac{(\sum_{cycl}(m_b + m_c - m_a)^2}{\sum_{cycl} m_a(m_b + m_c - m_a)} \geq \frac{\Big(\frac{3}{4}(a^2 + b^2 + c^2)\Big)^2}{\frac{3}{4}(a^2 + b^2 + c^2)} = \frac{3}{4}(a^2 + b^2 + c^2).$$

□

Solution 2 (by Soumava Chakraborty).

Let $m_b + m_c - m_a = x, m_c + m_a - m_b = y, m_a + m_b - m_c = z$. Note that $x, y, z > 0, m_a + m_b + m_c = x + y + z, m_a = \frac{y+z}{2}, m_b = \frac{z+x}{2}, m_c = \frac{x+y}{2}$. We thus have:

$$LHS = \frac{2x^3}{y + z} + \frac{2y^3}{z + x} + \frac{2z^3}{x + y}.$$

while

$$RHS = \frac{3}{2}\sum_{cycl} a^2 = \sum_{cycl} m_a^2$$

$$= \frac{1}{4}\sum_{cycl}(y+z)^2 = \frac{1}{2}\left(\sum_{cycl}x^2 + \sum_{cycl}xy\right)$$

such that the required inequality reduces to

$$\sum_{cycl}\frac{x^3}{y+z} \geq \frac{2}{4}\left(\sum_{cycl}x^2 + \sum_{cycl}xy\right)$$

Note that $LHS = \sum_{cycl}\frac{x}{y+z}\cdot x^2$. WLOG, $x \geq y \geq z$, then $x+y \geq x+z \geq y+z$, implying $\frac{x}{y+z} \geq \frac{y}{z+x} \geq \frac{z}{x+y}$. Now we are in a position to apply *Chebysev's inequality*:

$$LHS \geq \frac{1}{3}\sum_{cycl}\frac{x}{y+z}\cdot\sum_{cycl}x^2$$

$$= \frac{1}{3}\sum_{cycl}\frac{x^2}{xy+zx}\cdot\sum_{cycl}x^2 \geq \frac{1}{3}\cdot\frac{(\sum_{cycl}x)^2}{2\sum_{cycl}xy}\cdot\sum_{cycl}x^2.$$

Thus, suffice it to prove that

$$\frac{(\sum_{cycl}x^2 + 2\sum_{cycl}xy)(\sum_{cycl}x^2)}{6\sum_{cycl}xy} \geq \frac{1}{4}\left(\sum_{cycl}x^2 + \sum_{cycl}xy\right).$$

With $u = \sum_{cycl}x^2$ and $v = \sum_{cycl}xy$, the inequality rewrites as

$$\frac{(u+2v)u}{3v} \geq \frac{u+v}{2}.$$

This is successively equivalent to

$$\Leftrightarrow 2u^2 + 2uv \geq 3uv + 3v^2 \Leftrightarrow 2u^2 + uv - 3v^2 \geq 0$$

$$\Leftrightarrow (u-v)(2u+3v) \geq 0$$

because, *as we know,* $u = \sum_{cycl}x^2 \geq \sum_{cycl}xy = v$. □

Solution 3 (by Alexander Bogomolny).

If we take into account the *well known expressions for the medians*, we shall reduce the inequality to

$$\sum_{cycl}\frac{(m_b+m_c-m_a)^3}{m_a} \geq m_a^2 + m_b^2 + m_c^2.$$

For simplicity, let's use x, y, z for m_a, m_b, m_c. Thus the inequality to prove becomes

$$\sum_{cycl}\frac{(x+y-z)^3}{z} \geq x^2 + y^2 + z^2.$$

From here we follow in the footsteps of Solution 1. On one hand,

$$\sum_{cycl}(x+y-z)^2 = 3\sum_{cycl}x^2 - 2\sum_{cycl}xy \geq \sum_{cycl}x^2.$$

On the other hand,

$$\sum_{cycl}z(x+y-z) = 2\sum_{cycl}xy - \sum_{cycl}x^2 \leq \sum_{cycl}x^2.$$

It follows by Bergström inequality that

$$\sum_{cycl} \frac{(x+y-z)^3}{z} = \sum_{cycl} \frac{(x+y-z)^4}{z(x+y-z)}$$

$$\geq \frac{(\sum_{cycl}(x+y-z)^2)^2}{\sum_{cycl} z(x+y-z)} \geq \frac{(\sum_{cycl} x^2)^2}{\sum_{cycl} x^2} = \sum_{cycl} x^2.$$

□

Acknowledgment (by Alexander Bogomolny)
CutTheKnotMath facebook page. The problem was originally published at the ***Romanian Mathematical Magazine***. Solution 1 is by Soumitra Mandal ; Solution 2 is by Soumava Chakraborty.

198. An Inequality with Just Two Variable And an Integer

Prove that, for real $a, b > 0$ and integer n,

$$\left(\frac{a}{b^n} + \frac{b}{a^n}\right)\left(\frac{a^n}{b} + \frac{b^n}{a}\right)\left(\frac{a^n}{b^n} + \frac{b}{a}\right)\left(\frac{b^n}{a^n} + \frac{a}{b}\right) \geq \left(\sqrt{\left(\frac{a}{b}\right)^{n-1}} + \sqrt{\left(\frac{b}{a}\right)^{n-1}}\right)$$

Proposed by Daniel Sitaru

Solution (by Daniel Sitaru).

From means the inequality

(1) $$\frac{a}{b^n} + \frac{b}{a^n} \geq 2\sqrt{\frac{ab}{a^n b^n}}.$$

(2) $$\frac{a^n}{b^n} + \frac{b}{a} \geq 2\sqrt{\frac{a^n b}{b^n a}}.$$

(3) $$\frac{b^n}{a^n} + \frac{a}{b} \geq 2\sqrt{\frac{b^n}{a^n b}}$$

By multiplying the relationship (1), (2), (3):

(4) $$\left(\frac{a}{b^n} + \frac{b}{a^n}\right)\left(\frac{a^n}{b^n} + \frac{b}{a}\right)\left(\frac{b^n}{a^n} + \frac{a}{b}\right) \geq \frac{8}{\sqrt{a^{n-1} \cdot b^{n-1}}}.$$

We prove that

(5) $$\frac{a^n}{b} + \frac{b^n}{a} \geq a^{n-1} + b^{n-1}.$$

$$a^{n+1} + b^{n+1} \geq a^n b + ab^n$$

$$a^n(a-b) - b^n(a-b) \geq 0$$

$$(a-b)(a^n - b^n) \geq 0$$

$$(a-b)^2(a^{n-1} + a^{n-2}b + \cdot + b^{n-1}) \geq 0,$$

which is true. Now multiply the relationships (4), (5)

$$\left(\frac{a^n}{b} + \frac{b^n}{a}\right)\left(\frac{a}{b^n} + \frac{b}{a^n}\right)\left(\frac{a^n}{b^n} + \frac{b}{a}\right)\left(\frac{b^n}{a^n} + \frac{a}{b}\right) \geq \frac{8(a^{n-1} + b^{n-1})}{\sqrt{a^{n-1} \cdot b^{n-1}}}$$

$$= 8\left(\sqrt{\left(\frac{a}{b}\right)^{n-1}} + \sqrt{\left(\frac{b}{a}\right)^{n-1}}\right)$$

□

Acknowledgment (by Alexander Bogomolny)
This problem, along with a solution, was kindly communicated to me by Daniel Sitaru. Daniel has earlier published the problem at ***Romanian Mathematical Magazine***.

199. An Universal Inequality for Cevians

Let in $\Delta, p = \min\{a, b, c\}, q = \max\{a, b, c\}; M \in BC, N \in AC, P \in AB$.
Prove that:

$$3 + 9pqAM_2 + BN_2 + CP_2 \leq (p + q)(1AM + 1BM + 1CP)$$

Examples: For l_a, l_b, l_c the angle bisectors in ΔABC,

$$3 + 9pql_{2a} + l_{2b} + l_{2c} \leq (p + q)(1l_a + 1l_b + 1l_c).$$

For m_a, m_b, m_c the medians in ΔABC,

$$3 + 9pqm_{2a} + m_{2b} + m_{2c} \leq (p + q)(1m_a + 1m_b + 1m_c).$$

Proof.

Naturally, $AM, BN, CP \in [p, q]$ so that $(p - AM)(q - AM) \leq 0$, implying

$$pq - (p + q)AM + AM_2 \leq 0,$$

which is the same as

$$pqAM_2 + 1 \leq p + qAM.$$

Similar inequalities holds for BN and CP. Taking the sum of all three gives

$$3 + pq \sum_{cyc} 1AM_2 \leq (p + q) \sum_{cyc} 1AM.$$

By the *Harmonic Mean-Arithmetic Mean Inequality*,

$$9 \sum_{cyc} AM_2 \leq \sum_{cyc} 1AM_2$$

such that

$$3 + 9pq \sum_{cyc} AM_2 \leq 3 + pq \sum_{cyc} 1AM_2.$$

By the *transitivity* of the relation of inequality,

$$3 + 9pqAM_2 + BN_2 + CP_2 \leq (p + q)(1AM + 1BN + 1CP).$$

Warning:
The above is a fallacy invented by Daniel Sitaru and Leo Giugiuc and posted at the ***CutTheKnotMath facebook page***. As a hint of what is wrong with the proof, note that it would have been nice to mention the applicability of the inequality to the altitudes h_a, h_b, h_c. The omission was deliberate!
Another way to see that there is something wrong with the above is to consider the most symmetric configuration of an equilateral triangle in which M, N, P are the midpoints of the corresponding sides. □

200. An Inequality in Integers

The following inequality, due (1979) to professor Radu Gologan has posted at the ***CutTheKnotMath facebook page*** by Leo Giugiuc along with a ***solution*** by Daniel Sitaru and Leonard Giugiuc. Radu Gologan is the Romanian team leader for the IMO.

Let a and b be positive integers such that $\frac{a}{b} < \sqrt{7}$. Prove that $\frac{a}{b} + \frac{1}{ab} <$ sqrt7.

Solution.

$a^2 < 7b^2$ so that $a^2 \leq 7b^2 - 1$. In $\mathbb{Z}_7, a^2 \in \{0, 1, 2, 4\}$, making $a^2 = 7b^2 - 1$ impossible. Thus, necessarily, $a^2 \leq 7b^2 - 2$. But then, again, $a^2 = 7b^2 - 2$ is also impossible such that, in fact $a^2 \leq 7b^3 - 3$, or $a \leq \sqrt{7b^2 - 3}$.

Introduce function $f(x) = x + \frac{1}{x}$ which is monotone increasing for $x \geq 1$. It follows that

$$\left(\sqrt{7b^2 - 3} + \frac{1}{\sqrt{7b^2 - 3}}\right)^2 \geq \left(a + \frac{1}{a}\right)^2$$

which is equivalent to

$$7b^2 - 1 + \frac{1}{7b^2 - 3} \geq \left(a + \frac{1}{a}\right)^2.$$

In addition, since b is a positive integer, $1 > \frac{1}{7b^2 - 3}$, such that $7b^2 > \left(a + \frac{1}{a}\right)^2$.

In other words, $7 > \left(\frac{a}{b} + \frac{1}{ab}\right)^2$, i.e., $\frac{a}{b} + \frac{1}{ab} < \sqrt{7}$, as required. □

Its nice to be important but more important its to be nice.

At this paper works a TEAM.

This is RMM TEAM.

To be continued!

Daniel Sitaru