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Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\int_0^1 x^{2n}\sqrt{(1-x^2)^{2n+1}}\,dx}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India, Solution 2 by Ravi Prakash-New Delhi-India, Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco, Solution 4 by Geanina Tudose-Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$, **when** $x = 0, \theta = 0; x = 1, \theta = \frac{\pi}{2}$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\int_0^1 x^{2n}\sqrt{(1-x^2)^{2n+1}}\,dx} = \lim_{n\to\infty} \sqrt[n]{\int_0^{\frac{\pi}{2}} \sin^{2n}\theta\cos^{2n+2}\theta\,d\theta}$$

$$= \lim_{n\to\infty} \sqrt[n]{\frac{1}{2}\beta(n, n+1)} = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2}\cdot\frac{\Gamma(n)\Gamma(n+1)}{\Gamma(2n+1)}} \overset{CAUCHY-D'ALEMBERT}{=}$$

$$= \lim_{n\to\infty}\left(\frac{\Gamma(n+1)\Gamma(n+2)}{\Gamma(2n+3)}\cdot\frac{\Gamma(2n+1)}{\Gamma(n)\Gamma(n+1)}\right)$$

$$= \lim_{n\to\infty}\left(\frac{n(n+1)}{(2n+1)(2n+2)}\right) = \frac{1}{4} \quad \textit{(Ans:)}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\textit{Let } a_n = \int_0^1 x^{2n}(1-x^2)^{\frac{(2n+1)}{2}}dx$$

Put $x^2 = t$**, so that**

$$a_n=\frac{1}{2}\int_0^1 t^{\frac{2n-1}{2}}(1-t)^{\frac{(2n+1)}{2}}dt$$

$$=\frac{1}{2}\int_0^1 t^{n+\frac{1}{2}-1}(1-t)^{n+\frac{3}{2}-1}dt=\frac{1}{2}\beta\left(n+\frac{1}{2},n+\frac{3}{2}\right)$$

$$\lim_{n\to\infty}(a_n)^{\frac{1}{n}}=\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\beta\left(n+\frac{3}{2},n+\frac{5}{2}\right)}{\beta\left(n+\frac{1}{2},n+\frac{3}{2}\right)}$$

$$=\lim_{n\to\infty}\frac{\sqrt{n+\frac{3}{2}}\sqrt{n+\frac{5}{2}}}{\sqrt{2n+4}}\cdot\frac{\sqrt{2n+2}}{\sqrt{n+\frac{1}{2}}\sqrt{n+\frac{3}{2}}}$$

$$=\lim_{n\to\infty}\frac{\left(n+\frac{3}{2}\right)\left(n+\frac{1}{2}\right)}{(2n+3)(2n+2)}=\frac{1}{4}\lim_{n\to\infty}\frac{\left(1+\frac{1}{2n}\right)}{\left(1+\frac{1}{n}\right)}=\frac{1}{4}$$

Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco

$$I_n=\int_0^1 x^{2n}\sqrt{(1-x^2)^{2n+1}}\overset{x=\sin t}{=}\int_0^{\frac{\pi}{2}}\sin^{2n}t\sqrt{(1-\sin^2 t)^{2n+1}}\cos t\,dt$$

$$=\int_0^{\frac{\pi}{2}}\sin^{2n}t\times\cos^{2n+2}t\,dt=\int_0^{\frac{\pi}{2}}\sin^{2n}t\times\cos^{2n}t\,(1-\sin^2 t)dt$$

$$=\int_0^{\frac{\pi}{2}}\sin^{2n}t\times\cos^{2n}t\,dt-\int_0^{\frac{\pi}{2}}\sin^{2n+2}t\times\cos^{2n}t\,dt$$

$$\overset{t=\frac{\pi}{2}-u}{=} \frac{1}{2^{2n}}\int_0^{\frac{\pi}{2}} \sin^{2n} 2t\,dt - \int_0^{\frac{\pi}{2}} \cos^{2n+2} u \times \sin^{2n} u\,du$$

$$\rightarrow 2I = \frac{1}{2^n}\int_0^{\frac{\pi}{2}} \sin^{2n} 2t\,dt = \frac{1}{2^{n-1}}\int_0^{\frac{\pi}{2}} \sin^{2n} t\,dt = \frac{1}{2^{n-2}}\int_0^{\frac{\pi}{2}} \sin^{2n} t\,dt$$

$$\rightarrow I = \frac{1}{2^{n-1}}\int_0^{\frac{\pi}{2}} \sin^{2n} t\,dt = \frac{1}{2^{n-1}}\cdot\frac{\pi}{2}\cdot\frac{(2n)!}{2^{2n}(n!)^2} = \pi\cdot\frac{(2n)!}{2^{4n}(n!)^2}$$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{I_n} = \lim_{n\to\infty} \sqrt[n]{\pi\cdot\frac{(2n)!}{2^{4n}(n!)^2}} = \lim_{n\to\infty} \frac{\pi\cdot\frac{(2n+2)!}{2^{4n+4}\left((n+1)!\right)^2}}{\pi\cdot\frac{(2n)!}{2^{4n}(n!)^2}}$$

$$= \lim_{n\to\infty} \frac{(2n+2)(2n+1)}{16(n+1)^2} = \frac{1}{4}$$

Solution 4 by Geanina Tudose-Romania

Consider $\int_0^1 x^{2n}\sqrt{(1-x^2)^{3n+1}}\,dx$

Let $x = \sin x,\ \ x = 0 \Rightarrow \sin 0 = 0$

$$dx = \cos\alpha\ dx \quad x = 1 \Rightarrow \alpha = \frac{\pi}{2}$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2n} x\cdot\sqrt{(1-\sin^3\alpha)^{3n+1}}\cos\alpha\,dx = \int_0^{\frac{\pi}{2}} \sin^{2n}\alpha\cdot\cos^{2n+1}\alpha\cos x\,dx$$

$$= \int_0^{\frac{\pi}{2}} \sin^{2n}\alpha\cos^{2n+3}\alpha\,d\alpha = \int_0^{\frac{\pi}{2}} (\sin^2\alpha\cos^2\alpha)^n\cdot\cos^2\alpha\,d\alpha$$

$$= \int_0^{\frac{\pi}{2}} \left(\frac{\sin^3 2\alpha}{2^3}\right)^n \cdot \frac{\cos 2\alpha + 1}{2} d\alpha$$

$$= \underbrace{\frac{1}{2^{2n+1}}, \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \cdot \cos 2\alpha \, d\alpha}_{I_1} + \underbrace{\frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \, d\alpha}_{I_2}$$

$$I_1 = \frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \cdot \cos 2\alpha \, d\alpha$$

$$= \frac{1}{2^{2n+1}} \cdot \frac{1}{(2n+1) \cdot 2} \sin^{2n+1} 2\alpha \Big|_0^{\frac{\pi}{2}} = 0$$

$$I_2 = \frac{1}{2^{2n+1}} \int_0^{\frac{\pi}{2}} \sin^{2n} 2\alpha \, d\alpha = \frac{1}{2^{2n+2}} \int_0^{\pi} \sin^{2n} x \, dx$$

$$x = 2\alpha \Rightarrow d\alpha = 2d\alpha$$

$$= \frac{1}{2^{2n+2}} \left(\int_0^{\frac{\pi}{2}} \sin^{2n} \alpha \, d\alpha \underbrace{\int_{\frac{\pi}{2}}^{\pi} \sin^{2n} x \, dx}_{y = x - \frac{\pi}{2}} \right) = \frac{1}{2^{2n+2}} \left(\int_0^{\frac{\pi}{2}} \sin^{2n} x \, dx + \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx \right)$$

But $I_{2n} = \int_0^{\frac{\pi}{2}} \sin^2 x \, dx = \int_0^{\frac{\pi}{2}} \cos^{2n} x \, dx = \frac{(2n-1)!!}{(2n)!!} \cdot \frac{\pi}{2}$

Hence $I = I_2 = \frac{1}{2^{n+\alpha}} \;\; 8 \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \pi$

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\frac{1}{2^{2n+2}} \cdot \frac{(2n-1)!!}{(2n)!!} \cdot \pi}$$

Using Cauchy D'Alembert $\lim_{n\to\infty} \sqrt[n]{\frac{(2n-1)!!}{(2n)!!}} =$

$$= \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{(2n+1)!!}{(2n+2)!!}\cdot\frac{(2n)!!}{(2n-1)!!} = \lim_{n\to\infty}\frac{2n+1}{2n+2} = 1$$

Thus $\Omega = \lim_{n\to\infty}\sqrt[n]{\frac{1}{2^{2n+2}}}\cdot\sqrt[n]{\frac{(2n-1)!!}{(2n)!!}}\cdot\pi^{\frac{1}{n}} = \frac{1}{4}$