Romanian Mathematical Magazine
Web: http://www.ssmrmh.ro
The Author: This article is published with open access.

PROBLEM SP.077 RMM
NUMBER 6 AUTUMN 2017
ROMANIAN MATHEMATICAL MAGAZINE 2017

MARIN CHIRCIU

1. Let ABC be an acute triangle. Prove that

$$\boldsymbol{\frac{m_a}{h_a}\cdot\cos A+\frac{m_a}{h_c}\cdot\cos B+\frac{m_c}{h_c}\cdot\cos C\geq\frac{3}{2}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

Proof.

Using $m_a\geq\frac{b^2+c^2}{4R},h_a=\frac{bc}{2R}$ *and* $\cos A=\frac{b^2+c^2-a^2}{2bc}$
we obtain $\frac{m_a}{h_a}\cdot\cos A\geq\frac{(b^2+c^2)(b^2+c^2-a^2)}{4b^2c^2}$

It follows $\sum\frac{m_a}{h_a}\cdot\cos A\geq\sum\frac{(b^2+c^2)(b^2+c^2-a^2)}{4b^2c^2}=\frac{\sum a^2(b^2+c^2)(b^2+c^2-a^2)}{4a^2b^2c^2}=\frac{6a^2b^2c^2}{4a^2b^2c^2}=\frac{3}{2}$

Equality holds if and only if the triangle is equilateral.

□

Note.

From the above proof the condition of acute-angled triangle is not necessary.

Remark.

In the same way it can be proposed:

2. In ΔABC

$$\boldsymbol{\frac{m_a}{h_a}\cdot(\cos B+\cos C)+\frac{m_a}{h_b}\cdot(\cos C+\cos A)+\frac{m_c}{h_c}\cdot(\cos A+\cos B)\leq\frac{2R}{r}-1}$$

Proposed by Marin Chirciu - Romania

Proof.

We have

$$\sum\frac{m_a}{h_a}\cdot(\cos B+\cos C)=\sum\frac{m_a}{h_a}\cdot(\cos A+\cos B+\cos C-\cos A)=\sum\frac{m_a}{h_a}\sum\cos A-\sum\frac{m_a}{h_a}\cos A\leq$$
$$\leq\left(1+\frac{r}{R}\right)\sum\frac{m_a}{h_a}-\frac{3}{2}\leq\frac{3}{2}\sum\frac{m_a}{h_a}-\frac{3}{2}=\frac{3}{2}\left(\sum\frac{m_a}{h_a}-1\right)\leq\frac{3}{2}\left(\frac{4R+r}{3r}-1\right)=\frac{2R}{r}-1$$

where we've used:

$\frac{m_a}{h_a}\cdot\cos A\geq\frac{3}{2}$ *(inequality **1.**)*, $\sum\cos A=1+\frac{r}{R}\leq\frac{3}{2}$ *(Euler's inequality) and*

$\sum\frac{m_a}{h_a}\leq\frac{4R+r}{3r}$, *which follows from Cebyshev's inequality:*

The triplets (m_a,m_b,m_c) *and* $\left(\frac{1}{h_a},\frac{1}{h_b},\frac{1}{h_c}\right)$ *are reversed ordered, and* $\sum m_a\leq 4R+r$ *and* $\sum\frac{1}{h_a}=\frac{1}{r}$

$$\textit{wherefrom } \sum \frac{m_a}{h_a} \leq \frac{1}{3} \cdot \sum m_a \sum \frac{1}{h_a} \leq \frac{1}{3} \cdot (4R+r) \cdot \frac{1}{r} = \frac{4R+r}{3r}$$

The equality holds if and only if the triangle is equilateral.

□

MATHEMATICS DEPARTMENT, "THEODOR COSTESCU" NATIONAL ECONOMIC COLLEGE, DROBETA TURNU - SEVERIN, MEHEDINTI.

E-mail address: dansitaru63@yahoo.com