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1. Let ABC be an acute triangle. Prove that

$$\sum \sqrt{\cos A \sin B \sin C} \leq \frac{3}{2}\sqrt{\frac{3}{2}}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

Proof.

Using CBS inequality we obtain

$$\left(\sum \sqrt{\cos A \cdot \sin B \cdot \sin C}\right)^2 \leq \sum \cos A \cdot \sum \sin B \sin C = \left(1+\frac{r}{R}\right)\cdot \frac{p^2+r^2+4Rr}{4R^2} < \frac{3}{2}\cdot\frac{9}{4} = \left(\frac{3}{2}\sqrt{\frac{3}{2}}\right)^2,$$

where the last inequality follows from:

1) $1+\frac{r}{R} \leq \frac{3}{2} \Leftrightarrow R \geq 2r$ *(Euler's inequality).*

2) $\frac{p^2+r^2+4Rr}{4R^2} \leq \frac{9}{4} \Leftrightarrow p^2 \leq 9R^2 - 4Rr - r^2$ *, true from Gerretsen's inequality* $p^2 \leq 4R^2+4Rr+3r^2$ *. It remains to prove that:*

$$4R^2+4Rr+3r^2 \leq 9R^2-4Rr-r^2 \Leftrightarrow 5R^2-8Rr-4r^2 \geq 0 \Leftrightarrow (R-2r)(5R+2r) \geq 0,$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way it can be proposed:

2. In ΔABC

$$\sum \sqrt{\sin A \cos B \cos C} \leq \frac{3}{2}\sqrt{\frac{\sqrt{3}}{2}}$$

Proposed by Marin Chirciu

Proof.

Using CBS inequality we obtain

$$\left(\sum \sqrt{\sin A \cdot \cos B \cos C}\right)^2 \leq \sum \sin A \cdot \sum \cos B \cos C = \frac{rp}{2R^2}\cdot\frac{p^2+r^2-4R^2}{4R^2} \leq \frac{3\sqrt{3}}{8}\cdot\frac{3}{4} = \left(\frac{3}{2}\sqrt{\frac{\sqrt{3}}{2}}\right)^2$$

where the last inequality follows from:

1) $\frac{rp}{2R^2} \leq \frac{3\sqrt{3}}{8} \Leftrightarrow p \leq \frac{3R^2\sqrt{3}}{4r}$ *, true from Mitrinović's inequality* $p \leq \frac{3R\sqrt{3}}{2}$ *and Euler's inequality* $R \geq 2r$

2) $\frac{p^2+r^2-4R^2}{4R^2} \leq \frac{3}{4} \Leftrightarrow p^2 \leq 7R^2 - r^2$ *, true from Gerretsen's inequality* $p^2 \leq 4R^2 + 4Rr + 3r^2$. It remains to prove that:

$$4R^2 + 4Rr + 3r^2 \leq 7R^2 - r^2 \Leftrightarrow 3R^2 - 4Rr - 4r^2 \geq 0 \Leftrightarrow (R - 2r)(3R + 2r) \geq 0$$

obviously from Euler's inequality $R \geq 2r$.

Equality holds if and only if the triangle is equilateral.

□

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