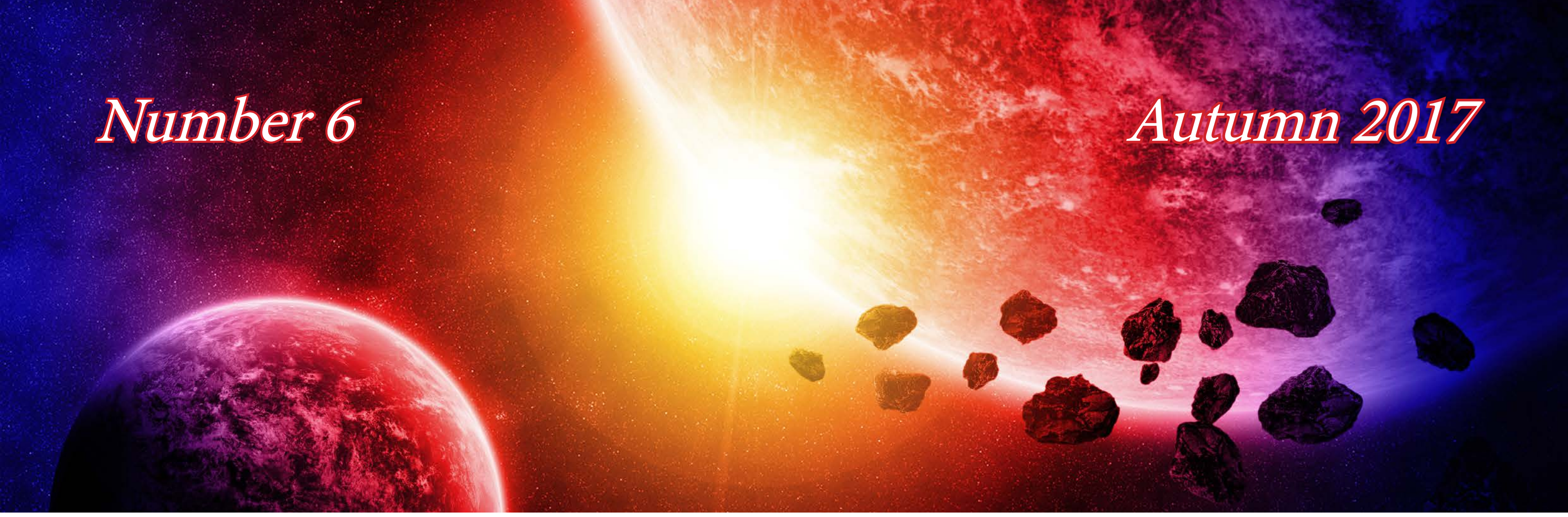

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PROBLEMS FOR JUNIORS

JP.076. Let ABC be an acute triangle. Prove that

$$(a\cot A)^a(b\cot B)^b(c\cot C)^c \leq (2r)^{a+b+c}$$

where $a = BC, b = CA, c = AB$, and r is the inradius.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.077. Let $a_1, a_2, \ldots, a_9$ be non-negative real numbers such that $a_1 + a_2 + \ldots + a_9 = 1$. Prove that for all $\lambda \geq 4$, the following inequality holds

$$\sqrt{\sum_{1\leq i\leq 9} a_i^2} + \lambda\sqrt{\sum_{1\leq i<j\leq 9} a_i a_j} \leq \frac{2\lambda + 1}{3}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.078. Let a, b, c be positive real numbers such that $a^2 = b^2 + c^2$. Prove that

$$ab + bc + ca + (\sqrt{2} - 1)\frac{abc}{a + b + c} \leq 2a^2.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.079. Prove the inequality holds for all positive real numbers a, b, c

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{4}{2a+3b+3c} + \frac{4}{2b+3c+3a} + \frac{4}{2c+3a+3b}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.080. Prove that in any triangle ABC,

$$\frac{a^2+b^2+c^2}{a+b+c}\left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c}\right) \geq 2\sqrt{3}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.081. If $x, y, z > 0$ then:

$$\sqrt{\frac{13x}{6x+7y}} + \sqrt{\frac{13y}{6y+7z}} + \sqrt{\frac{13z}{6z+7x}} \leq 3$$

Proposed by Marin Chirciu - Romania

JP.082. If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{a}{1+3b^4} + \frac{b}{1+3c^4} + \frac{c}{1+3a^4} \geq \frac{3}{4}$$

Proposed by Marin Chirciu - Romania

JP.083. In ΔABC the following relationship holds:

$$(a^{2m} + b^{2m} + c^{2m})\left(\frac{1}{(a+b)^{2n}} + \frac{1}{(b+c)^{2n}} + \frac{1}{(c+a)^{2n}}\right) \geq$$

$$\geq 3^{m-n+2} \cdot 4^{m-2n} \cdot r^{2(m-n)}; m, n \geq 1$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

JP.084. In ΔABC the following relationship holds:

$$\sum\left((a+b)\tan\frac{C}{2}\right)^m \cdot \sum \frac{1}{\left(\tan\frac{A}{2} + \tan\frac{B}{2}\right)^{2n}} \geq$$

$$\geq 3^{n+2} \cdot 4^{m-n} \cdot r^n; m \geq n \geq 1$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

JP.085. Let ABC denote a triangle, I its incentre, R its circumradius, r its inradius, and x, y and z the inradii of triangles IBC, ICA and IAB respectively. Prove that

$$\frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z} \leq \frac{4+3\sqrt{3}}{2r} + \frac{2}{R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

JP.086. Let a, b, c be the side lengths of a triangle ABC with incentre I, circumradius R and inradius r. Prove that

$$\frac{\sqrt{AI}}{a} + \frac{\sqrt{BI}}{b} + \frac{\sqrt{CI}}{c} \leq \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{R+r}}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

JP.087. Let ABC be an acute triangle. Prove that

$$\sqrt{\cos A \cdot \sin B \cdot \sin C} + \sqrt{\sin A \cdot \cos B \cdot \sin C} + \sqrt{\sin A \cdot \sin B \cdot \cos C} \leq \frac{3}{2}\sqrt{\frac{3}{2}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

JP.088. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3+b^3}{c^2+ab} + \frac{b^3+c^3}{a^2+bc} + \frac{c^3+a^3}{b^2+ca} \geq \frac{9abc}{ab+bc+ca}$$

Proposed by Nguyen Ngoc Tu – HaGiang –Vietnam

JP.089. Let a, b, c be positive real numbers, take $X = \frac{a}{b} + \frac{b}{a}, Y = \frac{b}{c} + \frac{c}{b}, Z = \frac{c}{a} + \frac{a}{c}$. Prove that

$$X + Y + Z \geq 2\sqrt[4]{(X^2 + Y^2 + Z^2 - 3)(X + Y + Z + 3)}$$

Proposed by Nguyen Ngoc Tu – HaGiang –Vietnam

JP.090. Let r and s be the inradius and the semiperimeter of a triangle ABC respectively. Prove that

$$\frac{1 + \cos A \cos B \cos C}{\sin A \sin B \sin C} \geq \frac{s}{3r}.$$

Proposed by Martin Lukarevski – Skopje - Macedonia

PROBLEMS FOR SENIORS

SP.076. Let a, b, c be the side - lengths of an acute triangle with perimeter 1. Prove that

$$E_1 \geq a^a b^b c^c \geq E_2$$

where

$$E_1 = \frac{(b + c - a)(c + a - b)(a + b - c)}{(b^2 + c^2 - a^2)^a (c^2 + a^2 - b^2)^b (a^2 + b^2 - c^2)^c},$$

and

$$E_2 = \frac{(b^2 + c^2 - a^2)^{b+c} (c^2 + a^2 - b^2)^{c+a} (a^2 + b^2 - c^2)^{a+b}}{(b + c - a)(c + a - b)(a + b - c)}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.077. Prove that in any acute triangle ABC the following inequality holds

$$\frac{m_a}{h_a} \cos A + \frac{m_a}{h_b} \cos B + \frac{m_c}{h_c} \cos C \geq \frac{3}{2}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.078. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that

$$a^{-a} b^{-b} c^{-c} + a^{-b} b^{-c} c^{-a} + a^{-c} b^{-a} c^{-b} \leq a^{-1} + b^{-1} + c^{-1}.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.079. Prove that for all positive real numbers a, b, c and integer $n \geq 3$, the following inequality holds

$$\frac{a^n + b^n + c^n}{9}\left(\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}\right) \geq \left(\frac{b+c}{6a} + \frac{c+a}{6b} + \frac{a+b}{6c}\right)^n$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.080. Prove that for all positive real numbers a, b, c the following inequality holds

$$\frac{(a+b)^2}{a^2 - ab + b^2} + \frac{(b+c)^2}{b^2 - bc + c^2} + \frac{(c+a)^2}{c^2 - ca + a^2} \geq \frac{9(a^2b + b^2c + c^2a + abc)}{a^3 + b^3 + c^3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

SP.081. Let a, b, c be positive real numbers and $k \geq 2$. Prove that

$$\sqrt{\frac{bc}{(b+ka)(c+ka)}} + \sqrt{\frac{ca}{(c+kb)(a+kb)}} + \sqrt{\frac{ab}{(a+kc)(b+kc)}} \geq \frac{3}{k+1}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.082. Let ABC be an equilateral triangle with side - length a and let M be any point inside the triangle. Prove that

$$\frac{a^2}{2} \geq xMA + yMB + zMC \geq 2(xy + yz + zx)$$

where x, y, z denote the distances from M to the sides BC, CA, AB, respectively.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.083. Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R. Prove that

$$\left(1 + \frac{1}{m_a}\right) \cdot \left(1 + \frac{1}{m_b}\right) \cdot \left(1 + \frac{1}{m_c}\right) \geq \left(1 + \frac{2}{3R}\right)^3.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

SP.084. Prove that if $n \in \mathbb{N}^*$ then:

$$2\int_0^1 \arctan(x^{n-1})\arctan(x^n)dx \leq \int_0^1 \arctan^2(x^n)dx + \frac{1}{2n-1}$$

Proposed by Daniel Sitaru - Romania

SP.085. Prove that if $a, b, \in (0, \infty); n \in \mathbb{N}^*$ then:

$$\left(\frac{a}{b^n} + \frac{b}{a^n}\right)\left(\frac{a^n}{b} + \frac{b^n}{a}\right)\left(\frac{a^n}{b^n} + \frac{b}{a}\right)\left(\frac{b^n}{a^n} + \frac{a}{b}\right) \geq 8\left(\sqrt{(\frac{a}{b})^{n-1}} + \sqrt{(\frac{b}{a})^{n-1}}\right)$$

Proposed by Daniel Sitaru - Romania

SP.086. Prove that if a, b, c are the lengths's sides in triangle ABC then:

$$\sin^2 a + \sin^2 b + \sin^2 c \geq 4 \sin s \sin(s-a) \sin(s-b) \sin(s-c)$$

Proposed by Daniel Sitaru - Romania

SP.087. Let z_1, z_2, z_3 be the affixes of A, B respectively C in acute-angled ΔABC.
Prove that:

$$\prod\left(\left|\frac{z_2 - z_3}{z_2 + z_3}\right| + \left|\frac{z_3 - z_1}{z_3 + z_1}\right|\right) \geq \frac{32sr^3}{(s^2 - (2R+r)^2)^2}$$

Proposed by Daniel Sitaru - Romania

SP.088. Let $a, b, c > 0$ such that $ab + bc + ca + abc = 4$.
Prove that

$$(a+1)\sqrt{(b+1)(c+1)} + (b+1)\sqrt{(c+1)(a+1)} + (c+1)\sqrt{(a+1)(b+1)} \geq$$
$$\geq a + b + c + 9$$

Proposed by Nguyen Ngoc Tu – HaGiang –Vietnam

SP.089. Let r_a, r_b, r_c be the exradii of a triangle ABC, h_a, h_b, h_c the altitudes and let R, r, s denote the circumradius, inradius and semiperimeter respectively. Prove that

$$\frac{r_a^2}{h_a} + \frac{r_b^2}{h_b} + \frac{r_c^2}{h_c} \geq \frac{2s^2}{3}\left(\frac{1}{r} - \frac{1}{R}\right)$$

Proposed by Martin Lukarevski – Skopje - Macedonia

SP.090.If $u, v > 0$, with $2u - v > 0$ and α, β, γ are the measures of the angles of triangle ABC, then

$$\sum_{cyc} \frac{\sin\alpha}{u \sin\beta + v\sqrt{\sin\alpha \sin\beta}} \geq \frac{3}{u+v}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

UNDERGRADUATE PROBLEMS

UP.076. Evaluate:

$$S = \sum_{n=1}^{\infty}\left(\frac{H_{2n+1}}{n^2}\right)$$

Proposed by Shivam Sharma – New Delhi – India

UP.077. Evaluate:

$$S = \prod_{n=1}^{\infty}\left(e\left(\frac{n}{n+1}\right)^n\sqrt{\frac{n}{n+1}}\right)$$

Proposed by Shivam Sharma – New Delhi – India

UP.078. Find:

$$\Omega = \lim_{n\to\infty} n\left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[2n]{(2n-1)!!}\right)\left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!}\right)$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

UP.079. If $x, y, z > 0$ and $b \geq a > 0$ then:

$$\ln\frac{(x+b)(y+b)(z+b)}{(x+a)(x+b)(x+c)} \geq \frac{15}{8}\ln\frac{b}{a} + \frac{1}{16}\left(\frac{1}{b^2} - \frac{1}{a^2}\right)(x^2+y^2+z^2)$$

Proposed by Mihály Bencze - Romania

UP.080. Let be: $f : (0,\infty) \to (0,\infty)$ a function such that:

$$\lim_{x\to\infty}\frac{f(x)}{x} = a \in (0,\infty) \text{ and}$$

$$\lim_{x\to\infty}\left(\frac{f(x+1)}{f(x)}\right)^x = b \in (0,\infty). \text{ Find:}$$

$$\Omega = \lim_{x\to\infty}\Big(f(x+1) - f(x)\Big)$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

UP.081. If

$$B_n(t) = n^{1-t}\left(\frac{(n+1)^{2t}}{(\sqrt[n+1]{(n+1)!})^t} - \frac{n^{2t}}{(\sqrt[n]{n!})^t}\right), \text{ with } t > 0, \text{ then compute}$$

$$\lim_{n\to\infty} B_n(t).$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

UP.082. Let $n \in N$. Calculate

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x\left(\cos x\cos^{2n+1}\left(\frac{\pi}{2}\sin x\right) + \sin x\cos^{2n+1}\left(\frac{\pi}{2}\cos x\right)\right)dx.$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

UP.083. Prove that in any triangle ABC the following relationship holds:

$$R\sum(b+c-2a)^2 \leq 4(R-2r)\sum a^2$$

Proposed by Daniel Sitaru - Romania

UP.084. Evaluate

$$I = \int_0^1 \int_0^1 \frac{(\ln(x)\ln(y))^s}{1-xy} dxdy$$

Proposed by Shivam Sharma – New Delhi – India

UP.085. Let k be positive integer. Calculate:

$$\lim_{x\to\infty}\left(\left(\Gamma(x+2)\right)^{\frac{k+1}{x+1}} - \left(\Gamma(x+1)\right)^{\frac{k+1}{x}}\right)\left(\Gamma(x+1)\right)^{-\frac{k}{x}},$$

where $\Gamma(x)$ is the Gamma function (or Euler's second integral).

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu - Romania

UP.086. Let $a > 0, b, c > 1$ and $f, g : R \to R$ be continuous and odd functions. Prove that:

$$\int_{-a}^{a} f(x)\ln\left(b^{g(x)} + c^{g(x)}\right)dx = \left(\ln(bc)\right)\int_0^a f(x)g(x)dx.$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu - Romania

UP.087. Let $a, b \in R, a < b$ and continuous functions $f, g, h : R \to R$ such that $f(a+b-x) = -f(x), g(a+b-x) = g(x)$, $h(a+b-x) = -h(x), \forall x \in R$. Prove that

$$\int_a^b f(x)\left(\arctan g(x)\right)\ln\left(1+e^{h(x)}\right)dx = \frac{1}{2}\int_a^b f(x)h(x)\arctan g(x)dx.$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu - Romania

UP.088. Let $f : R \to R$ be a continuous function such that $f(x) = f(1-x), \forall x \in R$. Prove that:

$$\int_0^1 \frac{\sqrt{1-x}+\sqrt{x}}{1+\sqrt{2x}} f(x)dx = \frac{\sqrt{2}}{2}\cdot\int_0^1 f(x)dx$$

Proposed by D.M. Bătineţu – Giurgiu; Neculai Stanciu - Romania

UP.089. Evaluate:

$$\int_0^1 \Big[\ln(x)\ln(1-x) + Li_2(x)\Big]\left(\frac{Li_2(x)}{x(1-x)} - \frac{\zeta(2)}{1-x}\right)dx$$

Proposed by Shivam Sharma – New Delhi – India

UP.090. Evaluate:

$$\int_0^1 (\ln(\Gamma(x)))(\sin(2k\pi x))dx, \quad k \geq 1$$

Proposed by Shivam Sharma – New Delhi – India

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