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SOLUTIONS

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JP.076. Let ABC be an acute triangle. Prove that:

$$(a \cot A)^a (b \cot B)^b (c \cot C)^c \leq (2r)^{a+b+c}$$

where $a = BC$, $b = CA$, $c = AB$, and r is the inradius.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Anh Tai Tran-Vietnam

$$\begin{aligned} \pi(a \cdot \cot a)^{\frac{a}{a+b+c}} &\leq 2\pi \\ LHS &\leq \frac{a^2 \cdot \cot a}{\sum a} = \frac{\sum a^2 [b^2 + c^2 - a^2]}{4 \sum a \cdot S} \quad (\text{By Extended AM-GM}) \\ \sum a^2 (b^2 + c^2 - a) &= (a + b - c)(b + c - a)(a + c - b) \cdot \sum a \\ \Rightarrow LHS &\leq \frac{\pi(a+b-c)}{4S} = \frac{S}{a+b+c} = 2\pi \quad (\text{Shown}) \end{aligned}$$

Solution 2 by Kevin Soto Palacios-Huarmey-Peru

Siendo ABC un triángulo acutángulo. Probar que

$$(a \cot A)^a (b \cot B)^b (c \cot C)^c \leq (2r)^{a+b+c}$$

Como ABC es un triángulo acutángulo $\rightarrow \cot A, \cot B, \cot C > 0$

Tener en cuenta lo siguiente en un triángulo ABC

$$\begin{aligned} \sin 2A + \sin 2B + \sin 2C &= 4 \sin A \sin B \sin C = \frac{4abc}{8R^3} = \frac{16pRr}{8R^3} = \frac{2pr}{R^2} \\ \Leftrightarrow \frac{R^2(\sin 2A + \sin 2B + \sin 2C)}{2p} &= r \end{aligned}$$

Aplicando la desigualdad ponderada $MA \geq MG$

$$\begin{aligned} \frac{(a \cot A)a + (b \cot B)b + c(\cot C)c}{a + b + c} &\geq \frac{a+b+c}{\sqrt{(a \cot A)^a (b \cot B)^b (c \cot C)^c}} \\ \Leftrightarrow \frac{2R^2(\sin 2A + \sin 2B + \sin 2C)}{2p} &\geq \frac{a+b+c}{\sqrt{(a \cot A)^a (b \cot B)^b (c \cot C)^c}} \\ \Leftrightarrow (2r)^{a+b+c} &\geq (a \cot A)^a (b \cot B)^b (c \cot C)^c \quad (\text{LQOD}) \end{aligned}$$

Solution 3 by Ravi Prakash-New Delhi-India

$$(a \cot A)^a (b \cot B)^b (c \cot C)^c \leq \left[\frac{a(a \cot A) + b(b \cot B) + c(\cot C)c}{a + b + c} \right]^{a+b+c}$$

But $a(a \cot A) + b(b \cot B) + c(\cot C)c$

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$$\begin{aligned}
 &= 2R^2(2 \sin A \cos A + 2 \sin B \cos B + 2 \sin C \cos C) \\
 &= 2R^2(\sin 2A + \sin 2B + \sin 2C) = 2R^2(4 \sin A \sin B \sin C) \\
 &= 8R^2 \left(\frac{a}{2R}\right) \left(\frac{b}{2R}\right) \left(\frac{c}{2R}\right) = \frac{abc}{R} = 4S \quad [S = \text{area of } \Delta ABC] \\
 \therefore (a \cot A)^a (b \cot B)^b (c \cot C)^c &\leq \left(\frac{4S}{2s}\right)^{a+b+c} = (2r)^{a+b+c}
 \end{aligned}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned}
 \prod_{cyc} (a \cot A)^a &\stackrel{AM \geq GM}{\geq} \left(\frac{a^2 \cot A + b^2 \cot B + c^2 \cot C}{a+b+c}\right)^{a+b+c} \\
 &= \left(\frac{\frac{R}{abc} \sum_{cyc} a^2 (b^2 + c^2 - a^2)}{a+b+c}\right)^{a+b+c} = \left(\frac{R}{abc(a+b+c)} \sum_{cyc} (b^2 c^2 + c^2 a^2 - a^4)\right)^{a+b+c} \\
 &= \left(\frac{R}{abc(a+b+c)} \left(2 \sum_{cyc} a^2 b^2 - \sum_{cyc} a^4\right)\right)^{a+b+c} = \left(\frac{R}{abc} \prod_{cyc} (a+b-c)\right)^{a+b+c} \\
 &= \left(\frac{R}{abc} \cdot 8 \prod_{cyc} (p-a)\right)^{a+b+c} = \left(\frac{8R}{4Rrp} \cdot pr^2\right)^{a+b+c} = (2r)^{a+b+c}
 \end{aligned}$$

JP.077. Let a_1, a_2, \dots, a_9 be non-negative real numbers such that

$a_1 + a_2 + \dots + a_9 = 1$. Prove that for all $\lambda \geq 4$, the following inequality holds

$$\sqrt{\sum_{1 \leq i \leq 9} a_i^2} + \lambda \sqrt{\sum_{1 \leq i < j \leq 9} a_i a_j} \leq \frac{2\lambda + 1}{3}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo $a_1, a_2, a_3 \dots a_9$ números reales no negativos de tal manera que

$$a_1 + a_2 + a_3 + \dots + a_9 = 1.$$

Probar que para todo $\lambda \geq 4$, la siguiente desigualdad

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$$\sqrt{\sum_{1 \leq i \leq 9} a_i^2} + \lambda \sqrt{\sum_{1 \leq i < j \leq 9} a_i a_j} \leq \frac{2\lambda + 1}{3}$$

Utilizando la desigualdad ponderada Jensen para una función concava

$$f(x) = \sqrt{x}, x \geq 0$$

$$\begin{aligned} & \sqrt{\left(\sum_{1 \leq i \leq 9} a_i^2\right) \cdot \frac{1}{9} + 6\lambda \left(\sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{1}{36}} \leq \\ & \leq (3 + 6\lambda) \sqrt{\left(\left(\sum_{1 \leq i \leq 9} a_i^2\right) \cdot \frac{3}{9} + \left(\sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{6\lambda}{36}\right) \cdot \frac{1}{3 + 6\lambda}} = \\ & = (3 + 6\lambda) \sqrt{\left(\left(\left(\sum_{1 \leq i \leq 9} a_i\right)^2 - 2 \sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{1}{3} + \left(\sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{\lambda}{6}\right) \cdot \frac{1}{3 + 6\lambda}} = \\ & = (3 + 6\lambda) \sqrt{\left(\left(1 - 2 \sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{1}{3} + \left(\sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{\lambda}{6}\right) \cdot \frac{1}{3 + 6\lambda}} = \\ & = (3 + 6\lambda) \sqrt{\left(\left(2 - 4 \sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{1}{6} + \left(\sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{\lambda}{6}\right) \cdot \frac{1}{3 + 6\lambda}} \end{aligned}$$

A lo cual es equivalente

$$\Rightarrow (3 + 6\lambda) \sqrt{\left(2 + (\lambda - 4) \sum_{1 \leq i < j \leq 9} a_i a_j\right) \cdot \frac{1}{6(3 + 6\lambda)}}$$

Aplicando Maclaurin's inequality

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$$\sum_{1 \leq i \leq 9} a_i \geq 9 \sqrt{\left(\sum_{1 \leq i < j \leq 9} a_i a_j \right) \cdot \frac{1}{36}} \Leftrightarrow \frac{1}{9} \geq \sqrt{\left(\sum_{1 \leq i < j \leq 9} a_i a_j \right) \cdot \frac{1}{36}} \Leftrightarrow$$

$$\Leftrightarrow \sum_{1 \leq i < j \leq 9} a_i a_j \leq \frac{36}{81} = \frac{4}{9}$$

Como $\rightarrow \lambda - 4 \geq 0$

$$(3 + 6\lambda) \sqrt{\left(2 + (\lambda - 4) \sum_{1 \leq i < j \leq 9} a_i a_j \right) \cdot \frac{1}{6(3 + 6\lambda)}} \leq (3 + 6\lambda) \sqrt{\left(2 + (\lambda - 4) \cdot \frac{4}{9} \right) \cdot \frac{1}{6(3 + 6\lambda)}} =$$

$$= (3 + 6\lambda) \sqrt{\frac{2(1+2\lambda)}{6 \cdot 9 \cdot 3(1+2\lambda)}} (3 + 6\lambda) \sqrt{\frac{1}{81}} = \frac{3+6\lambda}{9} = \frac{2\lambda+1}{3} \text{ (LOQD)}$$

La igualdad se alcanza cuando: $a_1 = a_2 = a_3 = \dots = a_9 = \frac{1}{9}$

Solution 2 by Ravi Prakash-New Delhi-India

$$\sum_{i=1}^a a_i = 1, a_i \geq 0 \Rightarrow \sum_{i=1}^a a_i^2 + 2 \sum_{i < j} a_i a_j = 1$$

Put

$$\sum_{i=1}^a a_i^2 = \sin^2 \theta, 2 \sum_{i < j} a_i a_j = \cos^2 \theta$$

$(0 \leq \theta \leq \frac{\pi}{2})$. Also,

$$\frac{1}{9} \sum_{i=1}^a a_i^2 \geq \left(\frac{1}{9} \sum_{i=1}^a a_i \right)^2 = \frac{1}{81} \Rightarrow \sum_{i=1}^a a_i^2 \geq \frac{1}{9}$$

$$\Rightarrow \frac{1}{9} \leq \sin^2 \theta \leq 1 \Rightarrow \frac{1}{3} \leq \sin \theta \leq 1. \text{ Now,}$$

$$\sqrt{\sum_{i=1}^a a_i^2} + \lambda \sqrt{\sum_{i < j} a_i a_j} = \sin \theta + \frac{\lambda}{\sqrt{2}} \cos \theta = f(\theta) \text{ (say)}$$

For $\lambda = u, f(\theta) \leq \sqrt{1+8} = 3 = \frac{2\lambda+1}{3}$. So assume $\lambda > 4$

$$f'(\theta) = \cos \theta - \frac{\lambda}{\sqrt{2}} \sin \theta$$

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$$f'(\theta) \geq 0 \Leftrightarrow \cos \theta \geq \frac{\lambda}{\sqrt{2}} \sin \theta \Leftrightarrow 1 - \sin^2 \theta \geq \frac{\lambda^2}{2} \sin^2 \theta \Leftrightarrow 1 \geq \left(\frac{\lambda^2}{2} + 1 \right) \sin^2 \theta$$

$$\Leftrightarrow \sin^2 \theta \leq \frac{2}{\lambda^2 + 2} < \frac{1}{9}. \text{ Not possible as } \sin^2 \theta \geq \frac{1}{9} \therefore f'(\theta) < 0$$

$$\Rightarrow f(\theta) \text{ decreases in } \left[\sin^{-1} \left(\frac{1}{3} \right), \frac{\pi}{2} \right] \Rightarrow \max f(\theta) = f \left(\sin^{-1} \left(\frac{1}{3} \right) \right)$$

$$= \frac{1}{3} + \frac{\lambda}{\sqrt{2}} \cdot \frac{2\sqrt{2}}{3} \Rightarrow f(\theta) \leq \frac{2\lambda + 1}{3}$$

Solution 3 by Anas Adlany-El Zemamra-Morocco

First, we notice that the given condition can be reduced by squaring into

$$\sum_{i=1}^9 a_i^2 + 2 \sum_{1 \leq i < j \leq 9} a_i a_j = 1$$

or

$$\sum_{1 \leq i < j \leq 9} a_i a_j = \frac{1 - \sum_{i=1}^9 a_i^2}{2}$$

and by setting $t = \sum_{i=1}^9 a_i^2$ the inequality can be rewritten as follows

$$\sqrt{t} + \lambda \sqrt{\frac{1-t}{2}} \leq \frac{2\lambda+1}{3}; \text{ with } t \in \left[\frac{1}{9}, 1 \right) \text{ because of } \forall i, a_i \leq 1 \text{ and } \sum_{i=1}^9 a_i^2 \geq \frac{(\sum_{i=1}^9 a_i)^2}{9} = \frac{1}{9}.$$

Now rearranging the inequality as write it as

$$\sqrt{t} - \frac{1}{3} - \frac{\lambda}{2} \left(\frac{t - \frac{1}{9}}{\sqrt{\frac{1-t}{2}} + \frac{2}{3}} \right) \leq 0 \Leftrightarrow \left(\sqrt{t} - \frac{1}{3} \right) \left(1 - \frac{\lambda}{2} \left(\frac{\sqrt{t} + \frac{1}{3}}{\sqrt{\frac{1-t}{2}} + \frac{2}{3}} \right) \right) \leq 0$$

hence, to finish the proof it suffices to show that

$$1 - \frac{\lambda}{2} \left(\frac{\sqrt{t} + \frac{1}{3}}{\sqrt{\frac{1-t}{2}} + \frac{2}{3}} \right) \leq 0 \Leftrightarrow 4 \left(\sqrt{\frac{1-t}{8}} + \frac{1}{3} \right) \leq \lambda \left(\sqrt{t} + \frac{1}{3} \right)$$

which is obviously true since $\lambda \geq 4$ and the range of t that gives

$$\frac{1-t}{8} \leq t \Leftrightarrow t \geq \frac{1}{9}; \text{ and we are done!}$$

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JP.078. Let a, b, c be positive real numbers such that $a^2 = b^2 + c^2$. Prove that:

$$ab + bc + ca + (\sqrt{2} - 1) \frac{abc}{a + b + c} \leq 2a^2$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam

If a, b, c be positive real numbers such that $a^2 = b^2 + c^2$. Prove that

$$ab + bc + ca + (\sqrt{2} - 1) \cdot \frac{abc}{a+b+c} \leq 2a^2 \quad (1)$$

We have

$$(1) \Rightarrow \frac{ab+bc+ca+(\sqrt{2}-1)\frac{abc}{a+b+c}}{a^2} \leq 2 \Rightarrow \frac{b}{a} + \frac{c}{a} + \frac{b}{a} \cdot \frac{c}{a} + (\sqrt{2} - 1) \cdot \frac{\frac{b}{a} \cdot \frac{c}{a}}{1+\frac{b}{a}+\frac{c}{a}} \leq 2 \quad (2)$$

$$\text{Put } \frac{b}{a} + \frac{c}{a} = x, \frac{b}{a} \cdot \frac{c}{a} = y. \text{ We have } a^2 = b^2 + c^2 \Rightarrow x^2 - 2y = 1 \Rightarrow y = \frac{x^2-1}{2}$$

$$\text{We have } (2) \Rightarrow x + y + (\sqrt{2} - 1) \cdot \frac{y}{1+x} \leq 2 \Rightarrow x + \frac{x^2-1}{2} + (\sqrt{2} - 1) \cdot \frac{2}{1+x} \leq 2 \Rightarrow$$

$$\Rightarrow \frac{x^2 + 2x - 1}{2} + \frac{\sqrt{2} - 1}{2} (x - 1) \leq 2 \Rightarrow \frac{x^2 + (1 + \sqrt{2})x - \sqrt{2}}{2} \leq 4$$

$$\Rightarrow x^2 + (1 + \sqrt{2})x - \sqrt{2} \leq 4 \Rightarrow x^2 + (1 + \sqrt{2})x - (4 + \sqrt{2}) \leq 0 \Rightarrow$$

$$\Rightarrow (x - \sqrt{2})[x + (1 + 2\sqrt{2})] \leq 0 \quad (3)$$

$$\text{By BCS, we have } a^2 = b^2 + c^2 \geq \frac{(b+c)^2}{2} \Rightarrow \frac{b+c}{a} \leq \sqrt{2} \Rightarrow x \leq \sqrt{2} \Rightarrow (3) \text{ true} \Rightarrow \text{QED}$$

$$\text{The equality occurs when } b = c = \frac{a\sqrt{2}}{2}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{Let } b = a \cos \theta, c = a \sin \theta$$

$$\left(0 < \theta < \frac{\pi}{2}\right)$$

$$\text{Now, } ab + bc + ca + (\sqrt{2} - 1) \frac{abc}{a+b+c} \leq 2a^2$$

$$\Leftrightarrow \cos \theta + \sin \theta + \cos \theta \sin \theta + \frac{(\sqrt{2} - 1) \cos \theta \sin \theta}{1 + \cos \theta + \sin \theta} \leq 2$$

LHS

$$E = \cos \theta + \sin \theta + \cos \theta \sin \theta + \frac{(\sqrt{2} - 1)(\cos \theta \sin \theta)}{1 + \cos \theta + \sin \theta}$$

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$$\begin{aligned}
 &= (\cos \theta + \sin \theta) + \frac{(\cos \theta + \sin \theta)^2 - 1}{2} + \frac{(\sqrt{2} - 1) \{(\cos \theta + \sin \theta)^2 - 1\}}{2 \cos \theta + \sin \theta + 1} \\
 &= t + \frac{1}{2}(t^2 - 1) + \frac{1}{2}(\sqrt{2} - 1)(t - 1) \text{ where } t = \cos \theta + \sin \theta \leq \sqrt{2} \\
 &\therefore E \leq \sqrt{2} + \frac{1}{2}(2 - 1) + \frac{1}{2}(\sqrt{2} - 1)(\sqrt{2} - 1) = 2
 \end{aligned}$$

Solution 3 by Anas Adlany-El Zemamra-Morocco

We have $ab + bc + ca + \frac{(\sqrt{2}-1)abc}{a+b+c} \leq 2a^2 \Leftrightarrow (\sqrt{2} - 1)abc \leq a^3 + b^3 + c^3 - 3abc$

$\Leftrightarrow (\sqrt{2} + 2)abc \leq a^3 + b^3 + c^3$. By the AM-GM inequality, we have

$2abc \leq a(b^2 + c^2) = a^3$. By Cauchy's inequality, we have $b^3 + c^3 \geq \frac{a^4}{b+c}$

Hence we just still have to prove that $\sqrt{2abc} \leq \frac{a^4}{b+c} \Leftrightarrow a^3 \geq \sqrt{2}bc(b+c)$

But $2bc \leq a^2$, hence it suffices to check that

$a^3 \geq \frac{\sqrt{2}}{2}a^2(b+c) \Leftrightarrow \sqrt{2}a \geq b+c \Leftrightarrow (b-c)^2 \geq 0$. Which is true. Therefore, as a

result we have proved that $(\sqrt{2} + 2)abc \leq a^3 + b^3 + c^3$, and we are done.

JP.079. Prove the inequality holds for all positive real numbers a, b, c

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{4}{2a+2b+3c} + \frac{4}{2b+2c+3a} + \frac{4}{2c+2a+3b}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar para todo los números R^+ la siguiente desigualdad

$$\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{4}{3a+3b+2c} + \frac{4}{3b+3c+2a} + \frac{4}{3c+3a+2b}$$

Como $a, b, c > 0$. Aplicando la desigualdad de Cauchy

$$\frac{1}{a+b} + \frac{1}{c+a} \geq \frac{4}{b+c+2a} \quad (A),$$

$$\frac{1}{b+c} + \frac{1}{a+b} \geq \frac{4}{c+a+2b} \quad (B),$$

$$\frac{1}{c+a} + \frac{1}{b+c} \geq \frac{4}{a+b+2c} \quad (C)$$

Sumando (A)+ (B)+ (C)

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$$\begin{aligned}
 &\Rightarrow \frac{2}{a+b} + \frac{2}{b+c} + \frac{2}{c+a} \geq \frac{4}{a+b+2c} + \frac{4}{b+c+2a} + \frac{4}{c+a+2b} \\
 &\Rightarrow \frac{1}{2(a+b)} + \frac{1}{2(b+c)} + \frac{1}{2(c+a)} \geq \frac{1}{a+b+2c} + \frac{1}{b+c+2a} + \frac{1}{c+a+2b} \\
 &\qquad\qquad\qquad \Rightarrow \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \\
 &\geq \left(\frac{1}{a+b+2c} + \frac{1}{2(a+b)} \right) + \left(\frac{1}{b+c+2a} + \frac{1}{2(b+c)} \right) + \left(\frac{1}{c+a+2b} + \frac{1}{2(c+a)} \right) \\
 &\geq \frac{4}{a+b+2c+2(a+b)} + \frac{4}{b+c+2a+2(b+c)} + \frac{4}{c+a+2b+2(c+a)} = \\
 &= \frac{4}{3a+3b+2c} + \frac{4}{3b+3c+2a} + \frac{4}{3c+3a+2b}
 \end{aligned}$$

Solution 2 by Nguyen Ngoc Tu-Ha Giang-Vietnam

We have: $\frac{3}{3a+3b+2c} = \frac{4}{2(a+b)+(a+c)+(b+c)} \leq \frac{4}{16} \left(\frac{2}{a+b} + \frac{1}{a+c} + \frac{1}{b+c} \right) = \frac{1}{4} \left(\frac{2}{a+b} + \frac{1}{a+c} + \frac{1}{b+c} \right)$

Similar $\frac{4}{3b+3c+2a} \leq \frac{1}{4} \left(\frac{2}{b+c} + \frac{1}{a+b} + \frac{1}{a+c} \right)$, $\frac{4}{3c+3a+2b} \leq \frac{1}{4} \left(\frac{2}{a+c} + \frac{1}{a+b} + \frac{1}{b+c} \right)$

Hence $\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{4}{3a+3b+2c} + \frac{4}{3b+3c+2a} + \frac{4}{3c+3a+2b}$

Solution 3 by Ravi Prakash-New Delhi-India

$$\frac{1}{a+b} + \frac{2}{b+c} + \frac{1}{c+a} \geq \frac{16}{a+b+2b+2c+c+a}$$

$$\Rightarrow \frac{1}{a+b} + \frac{2}{b+c} + \frac{1}{c+a} \geq \frac{16}{2a+3b+3c}. \text{ Similarly, } \frac{2}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{16}{3a+3b+2c}$$

and $\frac{1}{a+b} + \frac{1}{b+c} + \frac{2}{c+a} \geq \frac{16}{3a+2b+3c}$. *Adding, we get* $4 \left(\frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \right) \geq$

$$\geq 16 \left(\frac{1}{2a+3b+3c} + \frac{1}{3a+3b+2c} + \frac{1}{3a+2b+3c} \right)$$

$$\Rightarrow \frac{1}{a+b} + \frac{1}{b+c} + \frac{1}{c+a} \geq \frac{4}{2a+3b+3c} + \frac{4}{3a+3b+2c} + \frac{4}{3a+2b+3c}$$

Solution 4 by Anas Adlany-El Zemamra-Morocco

Since the inequality is homogenous, we could suppose WLOG of the problem that

$a + b + c = 1$ and the inequality to prove can be reduce into

$$\sum \frac{3a-1}{(a-1)(a-3)} \geq 0. \text{ For that, let } f \text{ be, defined on } (0, 1) \text{ by:}$$

$$f(x) = \frac{3x-1}{(x-1)(x-3)}; \text{ by differentiating we get } f'(x) = -\frac{3x^2-2x-5}{((x-1)(x-3))^2} \text{ on other time}$$

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$f''(x) = \frac{2(3x^3 - 3x^2 - 15x + 23)}{(x-1)(x-3)^3} > 0$ hence f is convex, then by Jensen's inequality we have

$$\sum \frac{3a-1}{(a-1)(a-3)} = \sum f(a) \geq 3f\left(\frac{1}{3}\right) = 0 \text{ and we are done!}$$

JP.080. Prove that in any triangle ABC :

$$\frac{a^2 + b^2 + c^2}{a + b + c} \left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \right) \geq 2\sqrt{3}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Kevin Soto Palacios-Huarmey-Peru

Tener presente en un triángulo ABC , lo siguiente:

$$m_a^2 = \frac{2b^2 + 2c^2 - a^2}{4} \Leftrightarrow m_a^2 + \frac{3a^2}{4} = \frac{2b^2 + 2c^2 + 2a^2}{4}. \text{ Por } MA \geq MG$$

$$\frac{2b^2 + 2c^2 + 2a^2}{4} = m_a^2 + \frac{3a^2}{4} \geq \frac{2\sqrt{3}m_a a}{2} \Leftrightarrow a^2 + b^2 + c^2 \geq 2\sqrt{3}m_a a$$

$$\text{Invirtiendo tenemos: } \frac{2\sqrt{3}a}{a^2 + b^2 + c^2} \leq \frac{1}{m_a} \dots (A)$$

$$\text{por lo tanto: } \frac{2\sqrt{3}b}{a^2 + b^2 + c^2} \leq \frac{1}{m_b} \dots (B) \text{ y } \frac{2\sqrt{3}c}{a^2 + b^2 + c^2} \leq \frac{1}{m_c} \dots (C)$$

Sumando: (A) + (B) + (C)

$$\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \geq \frac{2\sqrt{3}(a+b+c)}{a^2 + b^2 + c^2} \Leftrightarrow \left(\frac{a^2 + b^2 + c^2}{a+b+c} \right) \left(\frac{1}{m_a} + \frac{1}{m_b} + \frac{1}{m_c} \right) \geq 2\sqrt{3} \text{ (LQOD)}$$

JP.081. If $x, y, z > 0$ then:

$$\sqrt{\frac{13x}{6x + 7y}} + \sqrt{\frac{13y}{6y + 7z}} + \sqrt{\frac{13z}{6z + 7x}} \leq 3$$

Proposed by Marin Chirciu-Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

$$\text{Siendo } x, y, z > 0. \text{ Probar que: } A = \sqrt{\frac{13x}{6x+7y}} + \sqrt{\frac{13y}{6y+7z}} + \sqrt{\frac{13z}{6z+7x}} \leq 3. \text{ Aplicando la}$$

desigualdad de Cauchy

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$$\begin{aligned}
 A &\leq \sqrt{\left(\frac{13}{(6x+7y)(6y+7z)} + \frac{13}{(6y+7z)(6z+7x)} + \frac{13}{(6z+7x)(6x+7y)}\right) (x(6y+7z) + y(6z+7x) + z(6x+7y))} \\
 &\Leftrightarrow A \leq \sqrt{\left(\frac{13(6y+7z) + 13(6x+7y) + 13(6z+7x)}{(6x+7y)(6y+7z)(6z+7x)}\right) \cdot (13xy + 13yz + 13zx)} \\
 &\Leftrightarrow A \leq \sqrt{\frac{13(13x + 13y + 13z)}{(6x+7y)(6y+7z)(6z+7x)} \cdot (13xy + 13yz + 13zx)} = \\
 &= \sqrt{\frac{13^3(x+y+z)(xy+yz+zx)}{(6x+7y)(6y+7z)(6z+7x)}}. \text{ Es suficiente probar}
 \end{aligned}$$

$13^3(x+y+z)(xy+yz+zx) \leq 9(6x+7y)(6y+7z)(6z+7x)$. Ahora bien

$$\begin{aligned}
 (x+y+z)(xy+yz+zx) &= xy(x+y) + yz(y+z) + zx(z+x) + 3xyz \\
 \Rightarrow (x+y+z)(xy+yz+zx) &= x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 3xyz \\
 &= (6x+7y)(6y+7z)(6z+7x) = \\
 &= 6x \cdot 7y(6x+7y) + 6y \cdot 7z(6y+7z) + 6z \cdot 7x(6z+7x) + (6^3 + 7^3)xyz \\
 &\Rightarrow (6x+7y)(6y+7z)(6z+7x) = \\
 &= 252(x^2y + y^2z + z^2x) + 294(y^2x + z^2y + x^2z) + 559xyz. \text{ Por lo tanto} \\
 13^3(x^2y + y^2z + z^2x + y^2x + z^2y + x^2z + 3xyz) &\leq \\
 &\leq 9(252(x^2y + y^2z + z^2x) + 294(y^2x + z^2y + x^2z) + 559xyz) \\
 \Leftrightarrow 2197(x^2y + y^2z + z^2x) + 2197(y^2x + z^2y + x^2z) + 6591xyz &\leq \\
 &\leq 2268(x^2y + y^2z + z^2x) + 2646(y^2x + z^2y + x^2z) + 5031xyz \\
 \Leftrightarrow 1560xyz &\leq 71(x^2y + y^2z + z^2x) + 449(y^2x + z^2y + x^2z)
 \end{aligned}$$

Aplicando $MA \geq MG$

$$71(x^2y + y^2z + z^2x) + 449(y^2x + z^2y + x^2z) \geq 213xyz + 1347xyz = 1560xyz$$

(LQOD). Finalmente, concluimos que

$$A = \sqrt{\frac{13x}{6x+7y}} + \sqrt{\frac{13y}{6y+7z}} + \sqrt{\frac{13z}{6z+7x}} \leq \sqrt{\frac{13^3(x+y+z)(xy+yz+zx)}{(6x+7y)(6y+7z)(6z+7x)}} \leq \sqrt{9} = 3$$

Solution 2 by Anas Adlany-El Zemamra-Morocco

More generally, we prove that $\sum \sqrt{\frac{x}{mx+ny}} \leq \frac{3}{\sqrt{m+n}}$, where $(m, n) \in (\mathbb{N}^*)^2$.

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$$\begin{aligned} \sum \sqrt{\frac{x}{mx+ny}} &= \frac{\sum \sqrt{x(my+nz)(mz+nx)}}{\sqrt{\prod(mx+ny)}} \leq \sqrt{\frac{(\sum x(my+nz))(\sum mz+nx)}{\prod(mx+ny)}} \\ &= (m+n) \cdot \frac{\sqrt{(\sum xy)(\sum x)}}{\sqrt{\prod(mx+ny)}} = (m+n) \cdot \sqrt{\frac{xyz + \prod(x+y)}{\prod(mx+ny)}} \\ &= \frac{1}{\sqrt{m+n}} \cdot \sqrt{8\left(1 + \frac{xyz}{\prod(x+y)}\right)} \leq \frac{1}{\sqrt{m+n}} \cdot \sqrt{8\left(1 + \frac{1}{8}\right)} = \frac{3}{\sqrt{m+n}} \end{aligned}$$

where the last step come from Chebyshev's inequality, and we are done!

JP.082. If $a, b, c > 0; a + b + c = 3$ then:

$$\frac{a}{1+3b^4} + \frac{b}{1+3c^4} + \frac{c}{1+3a^4} \geq \frac{3}{4}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo $a, b, c > 0$ de tal manera que $a + b + c = 3$. Probar que

$$\begin{aligned} \frac{a}{1+3b^4} + \frac{b}{1+3c^4} + \frac{c}{1+3a^4} &\geq \frac{3}{4} \\ \left(\frac{a}{1+3b^4} - a\right) + \left(\frac{b}{1+3c^4} - b\right) + \left(\frac{c}{1+3a^4} - c\right) &\geq \frac{3}{4} - 3 \\ \Leftrightarrow -\frac{3ab^4}{1+3b^4} - \frac{3bc^4}{1+3c^4} - \frac{3a^4c}{1+3a^4} &\geq -\frac{9}{4} \Leftrightarrow \frac{3ab^4}{1+3b^4} + \frac{3bc^4}{1+3c^4} + \frac{3a^4c}{1+3a^4} \leq \frac{9}{4} \end{aligned}$$

Como $a, b, c > 0$. Aplicando $MA \geq MG$

$3a^4 + 1 = a^4 + a^4 + a^4 + 1 \geq 4a^3, 3b^4 + 1 \geq 4b^3, 3c^4 + 1 \geq 4c^3$. Por lo tanto

$$\begin{aligned} \Rightarrow \frac{3ab^4}{1+3b^4} + \frac{3bc^4}{1+3c^4} + \frac{3a^4c}{1+3a^4} &\leq \frac{3ab^4}{4b^3} + \frac{3bc^4}{4c^3} + \frac{3a^4c}{4a^3} = \frac{3}{4}(ab + bc + ca) \leq \\ &\leq \frac{3}{4} \cdot \frac{(a+b+c)^2}{3} = \frac{3}{4} \cdot 3 = \frac{9}{4} \end{aligned}$$

Solution 2 by Nguyen Minh Tri-Ho Chi Minh-Vietnam

If $a, b, c > 0, a + b + c = 3$ then

$$\frac{a}{1+3b^4} + \frac{b}{1+3c^4} + \frac{c}{1+3a^4} \geq \frac{3}{4} \Rightarrow ab + bc + ca \leq 3$$

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$$\begin{aligned} \Leftrightarrow & \frac{a(1+3b^4)-3ab^4}{1+3b^4} + \frac{b(1+3c^4)-3bc^4}{1+3c^4} + \frac{c(1+3a^4)-3ca^4}{1+3a^4} \geq \frac{3}{4} \\ \Leftrightarrow & a - \frac{3ab^4}{1+3b^4} + b - \frac{3bc^4}{1+3c^4} + c - \frac{3ca^4}{1+3a^4} \geq \frac{3}{4} \\ \Leftrightarrow & (a+b+c) - \frac{3ab^4}{1+3b^4} - \frac{3bc^4}{1+3c^4} - \frac{3ca^4}{1+3a^4} \geq \frac{3}{4} \\ \Leftrightarrow & 3 - \frac{3ab^4}{1+3b^4} - \frac{3bc^4}{1+3c^4} - \frac{3ca^4}{1+3a^4} \geq \frac{3}{4} \quad (1) \end{aligned}$$

Use Cauchy for 4 numbers, we have: $3b^4 + 1 = b^4 + b^4 + b^4 + 1 \geq 4\sqrt[4]{b^{12}} = 4b^3$

$$\begin{aligned} \Leftrightarrow & -\frac{3ab^4}{1+3b^4} \geq -\frac{3ab^4}{4b^3} = \frac{-3ab}{4}. \text{ Similarly } -\frac{3bc^4}{1+3c^4} \geq \frac{-3bc}{4}, \frac{-3ca^4}{1+3a^4} \geq \frac{-3ac}{4} \\ \Rightarrow & -\frac{3ab^4}{1+3b^4} - \frac{3bc^4}{1+3c^4} - \frac{3ca^4}{1+3a^4} \geq -\frac{3}{4}(ab+bc+ca) \text{ but } ab+bc+ca \leq 3 \\ \Rightarrow & -\frac{3ab^4}{1+3b^4} - \frac{3bc^4}{1+3c^4} - \frac{3ca^4}{1+3a^4} \geq -\frac{3}{4} \cdot 3 = -\frac{9}{4} \\ \Rightarrow & 3 - \frac{3ab^4}{1+3b^4} - \frac{3bc^4}{1+3c^4} - \frac{3ca^4}{1+3a^4} \geq \frac{3}{4} \Rightarrow (1) \text{ true} \Rightarrow \text{Q.E.D.} \end{aligned}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

Applying Tangent – Line Method,

Let $f(x) = \frac{1}{1+3x^4}$ for all $x > 0$, so the tangent to the curve $f(x)$ is

$$y = f(1) + f'(1)(x-1) = \frac{4-3x}{4}. \text{ So, we can write, } \frac{1}{1+3x^4} \geq \frac{4-3x}{4}$$

$$\Leftrightarrow (x-1)^2(3x^2+2x+1) \geq 0 \text{ for all } x > 0$$

$$\therefore \sum_{cyc} \frac{a}{1+3b^4} \geq \sum_{cyc} \frac{a(4-3b)}{4} = \sum_{cyc} a - \frac{3}{4} \sum_{cyc} ab \geq \sum_{cyc} a - \frac{1}{4} (\sum_{cyc} a)^2 = \frac{3}{4} \text{ (Proved)}$$

Solution 4 by Kunihiko Chikaya-Tokyo-Japan

$$\frac{a}{1+3b^4} + \frac{b}{1+3c^4} + \frac{c}{1+3a^4} \geq \frac{3}{4}; a+b+c=3, a, b, c \in \mathbb{R}^+.$$

$$\frac{1}{1+3b^4} \geq -\frac{3}{4} + 1 \Leftrightarrow 3b(3b^2+2b+1)(b-1)^2 \geq 0$$

$$\frac{a}{1+3b^4} \geq a \left(-\frac{3}{4}b + 1 \right)$$

$$\frac{b}{1+3c^4} \geq b \left(-\frac{3}{4}c + 1 \right)$$

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$$\frac{c}{1+3a^4} \geq c \left(-\frac{3}{4}c + 1 \right)$$

$$\therefore \text{LHS of (*)} = a + b + c - \frac{3}{4}(ab + bc + ca) \geq a + b + c - \frac{1}{4}(a + b + c)^2 = \frac{3}{4}$$

$$\text{Equality: } a = b = c = 1$$

JP.083. In ΔABC then following relationship holds:

$$\begin{aligned} (a^{2m} + b^{2m} + c^{2m}) \left(\frac{1}{(a+b)^{2n}} + \frac{1}{(b+c)^{2n}} + \frac{1}{(c+a)^{2n}} \right) &\geq \\ &\geq 3^{m-n+2} \cdot 4^{m-2n} \cdot r^{2(m-n)}; m, n \geq 1 \end{aligned}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu - Romania

Solution by SK Rejuan-West Bengal-India

For ΔABC : With the help & m -th power theorem we get,

$$\frac{\sum a^{2m}}{3} \geq \left(\frac{\sum a^2}{3} \right)^m \Rightarrow \sum a^{2m} \geq 3^{1-m} (\sum a^2)^m \quad (1)$$

By Cauchy inequality we get, $(a+b)^2 \leq 2(a^2+b^2) \Rightarrow (a+b)^{2n} \leq 2^n (a^2+b^2)^n$

$$\Rightarrow \sum \frac{1}{(a+b)^{2n}} \geq \frac{1}{2^n} \sum \frac{1}{(a^2+b^2)^n} \quad (2)$$

Again by m th power theorem we get, $\frac{\sum \left(\frac{1}{a^2+b^2} \right)^n}{3} \geq \left(\frac{\sum \left(\frac{1}{a^2+b^2} \right)}{3} \right)^n$

$$\Rightarrow \sum \left(\frac{1}{a^2+b^2} \right)^n \geq 3^{1-n} \left(\sum \frac{1}{a^2+b^2} \right)^n \Rightarrow \frac{1}{2^n} \sum \frac{1}{(a^2+b^2)^n} \geq \frac{3^{1-n}}{2^n} \left(\sum \frac{1}{a^2+b^2} \right)^n \quad (3)$$

Again by $AM \geq HM$ we get,

$$\sum \frac{1}{a^2+b^2} \geq \frac{9}{\sum (a^2+b^2)} = \frac{9}{2 \sum a^2} \Rightarrow \frac{3^{1-n}}{2^n} \times \left(\sum \frac{1}{a^2+b^2} \right)^n \geq \frac{3^{1-n}}{2^n} \left(\frac{9}{2 \sum a^2} \right)^n \quad (4)$$

Combining (2), (3), (4) we get

$$\sum \frac{1}{(a+b)^{2n}} \geq \frac{3^{1-n}}{2^n} \left(\frac{9}{2 \sum a^2} \right)^n \quad (5)$$

Multiplying (1) & (5) we get, $\sum a^{2m} \cdot \sum \frac{1}{(a+b)^{2n}} \geq 3^{1-m} (\sum a^2)^m \cdot \frac{3^{1-n}}{2^n} \left(\frac{9}{2 \sum a^2} \right)^n$

$$= 3^{2-m-n+2n} 2^{-2n} (\sum a^2)^{m-n} = 3^{n-m+2} \cdot 4^{-n} \cdot (\sum a^2)^{(m-n)} \quad (6)$$

Also we know that $\sum a^2 \geq 36r^2 \Rightarrow (\sum a^2)^{m-n} \geq (36)^{m-n} \cdot r^{2(m-n)}$

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$$\begin{aligned} \Rightarrow 3^{n-m+2} \cdot 4^{-n} \cdot \left(\sum a^2\right)^{m-n} &\geq 3^{n-m+2} \cdot 4^{-n} \cdot 36^{(m-n)} \cdot r^{2(m-n)} \\ &= 3^{(n-m+2+2m-2n)} \cdot 4^{(-n+m-n)} \cdot r^{2(m-n)} \\ &= 3^{(m-n+2)} \cdot 4^{(m-2n)} \cdot r^{2(m-n)} \quad (7) \end{aligned}$$

Combining (6) & (7) we get, $\sum a^{2m} \cdot \sum \frac{1}{(a+b)^{2n}} \geq 3^{(m-n+2)} \cdot 4^{(m-2n)} \cdot r^{2(m-n)}$
(proved)

JP.084. In ΔABC the following relationship holds:

$$\sum \left((a+b) \tan \frac{C}{2} \right)^m \cdot \sum \frac{1}{\left(\tan \frac{A}{2} + \tan \frac{B}{2} \right)^{2n}} \geq 3^{n+2} \cdot 4^{m-n} \cdot r^n; m \geq n \geq 1$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu - Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo $m \geq n \geq 1$. Probar en un triángulo la siguiente desigualdad

$$\begin{aligned} &\left((a+b) \tan \frac{C}{2} + (b+c) \tan \frac{A}{2} + (c+a) \tan \frac{C}{2} \right)^m \cdot \\ &\left(\frac{1}{\left(\tan \frac{A}{2} + \tan \frac{B}{2} \right)^{2n}} + \frac{1}{\left(\tan \frac{B}{2} + \tan \frac{C}{2} \right)^{2n}} + \frac{1}{\left(\tan \frac{C}{2} + \tan \frac{A}{2} \right)^{2n}} \right) \geq 3^{n+2} \cdot 4^{m-n} \cdot r^m \end{aligned}$$

Recordar las siguientes identidades y desigualdades en un ΔABC

$$\tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2} = \frac{r}{p}, R \geq 2r, abc = 4Rpr. \text{ Ahora bien, utilizando } MA \geq MG$$

$$\begin{aligned} (a+b) \tan \frac{C}{2} + (b+c) \tan \frac{A}{2} + (c+a) \tan \frac{C}{2} &\geq 3 \sqrt[3]{(a+b)(b+c)(c+a)} \cdot \frac{r}{p} \geq 3 \sqrt[3]{8abc} \cdot \frac{r}{p} = \\ &= 3 \sqrt[3]{32Rr^2} \geq 3 \sqrt[3]{64r^3} = 12r \end{aligned}$$

IRAN INEQUALITY

Siendo $x, y, z \geq 0$ se cumple la siguiente desigualdad

$$(xy + yz + zx) \left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \right) \geq \frac{9}{4} \quad (A)$$

Realizamos los siguientes cambios de variables

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$$x = \tan \frac{A}{2} > 0, y = \tan \frac{B}{2} > 0, z = \tan \frac{C}{2} > 0 \Leftrightarrow xy + yz + zx = 1$$

Por lo tanto tenemos en (A)

$$\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2} \geq \frac{9}{4}. \text{ De la desigualdad ponderada de Cauchy}$$

$$\left(\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}\right) \cdot 3^{n-1} \geq \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^n, \text{ donde } a, b, c > 0, n \geq 1$$

$$\text{Siendo } a = (x+y)^2, b = (y+z)^2, c = (z+x)^2 \Leftrightarrow x, y, z > 0$$

$$\begin{aligned} \Rightarrow \frac{1}{(x+y)^{2n}} + \frac{1}{(y+z)^{2n}} + \frac{1}{(z+x)^{2n}} &\geq \frac{3}{3^n} \left(\frac{1}{(x+y)^2} + \frac{1}{(y+z)^2} + \frac{1}{(z+x)^2}\right)^n \geq \\ &\geq \frac{3}{3^n} \cdot \frac{9^n}{4^n} = \frac{3^{n+1}}{4^n} \end{aligned}$$

Utilizando $\rightarrow m - n \geq 0 \wedge m \geq 1$. Se concluye que

$$LHS \geq (12r)^m \cdot \frac{3^{n+1}}{4^n} = 4^m \cdot 3^m \cdot r^m \cdot \frac{3^{n+1}}{4^n} = 4^{m-n} \cdot 3^{n+m+1} \cdot r^m \geq 4^{m-n} \cdot 3^{n+2} \cdot r^m$$

JP.085. Let ABC denote a triangle, I its incenter, R its circumradius, r its inradius and x, y and z the inradii of triangles $IBC, ICA,$ and IAB respectively. Prove that:

$$\frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z} \leq \frac{4 + 3\sqrt{3}}{2r} + \frac{2}{R}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

Lemma

Siendo I - Incentro. En un triángulo ABC , se cumple los siguiente

$$IA + IB + IC \leq 2(R + r)$$

Tener en cuenta las siguientes identidades y desigualdades en un ΔABC

$$IA = \frac{bc}{p} \cos \frac{A}{2} = \frac{bc}{p} \sqrt{\frac{p(p-a)}{bc}} = \sqrt{\frac{bc(p-a)}{p}},$$

$$IB = \sqrt{\frac{ca(p-b)}{p}}, IC = \sqrt{\frac{ab(p-c)}{p}}$$

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$$a + b + c = 2p, ab + bc + ca = p^2 + r^2 + 4Rr,$$

$$p^2 \leq 4R^2 + 4Rr + 3r^2 \quad (\text{Gerretsen Inequality})$$

Aplicando la desigualdad de Cauchy

$$\begin{aligned} IA + IB + IC &= \sqrt{\frac{bc(p-a)}{p}} + \sqrt{\frac{ca(p-b)}{p}} + \sqrt{\frac{ab(p-c)}{p}} \leq \\ &\leq \sqrt{\left(\frac{p-a}{p} + \frac{p-b}{p} + \frac{p-c}{p}\right)(bc + ca + ab)} = \sqrt{ab + bc + ca} = \\ &= \sqrt{p^2 + r^2 + 4Rr} \leq \sqrt{4R^2 + 4Rr + 3r^2 + r^2 + 4Rr} = \\ &= \sqrt{4(R^2 + 2Rr + r^2)} = \sqrt{4(R+r)^2} = 2(R+r) \end{aligned}$$

(LQOD)

Siendo ABC un triángulo, con I - Incentro, R - circunradio, r - inradio, además x, y, z son los inradios de triángulos IBC, ICA, IAB . Probar que

$$\frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z} \leq \frac{4 + 3\sqrt{3}}{2r} + \frac{2}{r}$$

Recordar las siguientes desigualdades conocidas en un triángulo ABC

$\frac{p}{R} \leq \frac{3\sqrt{3}}{2}$, $IA + IB + IC \leq 2(R+r)$. De las condiciones, se verifica lo siguiente

$$\frac{a}{x} = \frac{pa + ab \cos \frac{C}{2} + ca \cos \frac{B}{2}}{S} = \frac{pa + pIC + pIB}{S} = \frac{p(a + IC + IB)}{pr} = \frac{a + IC + IB}{r}$$

$$\frac{a}{x} = \frac{pa + ab \cos \frac{C}{2} + ca \cos \frac{B}{2}}{S} = \frac{pa + pIC + pIB}{S} = \frac{p(a + IC + IB)}{pr} = \frac{a + IC + IB}{r}$$

$$\frac{b}{y} = \frac{pb + bc \cos \frac{A}{2} + ab \cos \frac{C}{2}}{S} = \frac{pb + pIA + pIC}{S} = \frac{p(b + IA + IC)}{pr} = \frac{b + IA + IC}{r}$$

$$\frac{c}{z} = \frac{pc + ca \cos \frac{B}{2} + bc \cos \frac{A}{2}}{S} = \frac{pc + pIB + pIA}{S} = \frac{p(c + IB + IA)}{pr} = \frac{c + IB + IA}{r}$$

$$\text{Lo cual implica} \Rightarrow \frac{a}{x} + \frac{b}{y} + \frac{c}{z} = \frac{2p}{r} + \frac{2(IA+IB+IC)}{r} \Leftrightarrow$$

$$\Leftrightarrow \frac{\sin A}{x} + \frac{\sin B}{y} + \frac{\sin C}{z} = \frac{p}{Rr} + \frac{IA + IB + IC}{Rr} \leq \frac{3\sqrt{3}}{2r} + 2\left(\frac{1}{R} + \frac{1}{r}\right) = \frac{4 + 3\sqrt{3}}{2r} + \frac{2}{R}$$

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JP.086. Let a, b, c be the side lengths of a triangle ABC with incentre I , circumradius R and inradius r . Prove that

$$\frac{\sqrt{AI}}{a} + \frac{\sqrt{BI}}{b} + \frac{\sqrt{CI}}{c} \leq \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{R+r}}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Lemma

Siendo I - incentro. En un triángulo ABC , se cumple lo siguiente

$$IA + IB + IC \leq 2(R + r)$$

Solucion

Tener en cuenta las siguientes identidades y desigualdades en un ΔABC

$$IA = \frac{bc}{p} \cos \frac{A}{2} = \frac{bc}{p} \sqrt{\frac{p(p-a)}{bc}} = \sqrt{\frac{bc(p-a)}{p}}, IB = \sqrt{\frac{ca(p-b)}{p}}, IC = \sqrt{\frac{ab(p-c)}{p}}$$

$$a + b + c = 2p, ab + bc + ca = p^2 + r^2 + 4Rr, p^2 \leq 4R^2 + 4Rr + 3r^2 \text{ (Gerretsen$$

Inequality). Aplicando la desigualdad de Cauchy

$$\begin{aligned} IA + IB + IC &= \sqrt{\frac{bc(p-a)}{p}} + \sqrt{\frac{ca(p-b)}{p}} + \sqrt{\frac{ab(p-c)}{p}} \leq \\ &\leq \sqrt{\left(\frac{p-a}{p} + \frac{p-b}{p} + \frac{p-c}{p}\right)(bc + ca + ab)} = \sqrt{ab + bc + ca} = \\ &= \sqrt{p^2 + r^2 + 4Rr} \leq \sqrt{4R^2 + 4Rr + 3r^2 + r^2 + 4Rr} = \sqrt{4(R^2 + 2Rr + r^2)} = \\ &= \sqrt{4(R+r)^2} = 2(R+r) \text{ (LQOD)} \end{aligned}$$

Siendo a, b, c los lados de un triángulo ABC con Incentro I , circunradio R e inradio r .

$$\text{Probar que } \frac{\sqrt{AI}}{a} + \frac{\sqrt{BI}}{b} + \frac{\sqrt{CI}}{c} \leq \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{R+r}}{r}$$

Tener en cuenta las siguientes desigualdades conocidas en un triángulo ABC

$$AI + BI + CI \leq 2(R + r), \frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \leq \frac{1}{4r^2}. \text{ Aplicando la desigualdad de Cauchy}$$

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$$\frac{\sqrt{AI}}{a} + \frac{\sqrt{BI}}{b} + \frac{\sqrt{CI}}{c} \leq \sqrt{(AI + BI + CI) \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2} \right)} \leq \sqrt{2(R+r) \cdot \frac{1}{4r^2}} = \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{R+r}}{r}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} LHS &\stackrel{C-B-S}{\leq} \sqrt{\sum AI} \cdot \sqrt{\sum \frac{1}{a^2}} = \frac{1}{4Rrs} \cdot \sqrt{\sum AI} \sqrt{\sum a^2 b^2} \\ &\stackrel{Goldstone}{\leq} \frac{1}{4Rrs} \sqrt{\sum AI} \sqrt{4R^2 s^2} = \frac{2Rs}{4Rrs} \sqrt{\sum AI} = \frac{1}{2r} \sqrt{\sum AI} \stackrel{?}{\leq} \frac{\sqrt{2}}{2} \cdot \frac{\sqrt{R+r}}{r} \\ &\Leftrightarrow \sum AI \leq 2(R+r) \Leftrightarrow \frac{1}{\sin \frac{A}{2}} + \frac{1}{\sin \frac{B}{2}} + \frac{1}{\sin \frac{C}{2}} \leq \frac{2(R+r)}{r} \\ &\Leftrightarrow \frac{\sqrt{bc}}{\sqrt{(s-b)(s-c)}} + \frac{\sqrt{ca}}{\sqrt{(s-c)(s-a)}} + \frac{\sqrt{ab}}{\sqrt{(s-a)(s-b)}} \leq \frac{2(R+r)}{r} \\ &\Leftrightarrow \frac{\sqrt{bc(s-a)} + \sqrt{ca(s-b)} + \sqrt{ab(s-c)}}{\sqrt{(s-a)(s-b)(s-c)}} \leq \frac{2(R+r)}{r} \\ &\Leftrightarrow \frac{\sqrt{s}}{rs} \left\{ \sqrt{bc(s-a)} + \sqrt{ca(s-b)} + \sqrt{ab(s-c)} \right\} \leq \frac{2(R+r)}{r} \\ &\Leftrightarrow \sqrt{bc(s-a)} + \sqrt{ca(s-b)} + \sqrt{ab(s-c)} \leq 2\sqrt{s}(R+r) \quad (1) \\ LHS \text{ of } (1) &\stackrel{C-B-S}{\leq} \sqrt{\sum ab} \sqrt{3s-2s} = \sqrt{\sum ab} \sqrt{s} \stackrel{?}{\leq} 2\sqrt{s}(R+r) \\ &\Leftrightarrow \sum ab \leq 4(R+r)^2 \Leftrightarrow s^2 + 4Rr + r^2 \leq 4R^2 + 8Rr + 4r^2 \\ &\Leftrightarrow s^2 \leq 4R^2 + 4Rr + 3r^2 \rightarrow \text{true, by Gerretsen (proved)} \end{aligned}$$

JP.087. Let ABC be an acute triangle. Prove that

$$\sqrt{\cos A \cdot \sin B \cdot \sin C} + \sqrt{\sin A \cdot \cos B \cdot \sin C} + \sqrt{\sin A \cdot \sin B \cdot \sin C} \leq \frac{3}{2} \sqrt{\frac{3}{2}}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo acutángulo ABC

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$$\sum \sqrt{\cos A \sin B \sin C} \leq \frac{3}{2} \sqrt{\frac{3}{2}}$$

Dado que es triángulo acutángulo $\cos A, \cos B, \cos C > 0$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc} \Leftrightarrow 2 \sin B \sin C \cos A = \sin^2 B + \sin^2 C - \sin^2 A$$

$$\Leftrightarrow 2 \sin A \sin C \cos B = \sin^2 A + \sin^2 C - \sin^2 B \wedge 2 \sin A \sin B \cos C = \sin^2 A + \sin^2 B - \sin^2 C$$

$$\Leftrightarrow 2 \sum \sin B \sin C \cos A = \sin^2 A + \sin^2 B + \sin^2 C \Leftrightarrow$$

$$\Leftrightarrow \sum \sin B \sin C \cos A = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2}$$

$$\text{Además } \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2} = (1 + \cos A \cos B \cos C) \leq \frac{9}{8}$$

Aplicando la desigualdad de Cauchy se tiene lo siguiente

$$\begin{aligned} \sum \sqrt{\cos A \sin B \sin C} &= \sqrt{3(\sum \sin B \sin C \cos A)} = \sqrt{3 \left(\frac{\sin^2 A + \sin^2 B + \sin^2 C}{2} \right)} \leq \\ &\leq \sqrt{\frac{3 \cdot 9}{8}} = \frac{3}{2} \sqrt{\frac{3}{2}} \end{aligned}$$

Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand

$$\begin{aligned} -1 &= \cos(A + B + C) = \cos A \cos B \cos C - \sin A \sin B \cos C - \\ &\quad - \sin A \cos B \sin C - \cos A \sin B \sin C \end{aligned}$$

Hence $\sin A \sin B \cos C + \sin A \cos B \sin C + \cos A \sin B \sin C$

$$= 1 + \cos A \cos B \cos C \leq 1 + \frac{1}{8} = \frac{9}{8}$$

Because $\cos A + \cos B + \cos C \leq \frac{3}{2} \rightarrow \cos A \cos B \cos C \leq \frac{1}{8}$

Therefore $\sqrt{\sin A \sin B \cos C} + \sqrt{\sin A \cos B \sin C} + \sqrt{\cos A \sin B \sin C}$

$$\leq \sqrt{3(\sin A \sin B \cos C + \sin A \cos B \sin C + \cos A \sin B \sin C)}$$

$$\leq \sqrt{3 \left(\frac{9}{8} \right)} = \frac{3}{2} \sqrt{\frac{3}{2}}$$

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Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} \sum_{cyc} \sqrt{\sin A \cdot \sin B \cdot \cos C} &= \sum_{cyc} \sqrt{\frac{a}{2R} \cdot \frac{b}{2R} \cdot \left(\frac{b^2 + c^2 - a^2}{2ab}\right)} \\ &= \frac{1}{2\sqrt{2}R} \sum_{cyc} \sqrt{b^2 + c^2 - a^2} \leq \frac{1}{2\sqrt{2}R} \sqrt{3 \sum_{cyc} (b^2 + c^2 - a^2)} \\ & \quad [\because \sqrt{x} \text{ is concave hence by Jensen's Inequality}] \\ &= \frac{1}{2\sqrt{2}R} \sqrt{3 \sum_{cyc} a^2} \leq \frac{1}{2\sqrt{2}R} \sqrt{3 \cdot 9R^2} = \frac{3}{2} \sqrt{\frac{3}{2}} \text{ (Proved)} \end{aligned}$$

JP.088. Let a, b, c be positive real numbers. Prove that

$$\frac{a^3 + b^3}{c^2 + ab} + \frac{b^3 + c^3}{a^2 + bc} + \frac{c^3 + a^3}{b^2 + ca} \geq \frac{9abc}{ab + bc + ca}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo a, b, c números R^+ . Probar que $\frac{a^3+b^3}{c^2+ab} + \frac{b^3+c^3}{a^2+bc} + \frac{c^3+a^3}{b^2+ca} \geq \frac{9abc}{ab+bc+ca}$

Como $a, b, c > 0$. Aplicando la desigualdad de Cauchy

$$\frac{a^3+b^3}{c^2+ab} + \frac{b^3+c^3}{a^2+bc} + \frac{c^3+a^3}{b^2+ca} \geq \frac{a^3+b^3}{\sqrt{(c^2+a^2)(c^2+b^2)}} + \frac{b^3+c^3}{\sqrt{(a^2+b^2)(a^2+c^2)}} + \frac{c^3+a^3}{\sqrt{(b^2+c^2)(b^2+a^2)}}$$

Utilizando las siguientes desigualdades conocidas $\forall a, b, c > 0$

$$\Rightarrow a^3 + b^3 \geq ab(a + b) \Leftrightarrow 2(a^3 + b^3) \geq (a^2 + b^2)(a + b)$$

Análogamente para los siguientes términos

$$2(b^3 + c^3) \geq (b^2 + c^2)(b + c), 2(c^3 + a^3) \geq (c^2 + a^2)(c + a)$$

Luego por $MA \geq MG$

$$\begin{aligned} &\frac{a^3 + b^3}{\sqrt{(c^2 + a^2)(c^2 + b^2)}} + \frac{b^3 + c^3}{\sqrt{(a^2 + b^2)(a^2 + c^2)}} + \frac{c^3 + a^3}{\sqrt{(b^2 + c^2)(b^2 + a^2)}} \geq \\ &\geq 3 \sqrt[3]{\frac{(a^3 + b^3)(b^3 + c^3)(c^3 + a^3)}{(a^2 + b^2)(b^2 + c^2)(c^2 + a^2)}} \geq \end{aligned}$$

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$$\begin{aligned} &\geq 3 \sqrt[3]{\frac{(a+b)(b+c)(c+a)}{8}} \geq 3 \sqrt[3]{abc} = \frac{3 \sqrt[3]{abc} \cdot (ab+bc+ca)}{ab+bc+ca} \geq \\ &\geq \frac{3 \sqrt[3]{abc} \cdot 3 \sqrt[3]{a^2 b^2 c^2}}{ab+bc+ca} = \frac{9abc}{ab+bc+ca} \quad (\text{LQOD}) \end{aligned}$$

Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand

For $a, b, c > 0$, Prove that: $\frac{a^3+b^3}{c^2+ab} + \frac{b^3+c^3}{a^2+bc} + \frac{c^3+a^3}{b^2+ca} \geq \frac{9abc}{ab+bc+ca}$. We will show that

$$\frac{a^3}{c^2+ab} + \frac{b^3}{a^2+bc} + \frac{c^3}{b^2+ca} \geq \frac{9abc}{2(ab+bc+ca)} \quad (1)$$

$$\frac{a^3}{b^2+ca} + \frac{b^3}{c^2+ab} + \frac{c^3}{a^2+bc} \geq \frac{9abc}{2(ab+bc+ca)} \quad (2)$$

$$\frac{a^3}{c^2+ab} + \frac{b^3}{a^2+bc} + \frac{c^3}{b^2+ca} \geq \frac{9abc}{2(ab+bc+ca)} \quad (3)$$

$$\text{Iff } \frac{a^4}{c^2 a + a^2 b} + \frac{b^4}{a^2 b + b^2 c} + \frac{c^4}{b^2 c + c^2 a} \geq \frac{9abc}{2(ab+bc+ca)}$$

$$\text{Iff } \frac{(a^2+b^2+c^2)^2}{2(a^2 b + b^2 c + c^2 a)} \geq \frac{9(abc)}{2(ab+bc+ca)}$$

$$\text{Iff } \frac{27(a^2+b^2+c^2)^2}{b(a+b+c)^3} \geq \frac{9(abc)}{2(ab+bc+ca)}$$

$$\text{Iff } \frac{(a+b+c)}{9} \geq \frac{abc}{ab+bc+ca}$$

$$\text{Iff } \frac{(ab+bc+ca)(a+b+c)}{9(ab+bc+ca)} \geq \frac{abc}{ab+bc+ca} \text{ and it is to be true. Similarly (2) is to be true}$$

Therefore this problem is to be true.

Solution 3 by Dinh Tien Dung-Hanoi-Vietnam

$$\text{By Holder inequality we have: } \frac{a^3+b^3}{c^2+ab} + \frac{b^3+c^3}{a^2+bc} + \frac{c^3+a^3}{b^2+ca} \geq \frac{(\sqrt[3]{a^3+b^3} + \sqrt[3]{b^3+c^3} + \sqrt[3]{c^3+a^3})^3}{3(a^2+b^2+c^2+ab+bc+ca)}$$

$$\text{Since } 4(a^3 + b^3) \geq (a + b)^3, \text{ we have: } \frac{a^3+b^3}{c^2+ab} + \frac{b^3+c^3}{a^2+bc} + \frac{c^3+a^3}{b^2+ca} \geq \frac{2(a+b+c)^3}{3(a^2+b^2+c^2+ab+bc+ca)}$$

$$\text{We need to show that: } \frac{2(a+b+c)^3}{3(a^2+b^2+c^2+ab+bc+ca)} \geq \frac{9abc}{ab+bc+ca}. \text{ The inequality is equivalent to}$$

$$2(a+b+c)^3(ab+bc+ca) \geq 27abc(a^2+b^2+c^2+ab+bc+ca)$$

We assume that $a = \min\{a, b, c\}$ and let $a = b - p, a = c - q$ ($p, q \geq 0$)

The inequality is equivalent to

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$$27a^3(p^2 - pq + q^2) + 3a^2(5p^3 + 6p^2q + 6pq^2 + 5q^3) + \\ + a(4p^4 + 7p^3q + 33p^2q^2 + 7pq^3 + 4q^4) + 2pq(p^3 + 3p^2q + 3pq^2 + q^3) \geq 0$$

Equality if and only if $a = b = c$. Q.E.D.

JP.089. Let a, b, c be positive real numbers, take:

$$X = \frac{a}{b} + \frac{b}{a}, Y = \frac{b}{c} + \frac{c}{b}, Z = \frac{c}{a} + \frac{a}{c}.$$

Prove that: $X + Y + Z \geq 2\sqrt[4]{(X^2 + Y^2 + Z^2 - 3)(X + Y + Z + 3)}$

Proposed by Nguyen Ngoc Tu – Ha Giang – Vietnam

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo a, b, c números R^+ , de tal manera que

$$X = \frac{a}{b} + \frac{b}{a} > 0, Y = \frac{b}{c} + \frac{c}{b} > 0, Z = \frac{c}{a} + \frac{a}{c} > 0. \text{ Probar que}$$

$$X + Y + Z \geq 2\sqrt[4]{(X^2 + Y^2 + Z^2 - 3)(X + Y + Z + 3)}. \text{ Ahora bien}$$

$$X + Y + Z = \frac{a}{b} + \frac{b}{a} + \frac{b}{c} + \frac{c}{b} + \frac{c}{a} + \frac{a}{c} = \left(\frac{a+b}{c} + 1\right) + \left(\frac{b+c}{a} + 1\right) + \left(\frac{c+a}{b} + 1\right) - 3 =$$

$$= (a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 \Rightarrow X + Y + Z + 3 = (a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)$$

$$\Rightarrow X^2 + Y^2 + Z^2 = \left(\frac{a}{b} + \frac{b}{a}\right)^2 + \left(\frac{b}{c} + \frac{c}{b}\right)^2 + \left(\frac{c}{a} + \frac{a}{c}\right)^2 =$$

$$= \left(\frac{a^2}{b^2} + \frac{b^2}{a^2} + 2\right) + \left(\frac{b^2}{c^2} + \frac{c^2}{b^2} + 2\right) + \left(\frac{c^2}{a^2} + \frac{a^2}{c^2} + 2\right)$$

$$\Rightarrow X^2 + Y^2 + Z^2 - 3 = \left(\frac{a^2 + b^2}{c^2} + 1\right) + \left(\frac{b^2 + c^2}{a^2} + 1\right) + \left(\frac{c^2 + a^2}{b^2} + 1\right) =$$

$$= (a^2 + b^2 + c^2) \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right). \text{ La desigualdad propuesta es equivalente}$$

$$\Leftrightarrow (a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 \geq 2\sqrt[4]{(a^2 + b^2 + c^2) \left(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2}\right) (a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)}$$

Aplicando la desigualdad de Cauchy

$$(a+b+c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) - 3 = \sqrt{(a+b+c)^2 \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right)^2} - 3 =$$

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$$= \sqrt{(\sum a^2 + 2 \sum bc)(\sum \frac{1}{a^2} + 2 \sum \frac{1}{bc})} - 3 \geq$$

$$\geq \sqrt{(\sum a^2)(\sum \frac{1}{a^2})} + 2\sqrt{(\sum bc)(\sum \frac{1}{bc})} - 3 = \sqrt{(\sum a^2)(\sum \frac{1}{a^2})} + 2\sqrt{(\sum a)(\sum \frac{1}{a})} - 3$$

Utilizando MA ≥ MG

$$\sqrt{(\sum a^2)(\sum \frac{1}{a^2})} + \sqrt{(\sum a)(\sum \frac{1}{a})} + \sqrt{(\sum a)(\sum \frac{1}{a})} - 3 \geq$$

$$\geq \sqrt{(\sum a^2)(\sum \frac{1}{a^2})} + \sqrt{(\sum a)(\sum \frac{1}{a})} \geq$$

$$\geq 2^4 \sqrt{(a^2 + b^2 + c^2)(\frac{1}{a^2} + \frac{1}{b^2} + \frac{1}{c^2})(a + b + c)(\frac{1}{a} + \frac{1}{b} + \frac{1}{c})} \text{ (LQGD)}$$

Solution 2 by Richdad Phuc-Hanoi-Vietnam

We have $X = \frac{a}{b} + \frac{b}{a} \geq 2$ (AM-GM). *Similar* $Y \geq 2, Z \geq 2$

$$(X - 2)(Y - 2) \geq 0 \Rightarrow XY \geq 2(X + Y - 2)$$

similar $YZ \geq 2(Y + Z - 2)$ and $ZX \geq 2(Z + X - 2)$

Let $t = X + Y + Z$ ($t \geq 6$). *We get*

$$X^2 + Y^2 + Z^2 = (X + Y + Z)^2 - 2(XY + YZ + ZX) \leq t^2 - 8t + 24$$

we need to prove that

$$t^4 \geq 16(t^2 - 8t + 21)(t + 3) \Leftrightarrow (t - 6)(t^3 - 10t^2 + 20t + 168) \geq 0$$

$$(t - 6)[t(t - 6)^2 + 2(t - 4)^2 + 136] \geq 0 \text{ (true with } t \geq 6)$$

equality holds if $t = 6 \Leftrightarrow a = b = c$

JP.090. In ΔABC :

$$\frac{1 + \cos A \cos B \cos C}{\sin A \sin B \sin C} \geq \frac{s}{3r}$$

Proposed by Martin Lukarevski-Skopje-Macedonia

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo ABC

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$\frac{1+\cos A \cos B \cos C}{\sin A \sin B \sin C} \geq \frac{s}{3r}$. *Recordar las siguientes identidades en un triángulo ABC*

$$\sin^2 A + \sin^2 B + \sin^2 C = 2(1 + \cos A \cos B \cos C), \quad S = 2R^2 \sin A \sin B \sin C$$

$$\Rightarrow \cot A + \cot B + \cot C = \frac{\sin^2 A + \sin^2 B + \sin^2 C}{2 \sin A \sin B \sin C} = \frac{1 + \cos A \cos B \cos C}{\sin A \sin B \sin C} =$$

$$= \frac{a^2+b^2+c^2}{4S} = \frac{a^2+b^2+c^2}{4sr}. \text{ La desigualdad propuesta es equivalente}$$

$$\frac{a^2 + b^2 + c^2}{4sr} \geq \frac{s}{3r} \Leftrightarrow 3(a^2 + b^2 + c^2) \geq 4s^2 = (a + b + c)^2$$

(Válido por desigualdad de Cauchy)

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\cos A \cos B \cos C = \frac{b^2+c^2-a^2}{2bc} \cdot \frac{c^2+a^2-b^2}{2ca} \cdot \frac{a^2+b^2-c^2}{2ab} \quad (1)$$

$$\text{Numerator} = (\sum a^2 - 2a^2)(\sum a^2 - 2b^2)(\sum a^2 - 2c^2)$$

$$= (\sum a^2)^3 - 2(\sum a^2)^2(\sum a^2) + 4(\sum a^2)(\sum a^2 b^2) - 8a^2 b^2 c^2$$

$$= -(\sum a^2)^3 + 4(\sum a^2)\left\{(\sum ab)^2 - 2abc(2s)\right\} - 128R^2 r^2 s^2$$

$$= (\sum a^2)\left\{4(\sum ab)^2 - (\sum a^2)^2 - 16s abc\right\} - 128R^2 r^2 s^2$$

$$= 4(\sum a^2)\{(s^2 + 4Rr + r^2)^2 - (s^2 - 4Rr - r^2)^2 - 16Rrs^2\} - 128R^2 r^2 s^2$$

$$= 4(\sum a^2)\{2s^2(8Rr + 2r^2) - 16Rrs^2\} - 128R^2 r^2 s^2$$

$$= 32r^2 s^2 (s^2 - 4Rr - r^2) - 128R^2 r^2 s^2 \stackrel{(2)}{=} 32r^2 s^2 (s^2 - 4Rr - r^2 - 4R^2)$$

$$(1), (2) \Rightarrow \cos A \cos B \cos C = \frac{32r^2 s^2 (s^2 - 4R^2 - 4Rr - r^2)}{128R^2 r^2 s^2} \stackrel{(3)}{=} \frac{s^2 - 4R^2 - 4Rr - r^2}{4R^2}$$

$$\text{Again, } \sin A \sin B \sin C = \frac{abc}{8R^3} = \frac{4Rrs}{8R^3} \stackrel{(4)}{=} \frac{rs}{2R^2}$$

$$(3), (4) \Rightarrow LHS = \frac{4R^2 + s^2 - 4R^2 - 4Rr - r^2}{4R^2} \times \frac{2R^2}{rs} = \frac{s^2 - 4Rr - r^2}{2rs} \geq \frac{s}{3r}$$

$$\Leftrightarrow 3s^2 - 12Rr - 3r^2 \geq 2s^2 \Leftrightarrow s^2 \geq 12Rr + 3r^2 \quad (i)$$

Gerretsen $\Rightarrow s^2 \geq 16Rr - 5r^2 \therefore$ in order to prove (i), it suffices to prove:

$$16Rr - 5r^2 \geq 12Rr + 3r^2 \Leftrightarrow 4Rr \geq 8r^2 \Leftrightarrow R \geq 2r \rightarrow \text{true by Euler (Proved)}$$

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SP.076. Let a, b, c be the side – lengths of an acute triangle with perimeter 1. Prove that

$$E_1 \geq a^a b^b c^c \geq E_2$$

where

$$E_1 = \frac{(b+c-a)(c+a-b)(a+b-c)}{(b^2+c^2-a^2)^a (c^2+a^2-b^2)^b (a^2+b^2-c^2)^c}$$

and

$$E_2 = \frac{(b^2+c^2-a^2)^{b+c} (c^2+a^2-b^2)^{c+a} (a^2+b^2-c^2)^{a+b}}{(b+c-a)(c+a-b)(a+b-c)}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c los lados de un triángulo acutángulo con perímetro 1.

Probar que $E_1 \geq a^a b^b c^c \geq E_2$ donde $E_1 = \frac{(b+c-a)(c+a-b)(a+b-c)}{(b^2+c^2-a^2)^a (c^2+a^2-b^2)^b (a^2+b^2-c^2)^c}$

$E_2 = \frac{(b^2+c^2-a^2)^{b+c} (c^2+a^2-b^2)^{c+a} (a^2+b^2-c^2)^{a+b}}{(b+c-a)(c+a-b)(a+b-c)}$. Como es un triángulo acutángulo

→ $\cos A, \cos B, \cos C > 0$. Recordar las siguientes identidades en un triángulo ABC

$$b^2 + c^2 - a^2 = 2bc \cos A, c^2 + a^2 - b^2 = 2ca \cos B, a^2 + b^2 - c^2 = 2ab \cos C$$

$$\sin 2A + \sin 2B + \sin 2C = 4 \sin A \sin B \sin C, abc = 4RS,$$

$$16S^2 = (a+b+c)(b+c-a)(c+a-b)(b+a-c)$$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C$. Lo cual es equivalente

$$\frac{\sin A}{\cos A} + \frac{\sin B}{\cos B} + \frac{\sin C}{\cos C} = \frac{\sin A \sin B \sin C}{\cos A \cos B \cos C} \Leftrightarrow$$

$$\Leftrightarrow \sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B = \sin A \sin B \sin C$$

Probaremos lo siguiente $(b+c-a)(a+c-b)(b+a-c) \geq$

$$\geq (b^2+c^2-a^2)^a \cdot a^a \cdot (c^2+a^2-b^2)^b b^b (a^2+b^2-c^2)^c c^c$$

$$\Leftrightarrow (b+c-a)(a+c-b)(b+a-c) \geq (2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c$$

Recordar → $2p = a+b+c = 1$. Aplicando la desigualdad ponderada $MA \geq MG$

$$\frac{(2abc \cos A)a + (2abc \cos B)b + (2abc \cos C)c}{a+b+c} \geq$$

$$\geq \sqrt[1]{(2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c}$$

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$$\begin{aligned} &\Leftrightarrow 2abc(a \cos A + b \cos B + c \cos C) \geq (2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c \\ &\Leftrightarrow 2abcR(\sin 2A + \sin 2B + 2 \sin 2C) \geq (2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c \\ &\Leftrightarrow 2abc R(4 \sin A \sin B \sin C) \geq (2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c \\ &\Leftrightarrow 2abcR \left(\frac{abc}{2R^3}\right) \geq (2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c \\ &\Leftrightarrow \frac{a^2 b^2 c^2}{R^2} \geq (2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c \\ &\quad (2abc \cos A)^a (2abc \cos B)^b (2abc \cos C)^c \leq 16S^2 = \\ &= (a + b + c)(b + c - a)(a + c - b)(b + a - c) = (b + c - a)(a + c - b)(b + a - c) \end{aligned}$$

(LQQD). Por último, probaremos

$$\begin{aligned} &a^a b^b c^c (b + c - a)(a + c - b)(b + a - c) \geq \\ &\geq (b^2 + c^2 - a^2)^{b+c} (c^2 + a^2 - b^2)^{c+a} (a^2 + b^2 - c^2)^{a+b} \\ &\Leftrightarrow a^a b^b c^c (b + c - a)(a + c - b)(b + a - c) \geq \\ &\geq (2bc \cos A)^{b+c} (2ca \cos B)^{c+a} (2ab \cos C)^{a+b} \\ &\Leftrightarrow (b + c - a)(a + c - b)(b + a - c) \geq \\ &\left[\frac{(2ab \cos C)(2ca \cos B)}{a} \right]^a \left[\frac{(2bc \cos A)(2ab \cos C)}{b} \right]^b \left[\frac{(2ca \cos B)(2bc \cos A)}{c} \right]^c \\ &\Leftrightarrow (b + c - a)(a + c - b)(b + a - c) \geq \end{aligned}$$

$$\geq (4abc \cos B \cos C)^a (4abc \cos C \cos A)^b (4abc \cos A \cos B)^c$$

Utilizando la desigualdad ponderada $MA \geq MG$

$$\begin{aligned} &\frac{(4abc \cos B \cos C)a + (4abc \cos C \cos A)b + (4abc \cos A \cos B)c}{a + b + c} \geq \\ &\geq \sqrt[a+b+c]{(4abc \cos B \cos C)^a (4abc \cos C \cos A)^b (4abc \cos A \cos B)^c} \\ &\Leftrightarrow 4abc(a \cos B \cos C + b \cos C \cos A + c \cos A \cos B) \geq \\ &\geq (4abc \cos B \cos C)^a (4abc \cos C \cos A)^b (4abc \cos A \cos B)^c \\ &\Leftrightarrow 8Rabc (\sin A \cos B \cos C + \sin B \cos C \cos A + \sin C \cos A \cos B) \geq \\ &\geq (4abc \cos B \cos C)^a (4abc \cos C \cos A)^b (4abc \cos A \cos B)^c \\ &\Leftrightarrow Rabc (8 \sin A \sin B \sin C) \geq \\ &\geq (4abc \cos B \cos C)^a (4abc \cos C \cos A)^b (4abc \cos A \cos B)^c \end{aligned}$$

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$$\begin{aligned} \Leftrightarrow \frac{a^2 b^2 c^2}{R^2} &\geq (4abc \cos B \cos C)^a (4abc \cos C \cos A)^b (4abc \cos A \cos B)^c \\ \Leftrightarrow (4abc \cos B \cos C)^a (4abc \cos C \cos A)^b (4abc \cos A \cos B)^c &\leq 16S^2 = \\ = (a+b+c)(b+c-a)(a+c-b)(b+a-c) &= (b+c-a)(a+c-b)(b+a-c) \end{aligned}$$

SP.077. Prove that in any acute triangle ABC the following inequality holds

$$\frac{m_a}{h_a} \cos A + \frac{m_b}{h_b} \cos B + \frac{m_c}{h_c} \cos C \geq \frac{3}{2}$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Kevin Soto Palacios-Huarmey-Peru

Probar en un triángulo acutángulo ABC la siguiente desigualdad

$$\frac{m_a}{h_a} \cos A + \frac{m_b}{h_b} \cos B + \frac{m_c}{h_c} \cos C \geq \frac{3}{2}. \text{ Como es un triángulo acutángulo}$$

$\cos A, \cos B, \cos C > 0$. *Recordar la siguientes identidades y desigualdades en un*

$$\Delta ABC \quad h_a = \frac{bc}{2R}, \quad h_b = \frac{ca}{2R}, \quad h_c = \frac{ab}{2R}; \quad m_a \geq \frac{b^2+c^2}{4R}, \quad m_b \geq \frac{c^2+a^2}{4R}, \quad m_c \geq \frac{a^2+b^2}{4R}$$

$$\begin{aligned} \text{Lo cual implica } \Rightarrow \frac{m_a}{h_a} \cos A + \frac{m_b}{h_b} \cos B + \frac{m_c}{h_c} \cos C &\geq a \left(\frac{b^2+c^2}{2abc} \right) \cos A + b \left(\frac{c^2+a^2}{2abc} \right) \cos B + \\ &+ c \left(\frac{a^2+b^2}{2abc} \right) \cos C = \frac{3}{2}. \text{ Lo cual es cierto ya que} \end{aligned}$$

$$\Rightarrow a(b^2 + c^2) \cos A + b(c^2 + a^2) \cos B + c(a^2 + b^2) = 3abc$$

$$\Leftrightarrow a^2(b^2 + c^2)(2bc \cos A) + b^2(c^2 + a^2)(2ca \cos B) + c^2(a^2 + b^2)(2ab \cos C) = 6a^2b^2c^2$$

$$\Leftrightarrow a^2(b^2 + c^2)(b^2 + c^2 - a^2) + b^2(c^2 + a^2)(c^2 + a^2 - b^2) +$$

$$+ c^2(a^2 + b^2)(a^2 + b^2 - c^2) = a^2(b^2 + c^2)^2 - a^4(b^2 + c^2) +$$

$$+ b^2(c^2 + a^2)^2 - b^4(c^2 + a^2) + c^2(a^2 + b^2)^2 - c^4(a^2 + b^2) =$$

$$= a^2(b^4 + c^4) - a^4(b^2 + c^2) + b^2(c^4 + a^4) - b^4(c^2 + a^2) +$$

$$+ c^2(a^4 + b^4) - c^4(a^2 + b^2) + 6a^2b^2c^2 = 6a^2b^2c^2$$

SP.078. Let a, b, c be positive real numbers such that $a + b + c = 1$. Prove that:

$$a^{-a}b^{-b}c^{-c} + a^{-b}b^{-c}c^{-a} + a^{-c}b^{-a}c^{-b} \leq a^{-1} + b^{-1} + c^{-1}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

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Solution by proposer

Applying the Weighted AM-GM inequality we obtain

$$a^{-a}b^{-b}c^{-c} = \left(\frac{1}{a}\right)^a \left(\frac{1}{b}\right)^b \left(\frac{1}{c}\right)^c \leq a \cdot \frac{1}{a} + b \cdot \frac{1}{b} + c \cdot \frac{1}{c},$$

$$a^{-b}b^{-c}c^{-a} = \left(\frac{1}{a}\right)^b \left(\frac{1}{b}\right)^c \left(\frac{1}{c}\right)^a \leq b \cdot \frac{1}{a} + c \cdot \frac{1}{b} + a \cdot \frac{1}{c},$$

$$a^{-c}b^{-a}c^{-b} = \left(\frac{1}{a}\right)^c \left(\frac{1}{b}\right)^a \left(\frac{1}{c}\right)^b \leq c \cdot \frac{1}{a} + a \cdot \frac{1}{b} + b \cdot \frac{1}{c}.$$

Adding up these relations yields

$$a^{-a}b^{-b}c^{-c} + a^{-b}b^{-c}c^{-a} + a^{-c}b^{-a}c^{-b} \leq (a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) = a^{-1} + b^{-1} + c^{-1}$$

as desired.

SP.079. Prove that for all positive real numbers a, b, c and integer $n \geq 3$, the following inequality holds

$$\frac{a^n + b^n + c^n}{9} \left(\frac{1}{a^n} + \frac{1}{b^n} + \frac{1}{c^n}\right) \geq \left(\frac{b+c}{6a} + \frac{c+a}{6b} + \frac{a+b}{6c}\right)^n$$

Proposed by Hung Nguyen Viet-Hanoi-Vietnam

Solution by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} & \frac{a^n + b^n + c^n}{9} \left(\sum_{cyc} \frac{1}{a^n}\right) = \frac{1}{9} \sum_{cyc} \left(\frac{a^n + b^n}{2}\right) \cdot \left(\sum_{cyc} \frac{1}{a^n}\right) \\ & = \frac{1}{9} \sum_{cyc} \left(\frac{a+b}{2}\right)^n \left(\sum_{cyc} \frac{1}{a^n}\right) = \sum_{cyc} \left(\frac{a+b}{6}\right)^n \cdot \left(\sum_{cyc} \frac{1}{a^n}\right) \cdot 3^{n-2} \\ & = \sum_{cyc} \left(\frac{a+b}{6}\right)^n \cdot \left(\sum_{cyc} \frac{1}{a^n}\right) \cdot (1+1+1) \cdot (1+1+1) \dots ((n-2) \text{ times}) \\ & \stackrel{\text{HOLDER}}{\geq} \left(\sum_{cyc} \sqrt[n]{\left(\frac{a+b}{6}\right)^n \cdot \frac{1}{c^n} \cdot 1}\right)^n = \left(\sum_{cyc} \frac{a+b}{6c}\right)^n \end{aligned}$$

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SP.080. Prove that for all positive real numbers a, b, c the following inequality holds:

$$\frac{(a+b)^2}{a^2-ab+b^2} + \frac{(b+c)^2}{b^2-bc+c^2} + \frac{(c+a)^2}{c^2-ca+a^2} \geq \frac{9(a^2b+b^2c+c^2a+abc)}{a^3+b^3+c^3}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

To solve this problem we must need the following results

Lemma 1. For any positive real numbers u, v, w, x, y, z then $\frac{u^3}{x} + \frac{v^3}{y} + \frac{w^3}{z} \geq \frac{(u+v+w)^3}{3(x+y+z)}$

Proof. By Hölder's inequality we have

$$\left(\frac{u^3}{x} + \frac{v^3}{y} + \frac{w^3}{z}\right)(x+y+z)(1+1+1) \geq (u+v+w)^3 \text{ and the conclusion follows.}$$

Lemma 2. For all non-negative real numbers a, b, c then

$$a^2b + b^2c + c^2a + abc \leq \frac{4}{27}(a+b+c)^3.$$

Proof. Because the variables a, b, c are cyclic, without loss of generality we can suppose that b is between a and c . Then

$$a^2b + b^2c + c^2a + abc - (a^2b + c^2b + 2abc) = c(b-a)(b-c) \leq 0$$

$$\text{Consequently } a^2b + b^2c + c^2a + abc \leq (a^2b + c^2b + 2abc) = b(a+c)^2$$

On the other hand, applying the AM-GM inequality we get

$$b(a+c)^2 = \frac{1}{2}(2b)(a+c)(a+c) \leq \frac{1}{2}\left(\frac{2b+(a+c)+(a+c)}{3}\right)^3 = \frac{4}{27}(a+c+c)^3$$

Hence $a^2b + b^2c + c^2a + abc \leq \frac{4}{27}(a+b+c)^3$. Come back to the main problem

We use lemma 1 and lemma 2, respectively, to obtain $LHS = \frac{(b+c)^3}{b^3+c^3} + \frac{(c+a)^3}{c^3+a^3} + \frac{(a+b)^3}{a^3+b^3} \geq$

$$\geq \frac{(2a+2b+2c)^3}{3(2a^3+2b^3+2c^3)} = \frac{4(a+b+c)^3}{3(a^3+b^3+c^3)} \geq \frac{9(a^2b+b^2c+c^2a+abc)}{a^3+b^3+c^3} \text{ and we are done.}$$

SP.081. Let a, b, c be positive real numbers and $k \geq 2$. Prove that:

$$\sqrt{\frac{bc}{(b+ka)(c+ka)}} + \sqrt{\frac{ca}{(c+kb)(a+kb)}} + \sqrt{\frac{ab}{(a+kc)(b+kc)}} \geq \frac{3}{k+1}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

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Solution by proposer

Using the AM-GM inequality we obtain:

$$\sum_{cyc} \sqrt{\frac{bc}{(b+ka)(c+ka)}} = \sum_{cyc} \frac{bc}{\sqrt{(bc+kca)(bc+kab)}} \geq \sum_{cyc} \frac{2bc}{2bc+k(ca+ab)}$$

After setting $bc = x, ca = y, ab = z$, the required inequality reduces to:

$$\frac{2x}{2x+k(y+z)} + \frac{2y}{2y+k(z+x)} + \frac{2z}{2z+k(x+y)} \geq \frac{3}{k+1}$$

Applying the Cauchy-Schwarz inequality yields

$$\begin{aligned} \sum_{cyc} \frac{2x}{2x+k(y+z)} &= \sum_{cyc} \frac{2x^2}{2x^2+kx(y+z)} \geq \frac{2(x+y+z)^2}{2(x^2+y^2+z^2)+2k(xy+yz+zx)} = \\ &= \frac{(x+y+z)^2}{(x+y+z)^2+(k-2)(xy+yz+zx)} \geq \frac{(x+y+z)^2}{(x+y+z)^2+\frac{k-2}{3}(x+y+z)^2} = \\ &= \frac{1}{1+\frac{k-2}{3}} = \frac{3}{k+1}. \end{aligned}$$

This completes the proof. The equality holds when $a = b = c$.

SP.082. Let ABC be an equilateral triangle with side-length a and let M be any point inside the triangle. Prove that:

$$\frac{a^2}{2} \geq xMA + yMB + zMC \geq 2(xy + yz + zx)$$

where x, y, z denote the distances from M to the sides BC, CA, AB , respectively.

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

We first see easily that

$$x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC} = \vec{0} \quad (1)$$

Next, we have:

$$\begin{aligned} x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC &\geq x \cdot \overrightarrow{PA} \cdot \overrightarrow{MA} + y \cdot \overrightarrow{PB} \cdot \overrightarrow{MB} + z \cdot \overrightarrow{PC} \cdot \overrightarrow{MC} = \\ &= x(\overrightarrow{PM} + \overrightarrow{MA})\overrightarrow{MA} + y(\overrightarrow{PM} + \overrightarrow{MB})\overrightarrow{MB} + z(\overrightarrow{PM} + \overrightarrow{MC})\overrightarrow{MC} = \\ &= \overrightarrow{PM}(x\overrightarrow{MA} + y\overrightarrow{MB} + z\overrightarrow{MC}) + xMA^2 + yMB^2 + zMC^2 = xMA^2 + yMB^2 + zMC^2. \end{aligned}$$

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Now we square both sides of (1) to obtain:

$$x^2MA^2 + y^2MB^2 + z^2MC^2 + 2xy\overrightarrow{MA} \cdot \overrightarrow{MB} + 2yz\overrightarrow{MB} \cdot \overrightarrow{MC} + 2zx\overrightarrow{MC} \cdot \overrightarrow{MA} = 0$$

or $x^2MA^2 + y^2MB^2 + z^2MC^2 + xy(MA^2 + MB^2 - AB^2) + yz(MB^2 + MC^2 - BC^2) +$
 $+zx(MC^2 + MA^2 - CA^2) = 0$ or

$$(x + y + z)(xMA^2 + yMB^2 + zMC^2) = yza^2 + zxb^2 + xyc^2 \text{ or}$$

$$xMA^2 + yMB^2 + zMC^2 = \frac{(xy+yz+zx)a^2}{x+y+z} \text{ (since } a = b = c)$$

Furthermore, it's not difficult to observe that: $x + y + z = h = \frac{a\sqrt{3}}{2}$. Hence

$$xMA^2 + yMB^2 + zMC^2 = \frac{2a}{\sqrt{3}}(xy + yz + zx). \text{ Thus we have proved}$$

$$x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC \geq \frac{2a}{\sqrt{3}}(xy + yz + zx) \quad (2)$$

Also, using the AM-GM inequality we get:

$$xPA^2 + xMA^2 \geq 2xPA \cdot MA,$$

$$yPB^2 + yMB^2 \geq 2yPB \cdot MB,$$

$$zPC^2 + zMC^2 \geq 2zPC \cdot MC,$$

It follows that:

$$(xPA^2 + yPB^2 + zPC^2) + (xMA^2 + yMB^2 + zMC^2) \geq 2x \cdot PA \cdot MA + 2y \cdot PB \cdot MB +$$

$$+ 2z \cdot PC \cdot MC. \text{ On the other hand, according to the above proof, we have}$$

$$x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC \geq xMA^2 + yMB^2 + zMC^2$$

Adding up two last results we obtain

$$xPA^2 + yPB^2 + zPC^2 \geq x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC \quad (3)$$

Combining (2) and (3) gives us

$$xPA^2 + yPB^2 + zPC^2 \geq x \cdot PA \cdot MA + y \cdot PB \cdot MB + z \cdot PC \cdot MC \geq \frac{2a}{\sqrt{3}}(xy + yz + zx)$$

The equalities occur if and only if $P \equiv M$. From here we take again $P \equiv O$ which is the center of the equilateral triangle ABC then to get

$$(x + y + z)R \geq xMA + yMB + zMC \geq \frac{2a}{R\sqrt{3}}(xy + yz + zx)$$

Also, $x + y + z = \frac{a\sqrt{3}}{2}$ and $R = \frac{a}{\sqrt{3}}$. Therefore we find the desired result.

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SP.083. Let m_a, m_b, m_c be the lengths of the medians of a triangle with circumradius R .

Prove that:

$$\left(1 + \frac{1}{m_a}\right) \left(1 + \frac{1}{m_b}\right) \left(1 + \frac{1}{m_c}\right) \geq \left(1 + \frac{2}{3R}\right)^3$$

Proposed by George Apostolopoulos-Messolonghi-Greece

Solution by proposer

Applying the Cauchy – Schwarz Inequality, we have:

$$\frac{1}{m_a + x} + \frac{1}{m_b + x} + \frac{1}{m_c + x} \geq \frac{(1 + 1 + 1)^2}{m_a + m_b + m_c + 3x} = \frac{9}{m_a + m_b + m_c + 3x}, x \geq 0$$

It is well – known that $m_a + m_b + m_c \leq \frac{9R}{2}$. So, $\frac{1}{m_a+x} + \frac{1}{m_b+x} + \frac{1}{m_c+x} \geq \frac{9}{\frac{9R}{2}+3x} = \frac{3}{\frac{3R}{2}+x}$.

Now, $\int_0^1 \left(\frac{1}{m_a+x} + \frac{1}{m_b+x} + \frac{1}{m_c+x}\right) dx \geq \int_0^1 \frac{3}{\frac{3R}{2}+x} dx$. So,

$$[\ln(m_a + x) + \ln(m_b + x) + \ln(m_c + x)]_0^1 \geq 3 \left[\ln\left(\frac{3R}{2} + x\right) \right]_0^1 \Leftrightarrow$$

$$\Leftrightarrow \ln(m_a + 1) + \ln(m_b + 1) + \ln(m_c + 1) - \ln m_a - \ln m_b - \ln m_c \geq$$

$$\geq 3 \left(\ln\left(\frac{3R}{2} + 1\right) - \ln\frac{3R}{2} \right) \Leftrightarrow \ln \frac{m_a + 1}{m_a} + \ln \frac{m_b + 1}{m_b} + \ln \frac{m_c + 1}{m_c} \geq 3 \ln\left(1 + \frac{2}{3R}\right) \Leftrightarrow$$

$$\Leftrightarrow \ln\left(\frac{m_a+1}{m_a} \cdot \frac{m_b+1}{m_b} \cdot \frac{m_c+1}{m_c}\right) \geq \ln\left(1 + \frac{2}{3R}\right)^3. \text{ Namely}$$

$$\left(1 + \frac{1}{m_a}\right) \left(1 + \frac{1}{m_b}\right) \left(1 + \frac{1}{m_c}\right) \geq \left(1 + \frac{2}{3R}\right)^3. \text{ Equality holds when triangle ABC is}$$

equilateral.

SP.084. Prove that if $n \in \mathbb{N}^*$ then:

$$2 \int_0^1 \arctan(x^{n-1}) \arctan(x^n) dx \leq \int_0^1 \arctan^2(x^n) dx + \frac{1}{2n-1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$2 \int_0^1 \tan^{-1} x^{n-1} \tan^{-1} x^n dx \stackrel{AM \geq GM}{\geq} \int_0^1 (\tan^{-1} x^{n-1})^2 dx + \int_0^1 (\tan^{-1} x^n)^2 dx$$

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$$\leq \int_0^1 x^{2n-2} dx + \int_0^1 (\tan^{-1} x^n)^2 dx [\because \tan^{-1} x \leq x] = \frac{1}{2n-1} + \int_0^1 (\tan^{-1} x^n)^2 dx$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$\begin{aligned} 2 \int_0^1 a \tan(x^{n-1}) a \tan(x^n) dx &\stackrel{AM-GM}{\geq} \int_0^1 a \tan^2(x^{n-1}) dx + \int_0^1 a \tan^2(x^n) dx \\ &\stackrel{a \tan x \leq x}{\geq} \int_0^1 (x^{n-1})^2 dx + \int_0^1 a \tan^2(x^n) dx \leq \frac{1}{2n-1} + \int_0^1 a \tan^2(x^n) dx \end{aligned}$$

Solution 3 by proposer

$$\begin{aligned} &\int_0^1 (x^{n-1} - \arctan(x^n))^2 dx \geq 0 \\ &\int_0^1 x^{2n-2} dx - 2 \int_0^1 x^{n-1} \arctan(x^n) dx + \int_0^1 \arctan^2(x^n) dx \geq 0 \\ &x \in [0, 1] \Rightarrow x^{n-1} \in [0, 1] \\ &\tan x \geq x; \arctan x \leq x \Rightarrow \arctan(x^{n-1}) \leq x^{n-1} \\ &\frac{1}{2n-1} + \int_0^1 \arctan^2(x^n) dx \geq 2 \int_0^1 x^{n-1} \arctan(x^n) dx \\ &\geq 2 \int_0^1 \arctan(x^{n-1}) \arctan(x^n) dx \end{aligned}$$

SP.085. Prove that if $a, b \in (0, \infty)$; $n \in \mathbb{N}^*$ then:

$$\left(\frac{a}{b^n} + \frac{b}{a^n}\right) \left(\frac{a^n}{b} + \frac{b^n}{a}\right) \left(\frac{a^n}{b^n} + \frac{b}{a}\right) \left(\frac{b^n}{a^n} + \frac{a}{b}\right) \geq 8 \left(\sqrt{\left(\frac{a}{b}\right)^{n-1}} + \sqrt{\left(\frac{b}{a}\right)^{n-1}} \right)$$

Proposed by Daniel Sitaru – Romania

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Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$h(x) = \left(x^{n+1} + \frac{1}{x^{n+1}}\right) - \left(x^{n-1} + \frac{1}{x^{n-1}}\right) = (x^2 - 1) \left(x^{n-1} - \frac{1}{x^{n+1}}\right)$$

$$\text{if } x \geq 1 \rightarrow x^2 - 1 \geq 0 \text{ \& } x^{n-1} - \frac{1}{x^{n+1}} \geq 0$$

$$\text{if } x \leq 1 \rightarrow x^2 - 1 \leq 0 \text{ \& } x^{n-1} - \frac{1}{x^{n+1}} \leq 0 \rightarrow \forall x > 0 \quad h(x) \geq 0$$

$$f(t) = t^4 - 8t \quad ; \quad t \geq 2$$

$$f'(t) = 4t^3 - 8 \geq 0 \quad \forall t \geq 2 \Rightarrow f(t) \geq f(2) = 0$$

$$g(z) = f\left(1 + \frac{1}{z}\right) \stackrel{z + \frac{1}{z} \geq 2}{\geq} 0 \Leftrightarrow \left(z + \frac{1}{z}\right)^4 - 8\left(z + \frac{1}{z}\right) \geq 0$$

$$\Rightarrow A = \left(\frac{a}{b^n} + \frac{b}{a^n}\right) \left(\frac{a^n}{b} + \frac{b^n}{a}\right) \left(\frac{a^n}{b^n} + \frac{b}{a}\right) \left(\frac{b^n}{a^n} + \frac{a}{b}\right)$$

$$= \left(\frac{(a^{n+1} + b^{n+1})^2}{a^{n+1}b^{n+1}}\right)^2 = \left(\frac{a^{\frac{n+1}{2}}}{b^{\frac{n+1}{2}}} + \frac{b^{\frac{n+1}{2}}}{a^{\frac{n+1}{2}}}\right)^4$$

$$\stackrel{\sqrt{\frac{a}{b}}=x}{=} \left(x^{n+1} + \frac{1}{x^{n+1}}\right)^4 \stackrel{h(x) \geq 0}{\geq} \left(x^{n-1} + \frac{1}{x^{n-1}}\right)^4$$

$$\stackrel{g(x^{n-1}) \geq 0}{\geq} 8 \left(x^{n-1} + \frac{1}{x^{n-1}}\right) = 8 \left(\sqrt{\frac{a^{n-1}}{b^{n-1}}} + \sqrt{\frac{b^{n-1}}{a^{n-1}}}\right)$$

Solution by proposer

From means inequality:

$$\frac{a}{b^n} + \frac{b}{a^n} \geq 2\sqrt{\frac{ab}{a^n b^n}} \quad (1)$$

$$\frac{a^n}{b^n} + \frac{b}{a} \geq 2\sqrt{\frac{a^n b}{b^n a}} \quad (2)$$

$$\frac{b^n}{a^n} + \frac{a}{b} \geq 2\sqrt{\frac{b^n a}{a^n b}} \quad (3)$$

By multiplying the relationships (1); (2); (3):

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$$\left(\frac{a}{b^n} + \frac{b}{a^n}\right) \left(\frac{a^n}{b^n} + \frac{b^n}{a}\right) \left(\frac{b^n}{a^n} + \frac{a}{b}\right) \geq \frac{8}{\sqrt{a^{n-1} \cdot b^{n-1}}} \quad (4)$$

We prove that: $\frac{a^n}{b} + \frac{b^n}{a} \geq a^{n-1} + b^{n-1} \quad (5)$

$$a^{n+1} + b^{n+1} \geq a^n b + a b^n$$

$$a^n(a - b) - b^n(a - b) \geq 0$$

$$(a - b)(a^n - b^n) \geq 0$$

$$(a - b)^2(a^{n-1} + a^{n-2}b + \dots + b^{n-1}) \geq 0 \quad (\text{true})$$

We multiply the relationships (4); (5):

$$\begin{aligned} \left(\frac{a^n}{b} + \frac{b^n}{a}\right) \left(\frac{a}{b^n} + \frac{b}{a^n}\right) \left(\frac{a^n}{b^n} + \frac{b^n}{a}\right) \left(\frac{b^n}{a^n} + \frac{a}{b}\right) &\geq \frac{8(a^{n-1} + b^{n-1})}{\sqrt{a^{n-1} \cdot b^{n-1}}} = \\ &= 8 \left(\sqrt{\left(\frac{a}{b}\right)^{n-1}} + \sqrt{\left(\frac{b}{a}\right)^{n-1}} \right) \end{aligned}$$

SP.086. Prove that if a, b, c are the length's sides in ΔABC then:

$$\sin^2 a + \sin^2 b + \sin^2 c \geq 4 \sin s \cdot \sin(s - a) \cdot \sin(s - b) \cdot \sin(s - c)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ruanghaw Chaoka-Bangkok-Thailand

$$LHS = \frac{3 - \cos 2a - \cos 2b - \cos 2c}{2}$$

$$RHS = (2 \sin s \sin(s - a))(2 \sin(s - b) \sin(s - c))$$

$$= (\cos a - \cos(2s - a))(\cos(-b + c) - \cos a)$$

$$= \cos a \cos(-b + c) - \cos^2 a - \cos(2s - a) \cos(-b + c) + \cos(2s - a) \cos a$$

$$= \frac{1}{2}(\cos(a - b + c) + \cos(a + b - c) - 1 - \cos 2a - \cos 2c - \cos 2b + \cos 2s + \cos(-a + b + c))$$

$$LHS \geq RHS \Leftrightarrow 4 \geq \cos(a - b + c) + \cos(a + b - c) + \cos 2s + \cos(-a + b + c) \quad \text{True}$$

$$\because 1 \geq \cos x$$

$$\text{Hold at } a - b + c = 2k\pi, a + b - c = 2l\pi, 2s = 2m\pi,$$

$$-a + b + c = 2n\pi \Leftrightarrow a = p\pi, b = q\pi, c = r\pi$$

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Solution 2 by Geanina Tudose-Romania

$$\begin{aligned}
 2 \sin \frac{a+b+c}{2} \sin \frac{b+c-a}{2} &= \cos \frac{a+b+c-b-c+a}{2} = \\
 &= \cos \frac{a+b+c+b+c-a}{2} = \cos a - \cos(b+c) \\
 2 \sin \frac{a-b+c}{2} \sin \frac{a+b-c}{2} &= \cos \frac{a-b+c-a+b+c}{2} - \cos \frac{a-b+c+a+b-c}{2} = \cos(c-b) - \cos a. \text{ We} \\
 &\text{have } \sin^2 a + \sin^2 b + \sin^2 c \geq (\cos a - \cos(b+c))(\cos(c-b) - \cos a) \\
 &= 2 \cos a \cos b \cos c + \cos a \sin b \sin c - \cos^2 a - \cos^2 b \cos^2 c \\
 &- \cos b \sin b \cos c \sin c + \sin^2 b \sin^2 c - \sin b \cos a \sin c + \cos b \sin b \cos c \sin c \\
 &= 2 \cos a \cos b \cos c - 1 + \sin^2 a - (1 - \sin^2 b)(1 - \sin^2 c) + \sin^2 b \sin^2 c \\
 &\Leftrightarrow \sin^2 a + \sin^2 b + \sin^2 c \geq 2 \cos a \cos b \cos c - 2 + \sin^2 a + \sin^2 b + \sin^2 c \\
 &\Leftrightarrow 2 \cos a \cos b \cos c \leq 2. \text{ which is obviously true}
 \end{aligned}$$

Solution 3 by Kevin Soto Palacios-Huarmey-Peru

Siendo a, b, c los lados de un triángulo ABC . Probar que

$$\begin{aligned}
 2 \sin^2 a + 2 \sin^2 b + 2 \sin^2 c &\geq 8 \sin s \sin(s-a) \sin(s-b) \sin(s-c) \\
 1 - \cos(2a) + 1 - \cos(2b) + 1 - \cos(2c) &\geq 8 \sin s \sin(s-a) \sin(s-b) \sin(s-c) \\
 3 - \cos(2a) - \cos(2b) - \cos(2c) &\geq 8 \sin s \sin(s-a) \sin(s-b) \sin(s-c)
 \end{aligned}$$

Realizamos los siguientes cambios de variables

$$s - a = x > 0, s - b = y > 0, s - c = z > 0,$$

$$x + y = c, y + z = a, z + x = b, x + y + z = s$$

$$\sin^2(x+y) + \sin^2(y+z) + \sin^2(z+x) \geq 4 \sin(x+y+z) \sin x \sin y \sin z$$

Ahora bien, por transformaciones trigonométricas

$$\begin{aligned}
 E &= 4 \sin(x+y+z) \sin x \sin y \sin z = \\
 &= (\cos(y+z) - \cos(2x+y+z))(\cos(y-z) - \cos(y+z)) \\
 2E &= 2 \cos(y-z) \cos(y+z) - 2 \cos^2(y+z) - 2 \cos(2x+y+z) \cos(y-z) + \\
 &\quad + 2 \cos(2x+y+z) \cos(y+z) \\
 2E &= 2 \cos(y+z) [\cos(y-z) + \cos(2z+y+z)] - 1 - \cos(2y+2z) - \\
 &\quad - \cos(2z+2x) - \cos(2x+2y) \\
 2E &= 4 \cos(y+z) \cos(z+x) \cos(x+y) - 1 - \cos(2y+2z) -
 \end{aligned}$$

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$$- \cos(2z + 2x) - \cos(2x + 2y)$$

$$\begin{aligned} \text{Por lo tanto } & 3 - \cos(2y + 2z) - \cos(2z + 2x) - \cos(2x + 2y) \geq \\ & \geq 4 \cos(y + z) \cos(z + x) \cos(x + y) - 1 - \cos(2y + 2z) - \\ & \quad - \cos(2z + 2x) - \cos(2x + 2y) \end{aligned}$$

$$\Leftrightarrow 4 \geq 4 \cos(y + z) \cos(z + x) \cos(x + y) \Leftrightarrow 1 \geq \cos a \cos b \cos c$$

(lo cual es evidente)

Solution 4 by proposer

$$\begin{aligned} \sin^2 a + \sin^2 b + \sin^2 c &= 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \geq \\ &\geq 1 - \cos^2 a - \cos^2 b - \cos^2 c + 2 \cos a \cos b \cos c = \\ &= 1 - \sum \frac{1 + \cos 2a}{2} + [\cos(a + b) + \cos(a - b)] \cos c = \\ &= \frac{-1 - \sum \cos 2a}{2} + \frac{\cos(a + b + c) + \sum \cos(a + b - c)}{2} = \\ &= \frac{\cos(a + b + c)}{2} + \frac{1}{2} \sum (\cos(b + c - a) - \cos 2a) = \\ &= -\sin^2 \frac{a + b + c}{2} - \sum \sin \frac{a + b + c}{2} \sin \frac{b + c - 3a}{2} = \\ &= -\sin \frac{a + b + c}{2} \left[\left(\sin \frac{a + b + c}{2} + \sin \frac{b + c - 3a}{2} \right) + \left(\sin \frac{a + b - 3c}{2} + \frac{a - 3b + c}{2} \right) \right] = \\ &= 4 \sin \frac{a + b + c}{2} \sin \frac{b + c - a}{2} \sin \frac{a + c - b}{2} \sin \frac{a + b - c}{2} = \\ &= 4 \sin \frac{2s}{2} \sin \frac{2s - 2a}{2} \sin \frac{2s - 2b}{2} \sin \frac{2s - 2c}{2} = 4 \sin s \sin(s - a) \sin(s - b) \sin(s - c) \end{aligned}$$

SP.087. Let z_1, z_2, z_3 be the affixes of A, B respectively C in acute-angled ΔABC .

Prove that:

$$\prod \left(\left| \frac{z_2 - z_3}{z_2 + z_3} \right| + \left| \frac{z_3 - z_1}{z_3 + z_1} \right| \right) \geq \frac{32sr^3}{(s^2 - (2R + r)^2)^2}$$

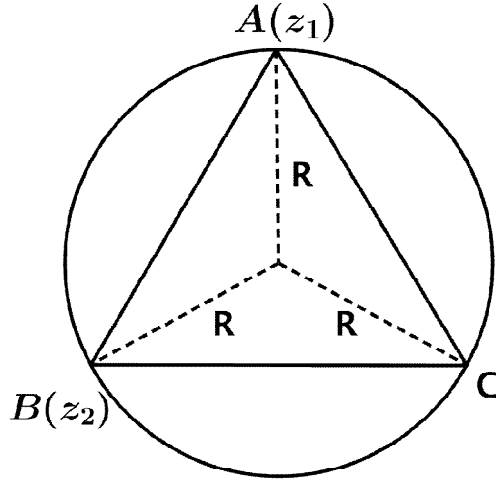
Proposed by Daniel Sitaru – Romania

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Solution by Ravi Prakash-New Delhi-India



Let's take O the circumcentre of ΔABC .

Let's take $z_1 = R(\cos \alpha + i \sin \alpha)$; $z_2 = R(\cos \beta + i \sin \beta)$; $z_3 = R(\cos \gamma + i \sin \gamma)$

$$\begin{aligned} \text{Now, } \frac{z_2 - z_3}{z_2 + z_3} &= \frac{1 - \frac{z_3}{z_2}}{1 + \frac{z_3}{z_2}} = \frac{1 - \cos(\gamma - \beta) - i \sin(\gamma - \beta)}{1 + \cos(\gamma - \beta) + i \sin(\gamma - \beta)} \Rightarrow \left| \frac{z_2 - z_3}{z_2 + z_3} \right|^2 = \frac{(1 - \cos(\gamma - \beta))^2 + \sin^2(\gamma - \beta)}{(1 + \cos(\gamma - \beta))^2 + \sin^2(\gamma - \beta)} \\ &= \frac{2[1 - \cos(\gamma - \beta)]}{2[1 + \cos(\gamma - \beta)]} = \frac{1 - \cos 2A}{1 + \cos 2A} = \tan^2 A \therefore \left| \frac{z_2 - z_3}{z_2 + z_3} \right| = \tan A. \text{ Similar for other expressions} \end{aligned}$$

$$\begin{aligned} \therefore LHS &= \prod (\tan A + \tan B) = \prod \frac{\sin(A + B)}{\cos A + \cos B} = \prod \frac{\sin C}{\cos A \cos B} \\ &= \frac{\sin A \sin B \sin C}{\cos^2 A \cos^2 B \cos^2 C} = \frac{16R^4 \sin A \sin B \sin C}{(4R^2 \cos A \cos B \cos C)^2} \quad (1) \end{aligned}$$

We now show that $4R^2 \cos A \cos B \cos C = s^2 - (2R + r)^2 = s^2 - (4R^2 + 4Rr + r^2)$

$$\begin{aligned} \text{Now, } 4Rr + r^2 &= \frac{abc}{\Delta} \cdot \frac{\Delta}{s} + \frac{\Delta^2}{s^2} \quad [\Delta = \text{area of } \Delta ABC] = \frac{abc}{s} + \frac{1}{s}(s - a)(s - b)(s - c) \\ &= \frac{1}{s}[abc + (s - a)(s - b)(s - c)] \end{aligned}$$

$$= \frac{1}{s}[abc + s^3 - (a + b + c)s^2 + (ab + bc + ca)s - abc] = -s^2 + ab + bc + ca$$

$$\therefore s^2 - (4R^2 + 4Rr + r^2) = s^2 - 4R^2 + s^2 - (ab + bc + ca)$$

$$= \frac{1}{2}(a + b + c)^2 - (ab + bc + ca) - (2R)^2 = \frac{1}{2}[a^2 + b^2 + c^2 - 2(2R)^2]$$

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$$\begin{aligned} &= \frac{1}{2}(2R)^2[\sin^2 A + \sin^2 B + \sin^2 C - 2] = 2R^2[-\cos^2 A - (\cos^2 B - \sin^2 C)] \\ &= 2R^2[-\cos^2 A - \cos(B+C)\cos(B-C)] = 2R^2[-\cos^2 A + \cos A \cos(B-C)] \\ &= 2R^2(\cos A)[\cos(B-C) + \cos(B+C)] = 4R^2 \cos A \cos B \cos C \quad (2) \end{aligned}$$

$$\begin{aligned} \text{Also, } 16R^4 \sin A \sin B \sin C &= 4R^2(2R \sin A)(2R \sin B) \sin C \\ &= 4R^2 ab \sin C = 4R^2(2\Delta) = 8R^2 \Delta = 8R^2(sr) \geq 8(2r)^2(sr) \\ \therefore 16R^4 \sin A \sin B \sin C &\geq 32sr^3 \quad (3) \end{aligned}$$

$$\text{From (1), (2), (3), we get } \prod \left(\frac{|z_2 - z_3|}{|z_2 + z_3|} + \frac{|z_1 - z_3|}{|z_1 + z_3|} \right) \geq \frac{32sr^3}{[s^2 - (2R+r)^2]^2}$$

SP.088. Let $a, b, c > 0$ such that $ab + bc + ca + abc = 4$.

Prove that:

$$\begin{aligned} (a+1)\sqrt{(b+1)(c+1)} + (b+1)\sqrt{(c+1)(a+1)} + (c+1)\sqrt{(a+1)(b+1)} &\geq \\ &\geq a + b + c + 9 \end{aligned}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution by Kevin Soto Palacios-Huarmey-Peru

$$\begin{aligned} \text{Siendo } a, b, c > 0 \text{ de tal manera que } ab + bc + ca + abc = 4. \text{ Probar que} \\ (a+1)\sqrt{(b+1)(c+1)} + (b+1)\sqrt{(c+1)(a+1)} + (c+1)\sqrt{(a+1)(b+1)} &\geq \\ &\geq a + b + c + 9 \end{aligned}$$

De la condición, realizamos las siguientes sustituciones

$$a = \frac{2x}{y+z} > 0, b = \frac{2y}{z+x} > 0, c = \frac{2z}{x+y} > 0, \text{ donde } x, y, z > 0$$

Aplicando la desigualdad de Cauchy y $MA \geq MG$

$$\begin{aligned} (a+1)\sqrt{(b+1)(c+1)} &= \left(\frac{(x+y)+(x+z)}{y+z} \right) \sqrt{\frac{(y+z)+(y+x)}{z+x} \cdot \frac{(y+z)+(z+x)}{x+y}} \geq \\ &\geq \left(\frac{(x+y)+(x+z)}{y+z} \right) \left(\frac{y+z}{\sqrt{(z+x)(x+y)}} + 1 \right) = \\ &= \frac{(x+y)+(x+z)}{y+z} \cdot \frac{y+z}{\sqrt{(z+x)(x+y)}} + \frac{2x}{y+z} + 1 \geq \frac{2x}{y+z} + 3 \quad (A) \end{aligned}$$

Analogamente para los siguientes términos

$$(b+1)\sqrt{(c+1)(a+1)} \geq \frac{2y}{z+x} + 3 \quad (B)$$

$$(c+1)\sqrt{(a+1)(b+1)} \geq \frac{2z}{x+y} + 3 \quad (C)$$

Sumando (A)+(B)+(C)

$$(a+1)\sqrt{(b+1)(c+1)} + (b+1)\sqrt{(c+1)(a+1)} + (c+1)\sqrt{(a+1)(b+1)} \geq$$

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$$\geq \frac{2x}{y+z} + \frac{2y}{z+x} + \frac{2z}{x+y} + 9 = a + b + c + 9$$

SP.089. Let r_a, r_b, r_c be the exradii of a triangle ABC , h_a, h_b, h_c the altitudes and let R, r, s denote the circumradius, inradius and semiperimeter respectively. Prove that

$$\frac{r_a^2}{h_a} + \frac{r_b^2}{h_b} + \frac{r_c^2}{h_c} \geq \frac{2s^2}{3} \left(\frac{1}{r} - \frac{1}{R} \right)$$

Proposed by Martin Lukarevski-Skopje

Solution by Soumava Chakraborty-Kolkata-India

WLOG, we may assume $a \geq b \geq c$. Then, $r_a^2 \geq r_b^2 \geq r_c^2$ and $\frac{1}{h_a} \geq \frac{1}{h_b} \geq \frac{1}{h_c}$

$$\therefore LHS \stackrel{\text{Chebyshev}}{\geq} \frac{1}{3} \left(\sum r_a^2 \right) \left(\sum \frac{1}{h_a} \right) = \frac{1}{3r} \{ (4R+r)^2 - 2s^2 \} \stackrel{?}{\geq} \frac{2s^2(R-r)}{3Rr}$$

$$\Leftrightarrow R(4R+r)^2 - 2Rs^2 \stackrel{?}{\geq} 2Rs^2 - 2rs^2 \Leftrightarrow (4R-2r)s^2 \stackrel{?}{\geq} R(4R+r)^2$$

$$\text{Now, Rouché} \Rightarrow (4R-2r)s^2 \leq (2R^2 + 10Rr - r^2)(4R-2r) +$$

$$+ 2(R-2r)(4R-2r)\sqrt{R^2 - 2Rr} \stackrel{?}{\leq} R(4R+r)^2$$

$$\Leftrightarrow (R-2r)(8R^2 - 12Rr + r^2) \stackrel{?}{\geq} 2(R-2r)(4R-2r)\sqrt{R^2 - 2Rr}$$

$\therefore R-2r \geq 0$ by Euler \therefore it suffices to prove: $8R^2 - 12Rr + r^2 > 4(2R-r)\sqrt{R^2 - 2Rr}$

$$[8R^2 - 12Rr + r^2 = (R-2r)(8r+4r) + 9r^2 > 0]$$

$$\Leftrightarrow (8R^2 - 12Rr + r^2)^2 - 16(2R-r)^2(R^2 - 2Rr) > 0$$

$$\Leftrightarrow 16R^2r^2 + 8Rr^3 + r^4 > 0 \rightarrow \text{true (Proved)}$$

SP.090. If $u, v > 0$, with $2u - v > 0$ and α, β, γ are the measures of the angles of triangle ABC , then

$$\sum_{cyc} \frac{\sin \alpha}{u \sin \beta + v \sqrt{\sin \alpha \sin \beta}} \geq \frac{3}{u+v}$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

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Solution by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned}
 & \sum_{cyc} \frac{\sin \alpha}{u \sin \beta + v \sqrt{\sin \alpha \sin \beta}} \stackrel{AM \geq GM}{\geq} 2 \sum_{cyc} \frac{\sin \alpha}{2u \sin \beta + v(\sin \alpha + \sin \beta)} \\
 & \geq 2 \sum_{cyc} \frac{\sin^2 \alpha}{v \sin^2 \alpha + (2u + v) \sin \alpha \sin \beta} \stackrel{Bergstrom}{\geq} \frac{2(\sin \alpha + \sin \beta + \sin \gamma)^2}{v \sum_{cyc} \alpha + (2u + v) \sum_{cyc} \sin \alpha \sin \beta} \\
 & = \frac{2(\sum_{cyc} \sin \alpha)^2}{v(\sum_{cyc} \sin \alpha)^2 + (2u - v) \sum_{cyc} \sin \alpha \sin \beta} \\
 & \geq 2 \frac{(\sum_{cyc} \sin \alpha)^2}{v(\sum_{cyc} \sin \alpha)^2 + \frac{2u - v}{3} (\sum_{cyc} \sin \alpha)^2} [\because 2u - v > 0] = \frac{3}{u + v}
 \end{aligned}$$

UP.076. Evaluate:

$$S = \sum_{n=1}^{\infty} \left(\frac{H_{2n} + 1}{n^2} \right)$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Ali Shather-Nasyria-Iraq

$$\begin{aligned}
 S &= \sum_{n=1}^{\infty} \frac{H_{2n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{H_{2n}}{n^2} + \sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)} = S_1 + S_2 \\
 S_1 &= \sum_{n=1}^{\infty} \frac{H_{2n}}{n^2} = 4 \sum_{n=1}^{\infty} \frac{H_{2n}}{(2n)^2} = 4 \sum_{n=1}^{\infty} \frac{H_n}{n^2} \left(\frac{1 + (-1)^n}{2} \right) \\
 &= 2 \sum_{n=1}^{\infty} \frac{H_n}{n^2} + 2 \sum_{n=1}^{\infty} (-1)^n \frac{H_n}{n^2} = 4\xi(3) - \frac{5}{4}\xi(3) = \frac{11}{4}\xi(3) \\
 S_2 &= \sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2} + 4 \sum_{n=1}^{\infty} \frac{1}{2n+1} - 2 \sum_{n=1}^{\infty} \frac{1}{n} \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + \frac{1}{2}} \right) = \xi(2) - 2H_{\frac{1}{2}} \\
 &= \xi(2) - 2(2 - 2 \ln 2) = \xi(2) - 4 + 4 \ln 2
 \end{aligned}$$

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$$\therefore S = S_1 + S_2 = \frac{11}{4} \xi(3) + \xi(2) + 4 \ln 2 - 4$$

Solution 2 by Khalef Ruhemi-Jarash-Jordan

$$\text{Let } S := \sum_{n=1}^{\infty} \frac{H_{2n+1}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \left(H_{2n} + \frac{1}{2n+1} \right)$$

$$= \sum_{n=1}^{\infty} \underbrace{\frac{1}{n^2(2n+1)}}_{I_1} + \sum_{n=1}^{\infty} \underbrace{\frac{H_{2n}}{n^2}}_{I_2}$$

$$\begin{aligned} I_2 &= \sum_{n=1}^{\infty} \frac{H_{2n}}{n^2} = \sum_{n=1}^{\infty} \frac{1}{n^2} \cdot \int_0^1 \left(\frac{1-x^{2n}}{1-x} \right) dx = \int_0^1 \left(\frac{1}{1-x} \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{x^{2n}}{n^2} \right) dx \\ &= \int_0^1 \left(\frac{1}{1-x} \right) \cdot \left(Li_2(1) - Li_2(x^2) \right) dx \end{aligned}$$

$$= \left(Li_2(1) - Li_2(x^2) \right) \ln(1-x) \Big|_0^1 + \int_0^1 \ln(1-x) \cdot \left(\ln \frac{(1-x^2)}{x^2} \cdot 2x \right) dx$$

$$= \int_0^1 2 \ln \frac{(1-x) \ln(1-x^2)}{x} dx = 2 \cdot \int_0^1 \ln \frac{(1-x) \cdot (\ln(1-x) + \ln(1+x))}{x} dx$$

$$= 2 \cdot \int_0^1 \ln^2 \frac{(1-x)}{x} dx + 2 \int_0^1 \ln \frac{(1-x) \ln(1+x)}{x} dx$$

$$= 2 \int_0^1 \ln^2 \frac{(1-x)}{x} \cdot dx + \frac{1}{2} \int_0^1 \underbrace{\frac{\ln^2(1-x^2)}{x}}_{\substack{\text{let } x^2=y \\ \frac{dx}{x} = \frac{dy}{2y}}} dx - \frac{1}{2} \int_0^1 \underbrace{\frac{\ln^2 \left(\frac{1-x}{1+x} \right)}{x}}_{\substack{\text{let } \frac{1-x}{1+x}=y \\ x = \frac{1-y}{1+y} \\ dx = -\frac{2dy}{(1+y)^2}}} dx$$

$$\therefore I_2 = 2 \cdot \int_0^1 \frac{\ln^2(1-x)}{x} + \frac{1}{4} \int_0^1 \frac{\ln^2(1-x)}{x} - \int_0^1 \frac{\ln^2 x dx}{(1-x)(1+x)}$$

$$= \frac{9}{4} \int_0^1 \frac{\ln^2 x}{1-x} - \int_0^1 \frac{\ln^2 x dx}{(1-x)(1+x)}$$

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$$\begin{aligned}
 &= \frac{9}{4} \int_0^1 \frac{\ln^2 x \, dx}{1-x} - \frac{1}{2} \int_0^1 \frac{\ln^2 x \, dx}{1-x} - \frac{1}{2} \int_0^1 \frac{\ln^2 x \, dx}{1+x} \\
 \therefore I_2 &= \left(\frac{7}{4}\right) \int_0^1 \frac{\ln^2 x \, dx}{1-x} - \left(\frac{1}{2}\right) \int_0^1 \frac{\ln^2 x \, dx}{1+x} \\
 &= \left(\frac{7}{4}\right) \cdot \left[\ln^2 x \ln(1-x) \Big|_1^0 + \int_0^1 2 \ln \frac{(1-x)}{x} \ln x \, dx \right] \\
 &\quad - \left(\frac{1}{2}\right) \left[\ln^2 x \ln(1+x) \Big|_0^1 - \int_0^1 2 \ln \frac{(1+x)}{x} \ln x \, dx \right] \\
 \therefore I_2 &= \left(\frac{7}{2}\right) \int_0^1 \ln x \cdot \ln \frac{(1-x)}{x} \, dx + \int_0^1 \ln x \frac{\ln(1+x)}{x} \, dx \\
 &= \left(\frac{7}{2}\right) \left[\ln x \cdot Li_2(x) \Big|_1^0 + \int_0^1 \frac{Li_2(x)}{x} \, dx \right] + \ln(x) Li_2(-x) \Big|_0^1 + \int_0^1 Li_2 \frac{(-x)}{x} \, dx \\
 &= \left(\frac{7}{2}\right) \int_0^1 \frac{Li_2(x)}{x} \, dx + \int_0^1 Li_2 \frac{(-x)}{x} \, dx = \left(\frac{7}{2}\right) \int_0^1 \frac{Li_2(x) \, dx}{x} + \int_0^{-1} \frac{Li_2(x)}{x} \, dx \\
 &\quad \text{let } -x=y \\
 &\quad \text{dx}=-dy \\
 &= \left(\frac{7}{2}\right) Li_3(1) + Li_3(-1) = \left(\frac{7}{2}\right) \varphi(3) - \frac{3}{4} \varphi(3) = \frac{11}{4} \varphi(3) \Rightarrow I_2 = \frac{11}{4} \varphi(3)
 \end{aligned}$$

$$\begin{aligned}
 \text{Let } I_1 &:= \sum_{n=1}^{\infty} \frac{1}{n^2(2n+1)} = \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right) \\
 &= \sum_{n=1}^{\infty} \frac{1}{n^2} - 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right) \therefore I_1 = \varphi(2) - 2 \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+\frac{1}{2}} \right)
 \end{aligned}$$

$$\begin{aligned}
 \text{But } \int_0^1 \underbrace{\left(\frac{1-x^2}{1-x} \right)}_{\substack{\text{let } x=y^2 \\ dx=2ydy}} \cdot dx &= \sum_{n=1}^{\infty} \frac{1}{n} - \frac{1}{n+\frac{1}{2}}
 \end{aligned}$$

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$$\begin{aligned} \therefore \sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n + \frac{1}{2}} \right) &= 2 \sum_{n=1}^{\infty} \frac{(1-x)}{(1-x^2)} \cdot x \, dx \\ &= 2 \cdot \int_0^1 \frac{x \, dx}{1+x} = 2 \int_0^1 \left(1 - \frac{1}{1+x} \right) dx = 2(x - \ln(1+x)) \Big|_0^1 = 2(1 - \ln(2)) \\ \therefore I_1 &= \varphi(2) - 4(1 - \ln(2)) \therefore S = I_1 + I_2 = \frac{11}{4} \varphi(3) + \varphi(2) + 4 \ln(2) - 4 \\ \therefore \sum_{n=1}^{\infty} \frac{H_{2n+1}}{n^2} &= \frac{11}{4} \varphi(3) + \frac{\pi^2}{6} + 4 \ln(2) - 4 \end{aligned}$$

UP.077. Evaluate:

$$S = \prod_{n=1}^{\infty} \left(e \left(\frac{n}{n+1} \right)^n \sqrt{\frac{n}{n+1}} \right)$$

Proposed by Shivam Sharma-New Delhi-India

Solution by Abdelhak Maoukuf-Casablanca-Morocco

$$\begin{aligned} L &= \prod_{n=1}^{\infty} \left(e \left(\frac{n}{n+1} \right)^n \sqrt{\frac{n}{n+1}} \right) = \lim_{p \rightarrow \infty} \prod_{n=1}^p \left(e \left(\frac{n}{n+1} \right)^n \sqrt{\frac{n}{n+1}} \right) \\ &= \lim_{p \rightarrow \infty} e^p \prod_{n=1}^p \left(\frac{n^n}{(n+1)^{n+1}} \sqrt{n(n+1)} \right) = \lim_{p \rightarrow \infty} e^p \prod_{n=1}^p \left(\frac{n^n}{(n+1)^{n+1}} \right) \prod_{n=1}^p \left(\sqrt{n(n+1)} \right) \\ &= \lim_{p \rightarrow \infty} e^p \frac{1^1}{(p+1)^{p+1}} \sqrt{p!(p+1)!} = \lim_{n \rightarrow \infty} \frac{e^p p!}{(p+1)^{p+\frac{1}{2}}} = \lim_{p \rightarrow \infty} \frac{e^p \left(\frac{p}{e} \right)^p \sqrt{2\pi p}}{(p+1)^{p+\frac{1}{2}}} \\ &= \lim_{p \rightarrow \infty} \sqrt{2\pi \frac{p}{p+1}} \left(\frac{p}{p+1} \right)^p = \lim_{p \rightarrow \infty} \sqrt{2\pi \frac{p}{p+1}} \left(\frac{1}{\left(1 + \frac{1}{p} \right)^p} \right) = \frac{\sqrt{2\pi}}{e} \therefore \lim_{p \rightarrow \infty} \left(1 + \frac{1}{p} \right)^p = e \end{aligned}$$

UP.078. Find:

$$\Omega = \lim_{n \rightarrow \infty} n \left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[2n]{(2n-1)!!} \right) \left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!} \right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

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Solution 1 by Kays Tomy-Nador-Tunisia

$$\Omega = n \left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[n]{(2n-1)!!} \right) \left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!} \right)$$

$$\Omega \rightarrow \infty$$

First term of limit $\left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!} \right) \sim \left(\exp \frac{\ln(2n+1)!}{2n+2} - \exp \frac{\ln(n!)}{2n} \right)$

$$\sim \left[\exp \left(\frac{(2n+2) \ln(2n+2) - (3n+2)}{2n+2} \right) - \exp \left(\frac{n \ln(n) - n}{2n} \right) \right]$$

$$\sim n \left[\exp \left(\frac{\ln(n+1) - 1}{2} \right) - \exp \left(\frac{\ln(n) - 1}{2} \right) \right]$$

$$\sim n \left(\sqrt{\frac{n+1}{e}} - \sqrt{\frac{n}{e}} \right) \sim \frac{1}{\sqrt{e}} \left(\frac{n}{\sqrt{n+1} + \sqrt{n}} \right) \sim \frac{1}{2} \sqrt{\frac{n}{e}} \rightarrow (*)$$

Now let us denote by $\theta_n = \frac{\ln(2n+1)!!}{2n+2} - \frac{\ln(2n-1)!!}{2n}$

$$\Rightarrow \theta_n \sim \frac{1}{2} \left\{ \frac{\ln(2n+1)!!}{n} - \frac{\ln(2n-1)!!}{n} \right\} \Rightarrow \theta_n \sim \frac{1}{2} \ln \left(\sqrt[n]{\frac{(2n+1)!!}{(2n-1)!!}} \right)$$

And as we know that $m! \cdot n! \leq (mn)!$. Whenever $m, n \geq 2$. Then

$$(2n)! \leq (2n+1)! (2n)! \leq ((2n+1)2n)!$$

$$\leq \frac{((2n+1)2n)! (2n-1)!!}{(2n-1)!!} \leq \frac{((2n+1)2n(2n-1))!}{(2n-1)!!} \leq \frac{(2n+1)!!}{(2n-1)!!}$$

Then it follows that $\frac{1}{2} \ln \sqrt[n]{(2n)!} \leq \frac{1}{2} \ln \sqrt[n]{\frac{(2n+1)!!}{(2n-1)!!}} \sim \theta_n$

$$\Rightarrow \frac{1 \ln(2n)!}{2n} \leq \theta_n \Rightarrow \frac{1}{2} \frac{2n \ln(2n) - 2n}{n} \sim \frac{1 \ln(2n)!}{2n} \leq \theta_n$$

$$\Rightarrow \underbrace{\ln(2n) - 1}_{\rightarrow \infty} \leq \theta_n \Rightarrow \theta_n \rightarrow \infty$$

Now lets look for the limit of $\sqrt[2n+2]{(2n+1)!!} - \sqrt[2n]{(2n-1)!!}$ (**)

By combining (*) and (**) we get

$$\Omega = n \left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[n]{(2n-1)!!} \right) \left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!} \right) \rightarrow \infty$$

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Solution 2 by Shivam Sharma-New Delhi-India

$$\Omega = \lim_{n \rightarrow \infty} n \left(((2n+1)!!)^{\frac{1}{2n+2}} - ((2n-1)!!)^{\frac{1}{2n}} \right) \left(((n+1)!)^{\frac{1}{2n+2}} - (n!)^{\frac{1}{2n}} \right)$$

As we know, $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$ & $(2n-1)!! = \frac{(2n)!}{2^n n!}$

Using this, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} n \left(\left(\frac{(2n+1)!}{2^n n!} \right)^{\frac{1}{2n+2}} - \left(\frac{(2n)!}{2^n n!} \right)^{\frac{1}{2n}} \right) \left(((n+1)!)^{\frac{1}{2n+2}} - (n!)^{\frac{1}{2n}} \right)$$

As we know, the Stirling's formula, we get $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. Using this, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} n \left[\frac{\left(\left(\frac{2n+1}{e} \right)^{2n+1} \sqrt{2\pi(2n+1)} \right)^{\frac{1}{2n+2}} - \left(\left(\frac{2n}{e} \right)^{2n} \right)^{\frac{1}{2n}}}{\left(2^n \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \right)^{\frac{1}{2n+2}}} - \frac{\left(\left(\frac{2n}{e} \right)^{2n} \right)^{\frac{1}{2n}}}{\left(2^n \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \right)} \right]$$

$$\left[\left(\left(\frac{n+1}{e} \right)^{n+1} \sqrt{2\pi(n+1)} \right)^{\frac{1}{2n+2}} - \left(\left(\frac{n}{e} \right)^n \sqrt{2\pi n} \right)^{\frac{1}{2n}} \right]$$

As we can see many terms are cancelling, Now, applying ratio test, we get. Our limit,

$$\Omega \rightarrow \infty. \text{ Hence } \Omega = \infty$$

Solution 3 by Abdelhak Maoukuf-Casablanca-Morocco

$$\Omega = \lim_{+\infty} n \left(\sqrt[2n+2]{(2n+1)!!} - \sqrt[2n]{(2n-1)!!} \right) \left(\sqrt[2n+2]{(n+1)!} - \sqrt[2n]{n!} \right)$$

$$= \lim_{+\infty} n \left(\sqrt[2n]{\frac{(2n+1)!}{2^n \times n!}} - \sqrt[2n]{\frac{(2n-1)!}{2^{n-1}(n-1)!}} \right) \left(\sqrt[2n]{(n+1)!} - \sqrt[2n]{n!} \right)$$

$$= \lim_{+\infty} n \left(\frac{(2n-1)!}{2^{n-1}(n-1)!} \right)^{\frac{1}{2n}} (n!)^{\frac{1}{2n}} \left(\left(\frac{(2n+1)!}{2^n n!} \right) \times \left(\frac{2^{n-1}(n-1)!}{(2n-1)!} \right)^{\frac{1}{2n}} \right)^{\frac{1}{2n}}$$

$$\cdot \left((n+1)^{\frac{1}{2n}} \times \frac{1}{(n!)^{\frac{1}{2n}}} - 1 \right)$$

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$$\begin{aligned}
 &= \lim_{+\infty} n \left(\frac{(2n-1)! \times n}{2^{n-1}} \right)^{\frac{1}{2n}} \left(\left(\frac{2n \times (2n+1)}{2 \times n} \right)^{\frac{1}{2n}} - 1 \right) \left((n+1)^{\frac{1}{2n}} - 1 \right) \\
 &= \lim_{+\infty} n \left(\frac{(2n)!}{2^n} \right)^{\frac{1}{2n}} \left(\sqrt[2n]{2n+1} - 1 \right) \left(\sqrt[2n]{n+1} - 1 \right) \\
 &= \lim_{+\infty} \frac{n}{\sqrt{2}} \cdot \left(\left(\frac{2n}{e} \right)^{2n} \sqrt{4\pi n} \right)^{\frac{1}{2n}} \left(\frac{e^{\frac{\ln(2n+1)}{2n}} - 1}{\frac{\ln(2n+1)}{2n}} \right) \left(\frac{e^{\frac{\ln(n+1)}{2n}} - 1}{\frac{\ln(n+1)}{2n}} \right) \times \\
 &\quad \times \frac{\ln(2n+1) \ln(n+1)}{4n^2} \\
 &\sim \lim_{+\infty} \frac{n}{\sqrt{2}} \times \left(\frac{2n}{e} \right) \times \frac{\ln(2n+1) \ln(n+1)}{4n^2} = \lim_{+\infty} \frac{\sqrt{2}}{4e} \times \ln(2n+1) \ln(n+1) \rightarrow +\infty
 \end{aligned}$$

UP.079. If $x, y, z > 0$ and $b \geq a > 0$ then:

$$\ln \frac{(x+b)(y+b)(z+b)}{(x+a)(x+b)(x+c)} \geq \frac{15}{8} \ln \frac{b}{a} + \frac{1}{16} \left(\frac{1}{b^2} - \frac{1}{a^2} \right) (x^2 + y^2 + z^2)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\begin{aligned}
 &\text{If } x, t > 0 \text{ then } \frac{t^3}{x+t} \geq \frac{5t^2-x^2}{8} \Leftrightarrow (x-t)^2(3t+x) \geq 0; \frac{1}{x+t} \geq \frac{5}{8t} - \frac{x^2}{8t^3} \\
 &\int_a^b \frac{dt}{x+t} \geq \int_a^b \left(\frac{5}{8t} - \frac{x^2}{8t^3} \right) dt \Leftrightarrow \ln \frac{b+x}{a+x} \geq \frac{5}{8} \ln \frac{b}{a} + \frac{x^2}{16} \left(\frac{1}{b^2} - \frac{1}{a^2} \right) \Rightarrow \\
 &\sum_{\text{cyclic}} \ln \frac{b+x}{a+x} \geq \sum_{\text{cyclic}} \left(\frac{5}{8} \ln \frac{b}{a} + \frac{x^2}{16} \left(\frac{1}{b^2} - \frac{1}{a^2} \right) \right) = \frac{15}{8} \ln \frac{b}{a} + \frac{1}{16} \left(\frac{1}{b^2} - \frac{1}{a^2} \right) (x^2 + y^2 + z^2)
 \end{aligned}$$

UP.080. Let be: $f: (0, \infty) \rightarrow (0, \infty)$ a function such that:

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = a \in (0, \infty) \text{ and } \lim_{x \rightarrow \infty} \left(\frac{f(x+1)}{f(x)} \right)^x = b \in (0, \infty). \text{ Find:}$$

$$\Omega = \lim_{x \rightarrow \infty} (f(x+1) - f(x))$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

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Solution 1 by Remus Florin Stanca-Romania

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{f(x+1)}{f(x)} &= \lim_{x \rightarrow \infty} \frac{f(x+1)}{x+1} \cdot \frac{x}{f(x)} \cdot \frac{x+1}{x} = 1 \\ \Rightarrow \lim_{x \rightarrow \infty} \left(\frac{f(x+1)}{f(x)} \right)^x &= \lim_{x \rightarrow \infty} \left(\frac{f(x+1) - f(x)}{f(x)} + 1 \right)^{\frac{f(x)}{f(x+1)-f(x)} \cdot x \cdot \frac{f(x+1)-f(x)}{f(x)}} = \\ &= e^{\lim_{x \rightarrow \infty} \frac{x}{f(x)} (f(x+1)-f(x))} = b \Rightarrow e^{\frac{\Omega}{a}} = b \Rightarrow \frac{\Omega}{a} = \ln(b) \Rightarrow \Omega = a \ln(b). \end{aligned}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{x \rightarrow \infty} \frac{f(x)}{x} = a \Rightarrow \lim_{n \in \mathbb{N}} \frac{f(n)}{n} = a \text{ and } \lim_{x \rightarrow \infty} \left(\frac{f(x+1)}{f(x)} \right)^x = b \Rightarrow \lim_{n \in \mathbb{N}} \left(\frac{f(n+1)}{f(n)} \right)^n = b$$

$$\lim_{x \rightarrow \infty} (f(x+1) - f(x)) = \lim_{\substack{x \rightarrow \infty \\ n \in \mathbb{N}}} (f(n+1) - f(n)) = \lim_{x \rightarrow \infty} \left(\frac{f(n)}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n \right)$$

$$\text{where } u_n = \frac{f(n+1)}{f(n)} \text{ for all } n \in \mathbb{N}$$

$$\left[\begin{array}{l} \lim_{x \rightarrow \infty} u_n = \lim_{x \rightarrow \infty} \left(\frac{f(n+1)}{n+1} \cdot \frac{n}{f(n)} \cdot \left(1 + \frac{1}{n} \right) \right) = 1 \\ \text{hence, } \frac{u_n - 1}{\ln u_n} \rightarrow 1 \text{ as } n \rightarrow \infty \end{array} \right]$$

$$\therefore \lim_{x \rightarrow \infty} (f(x+1) - f(x)) = a \ln b$$

UP.081. If

$$B_n(t) = n^{1-t} \left(\frac{(n+1)^{2t}}{(n+1)\sqrt{(n+1)!}^t} - \frac{n^{2t}}{(n\sqrt{n!})^t} \right), \text{ with } t > 0, \text{ then compute}$$

$$\lim_{n \rightarrow +\infty} B_n(t).$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Abdelhak Maoukouf-Casablanca-Morocco

$$B_n(t) = n^{1-t} \left(\frac{(n+1)^{2t}}{(n+1)\sqrt{(n+1)!}^t} - \frac{n^{2t}}{\sqrt[n]{n!}^t} \right)$$

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$$\begin{aligned}
 B_{n \rightarrow +\infty}(t) &\sim n^{1-t} \left(\frac{(n+1)^{2t}}{\left(\left(\frac{n+1}{e}\right)^{n+1} \sqrt{2\pi(n+1)}\right)^{\frac{t}{n+1}}} - \frac{n^{2t}}{\left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{t}{n}}} \right) \\
 &\sim n^{1-t} \left(\frac{(n+1)^{2t}}{\left(\frac{n+1}{e}\right)^t} - \frac{n^{2t}}{\left(\frac{n}{e}\right)^t} \right) \sim n^{1-t} ((n+1)^t e^t - n^t e^t) \\
 &\sim n^{1-t} x n^t x e^t \left(\left(\frac{n+1}{n}\right)^t - 1 \right) \sim n e^t \ln \left(\left(\frac{n+1}{n}\right)^t \right) \\
 &\sim n t e^t \ln \left(\frac{n+1}{n} \right) \sim n t e^t \left(\frac{n+1}{n} - 1 \right) \sim t e^t \Rightarrow \lim_{n \rightarrow +\infty} B_n(t) = t e^t
 \end{aligned}$$

Solution 2 by Kays Tomy-Nador-Tunisia

$$B_n(t) = n^{1-t} \left(\frac{(n+1)^{2t}}{\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^t} - \frac{n^{2t}}{\left(\frac{n}{\sqrt[n]{n!}}\right)^t} \right)$$

First of all, let us recall Stirling formula $\ln(n!) \sim n \ln(n) - n$. Then it follows

$$\begin{aligned}
 \left(\frac{n}{\sqrt[n]{n!}}\right)^t &= \left(\exp\left(\frac{\ln(n!)}{n}\right)\right)^t \sim \left(\exp\left(\frac{n \ln(n) - n}{n}\right)\right)^t \\
 &\sim (\exp(\ln(n) - 1))^t \sim (\exp(\ln(n)))^t e^{-t} \sim n^t \cdot e^{-t}
 \end{aligned}$$

likewise we get $\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^t \sim (n+1)^t \cdot e^{-t}$ then

$$\begin{aligned}
 B_n(t) &\sim n^{1-t} \left(\frac{(n+1)^{2t}}{\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^t} - \frac{n^{2t}}{\left(\frac{n}{\sqrt[n]{n!}}\right)^t} \right) \cdot e^t \sim n^{1-t} ((n+1)^t - n^t) e^t \\
 &\sim n \left(\left(1 + \frac{1}{n}\right)^t - 1 \right) e^t \sim n \left(1 + \frac{1}{n} - 1\right) \sim t \cdot e^t. \text{ Finally we get } B_n(t) \rightarrow_{\infty} t e^t = t e^t
 \end{aligned}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned}
 B_n(t) &= n^{1-t} \left(\frac{(n+1)^{2t}}{\left(\frac{n+1}{\sqrt[n+1]{(n+1)!}}\right)^t} - \frac{n^{2t}}{\left(\frac{n}{\sqrt[n]{n!}}\right)^t} \right) \\
 &= \left(\left(\frac{n}{\sqrt[n]{n!}}\right)^t \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n \right) \text{ where } u_n = \left(\frac{(n+1)^2}{n+1 \sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n]{n!}}{n^2} \right)^t \quad \forall n \in \mathbb{N}
 \end{aligned}$$

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$$\lim_{n \rightarrow \infty} \frac{n}{\sqrt[n]{n!}} \stackrel{\text{CAUCHY-D'ALEMBERT}}{=} \lim_{n \rightarrow \infty} \left(\frac{(n+1)^{n+1}}{(n+1)!} \cdot \frac{n!}{n^n} \right) = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n} \right)^n = e$$

$$\lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{n+1}{\sqrt[n+1]{(n+1)!}} \cdot \frac{\sqrt[n]{n!}}{n} \cdot \left(1 + \frac{1}{n} \right)^t \right) = 1, \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} = 1$$

$$\lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\frac{\sqrt[n+1]{(n+1)!}}{n+1} \cdot \left(1 + \frac{1}{n} \right) \cdot \left(1 + \frac{1}{n} \right)^{2n} \right)^t = e^t$$

$$\therefore \lim_{n \rightarrow \infty} B_n(t) = \lim_{n \rightarrow \infty} \left(\left(\frac{n}{\sqrt[n]{n!}} \right)^t \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n \right) = e^t \cdot \ln e^t = te^t \quad (\text{Ans :})$$

UP.082. Let $n \in \mathbb{N}$. Calculate

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x \left(\cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) + \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kays Tomy-Nador-Tunisia

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x \left(\cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) + \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx$$

$$\text{Let } J_n = \int_0^{\frac{\pi}{2}} \sin^2 x \cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) dx.$$

$$\text{And } T_n = \int_0^{\frac{\pi}{2}} \sin^2 x \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) dx$$

$\Rightarrow I_n = J_n + T_n$. As $\cos^2 x + \sin^2 x = 1$. Then we have

$$(*) \begin{cases} J_n = \int_0^{\frac{\pi}{2}} \cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) dx - \int_0^{\frac{\pi}{2}} \cos^3 x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) dx \\ T_n = \int_0^{\frac{\pi}{2}} \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) dx - \int_0^{\frac{\pi}{2}} \cos^2 x \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) dx \end{cases}$$

$$\text{Let us denote } K_n = \int_0^{\frac{\pi}{2}} \cos^3 x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) dx$$

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By substitution variable $x = \frac{\pi}{2} - U \Rightarrow \sin x = \cos U$ and $\cos x = \sin u$ and $dx = -du$

$$\Rightarrow K_n = - \int_{\frac{\pi}{2}}^0 \sin^3 u \cos^{2n} \left(\frac{\pi}{2} \cos u \right) du$$

$$\Rightarrow K_n = \int_0^{\frac{\pi}{2}} (1 - \cos^2 u) \sin(u) \cos^{2n+1} \left(\frac{\pi}{2} \cos(u) \right) du = T_n (**)$$

Then combining (*) and (**) we get $I_n = \int_0^{\frac{\pi}{2}} \cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) dx$

substituting $U = \frac{\pi}{2} \sin x \Rightarrow du = \frac{\pi}{2} \cos x dx \Rightarrow I_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^{2n+1}(u) du$

Then $I_n = \frac{2}{\pi} W_{2n+1}$; with w_{2n+1} is the Wallis integral for der $2n + 1$

We know that $w_{2n+1} = \frac{2^{2n}(n)!^2}{(2n+1)!}$. Finally we get $I_n = \frac{2^{2n+1}(n)!^2}{\pi (2n+1)!}$

Solution 2 by Shivam Sharma-New Delhi-India

As we know, the following Lemma,

If $f(x)$ is a continuous function defined on $[a, b]$, then,

$$\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$$

Using the above Lemma, we get,

$$2I_n = \int_0^{\frac{\pi}{2}} \left[\sin^2(x) \left(\cos(x) \cos^{2n+1} \left(\frac{\pi}{2} \sin(x) \right) + \sin(x) \cos^{2n+1} \left(\frac{\pi}{2} \cos(x) \right) \right) + \cos^2(x) \left(\sin(x) \cos^{2n+1} \left(\frac{\pi}{2} \cos(x) \right) + \cos(x) \cos^{2n+1} \left(\frac{\pi}{2} \sin(x) \right) \right) \right] dx$$

Applying the above Lemma again, we get,

$$2I_n = 2 \int_0^{\frac{\pi}{2}} \sin(x) \cos^{2n+1} \left(\frac{\pi}{2} \cos(x) \right) dx \text{ (OR) } I_n = \int_0^{\frac{\pi}{2}} \sin(x) \cos^{2n+1} \left(\frac{\pi}{2} \cos(x) \right) dx$$

Let, $\frac{\pi}{2} \cos(x) = u$

$$\Rightarrow \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^{2n+1}(u) du \Rightarrow \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^{2n+1}(u) du \Rightarrow \frac{2}{\pi} \cdot \frac{2}{2} \int_0^{\frac{\pi}{2}} \cos^{2n+1}(u) du$$

$$\Rightarrow \frac{2}{\pi} \cdot \frac{1}{2} B \left(n + 1, \frac{1}{2} \right) \Rightarrow \frac{1}{\pi} B \left(n + 1, \frac{1}{2} \right)$$

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(OR) $I_n = \frac{2^{2n+1}(n!)^2}{\pi(2n+1)!}$ $n > 0$ (Answer). For $n > 0$, we get, $I_n = \frac{2^{2n+1}(n!)^2}{\pi(2n+1)!}$

(Answer). For $n = 0$, we get, $I_n = \frac{2}{\pi}$ (Answer)

Solution 3 by Ravi Prakash-New Delhi-India

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x \left(\cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) + \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx \quad (1)$$

Using $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, we get (2)

$$I_n = \int_0^{\frac{\pi}{2}} \cos^2 x \left\{ \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) + \cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) \right\} dx \quad (3)$$

Adding (1) and (3), we get

$$2I_n = \int_0^{\frac{\pi}{2}} \left(\sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx + \int_0^{\frac{\pi}{2}} \cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) dx$$

Using (2) on 2nd integral, we get

$$2I_n = \int_0^{\frac{\pi}{2}} \left(\sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx + \int_0^{\frac{\pi}{2}} \left(\sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx$$

$$\therefore I_n = \int_0^{\frac{\pi}{2}} \left(\sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx$$

Put $\frac{\pi}{2} \cos x = \theta$; $-\frac{\pi}{2} \sin x dx = d\theta$; $x = 0, \theta = \frac{\pi}{2}$; $x = \frac{\pi}{2}, \theta = 0$

$$\therefore I_n = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} \cos^{2n+1} \theta d\theta$$

$$= \frac{2}{\pi} \cdot \frac{2n}{2n+1} \cdot \frac{2n-2}{2n-1} \cdots \frac{2}{3} = \frac{(2^n(n!))^2}{(2n+1)!} \cdot \frac{2}{\pi} = \frac{2^{2n+1}(n!)^2}{(2n+1)! \pi}$$

Solution 4 by Abdelhak Maoukuf-Casablanca-Morocco

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x \left(\cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) + \sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx$$

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$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x \left(\cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) \right) + \int_0^{\frac{\pi}{2}} \sin^2 x \left(\sin x \cos^{2n+1} \left(\frac{\pi}{2} \cos x \right) \right) dx$$

$$I_n = \int_0^{\frac{\pi}{2}} \sin^2 x \left(\cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) \right) + \int_0^{\frac{\pi}{2}} \cos^2 x \left(\cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) \right) dx$$

$$I_n = \int_0^{\frac{\pi}{2}} \cos x \cos^{2n+1} \left(\frac{\pi}{2} \sin x \right) dx$$

$$\frac{\pi}{2} \sin x = t \Rightarrow \cos x dx = \frac{2}{\pi} dt \quad (x = 0 \Leftrightarrow t = 0 \text{ \& } x = \frac{\pi}{2} \Leftrightarrow t = \frac{\pi}{2})$$

$$I_n = \int_0^{\frac{\pi}{2}} \frac{2}{\pi} \cos^{2n+1} t dt = \frac{2}{\pi} \int_0^{\frac{\pi}{2}} (1 - \sin^2 t) \cos^{2n-1} t dt$$

$$I_n = I_{n-1} - \frac{2}{\pi} \left(\left[\frac{-1}{2n} \cos^{2n} t \sin t \right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} \frac{-1}{2n} \cos^{2n+1} t dt \right) = I_{n-1} - \frac{1}{2n} I_n$$

$$\Rightarrow \frac{I_n}{I_{n-1}} = \frac{2n}{2n+1} \Rightarrow \prod_{n=1}^p \frac{I_n}{I_{n-1}} = \prod_{n=1}^p \frac{2n}{2n+1} \times \frac{2n}{2n}$$

$$\Rightarrow \frac{I_p}{I_0} = \frac{2^{2p} (p!)^2}{(2p+1)!} \text{ \& } I_0 = \int_0^{\frac{\pi}{2}} \frac{2}{\pi} \cos x dx = \frac{2}{\pi}; \quad I_n = \frac{2}{\pi} \cdot \frac{2^{2n} (n!)^2}{(2n+1)!}$$

UP.083. Prove that in any triangle ABC the following relationship holds:

$$R \sum (b + c - 2a)^2 \leq 4(R - 2r) \sum a^2$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} \sum (b + c - 2a)^2 &= b^2 + c^2 + 4a^2 + 2bc - 4ca - 4ab + c^2 + a^2 + 4b^2 + \\ &+ 2ca - 4ab - 4bc + a^2 + b^2 + 4c^2 + 2ab - 4bc - 4ca \end{aligned}$$

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$$\stackrel{(1)}{=} 6 \sum a^2 - 6 \sum ab$$

$$(1) \Rightarrow \text{given inequality is } R(6 \sum a^2 - 6 \sum ab) \leq 4(R - 2r) \sum a^2$$

$$\Leftrightarrow (R + 4r) \sum a^2 \leq 3R \left(\sum ab \right) \Leftrightarrow (2R + 8r)(s^2 - 4Rr - r^2) \leq 3R(s^2 + 4Rr + r^2)$$

$$\Leftrightarrow Rs^2 + (5R + 8r)(4R + r)r \geq 8rs^2 \quad (2)$$

$$\text{Now, LHS of (2)} \stackrel{\text{Gerretsen}}{\geq} Rr(16R - 5r) + (5R + 8r)(4R + r)r \stackrel{?}{\geq} 8rs^2$$

$$\Leftrightarrow R(16R - 5r) + (5R + 8r)(4R + r) \stackrel{?}{\geq} 8s^2 \Leftrightarrow 8s^2 \stackrel{?}{\leq} 36R^2 + 32Rr + 8r^2$$

$$\Leftrightarrow 2s^2 \stackrel{?}{\leq} 9R^2 + 8Rr + 2r^2 \quad (3)$$

$$\text{Now, LHS of (3)} \stackrel{\text{Gerretsen}}{\leq} 8R^2 + 8Rr + 6r^2 \stackrel{?}{\leq} 9R^2 + 8Rr + 2r^2$$

$$\Leftrightarrow R^2 \stackrel{?}{\geq} 4r^2 \Leftrightarrow R \stackrel{?}{\geq} 2r \rightarrow \text{true by Euler (Proved)}$$

UP.084. Evaluate

$$I = \int_0^1 \int_0^1 \frac{(\ln(x) \ln(y))^s}{1 - xy} dx dy$$

Proposed by Shivam Sharma-New Delhi-India

Solution by Khalef Ruhemi-Jarash-Jordan

$$\text{Evaluate } I := \int_0^1 \int_0^1 \frac{(\ln(x) \ln(y))^s}{1 - xy} \cdot dx \cdot dy \quad (*)$$

$$I = \int_0^1 \int_0^1 \frac{\left(\ln\left(\frac{1}{x}\right) \cdot \ln\left(\frac{1}{y}\right)\right)^s}{1 - xy} \cdot dx \cdot dy = \int_0^1 \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s \cdot \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot \sum_{n=0}^{\infty} x^n \cdot y^n \cdot dx \cdot dy$$

$$= \sum_{n=0}^{\infty} \int_0^1 \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s \cdot x^n \cdot \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot y^n \cdot dx \cdot dy$$

$$= \sum_{n=0}^{\infty} \left(\int_0^1 \left(\ln\left(\frac{1}{y}\right)\right)^s \cdot y^n \cdot dy \cdot \int_0^1 \left(\ln\left(\frac{1}{x}\right)\right)^s \cdot x^n \cdot dx \right)$$

$$= \sum_{n=0}^{\infty} \left(\int_0^1 x^n \cdot \left(\ln\left(\frac{1}{x}\right)\right)^s dx \right)^2 = I \quad (1)$$

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To find $\int_0^1 x^n \left(\ln\left(\frac{1}{x}\right)\right)^s dx := A$, let $\ln\left(\frac{1}{x}\right) = v \Rightarrow x = e^{-v} \Rightarrow dx = -e^{-v} dv$

$$\therefore A = \int_0^{\infty} v^s \cdot e^{-(1+n)v} dv = \frac{\Gamma(1+s)}{(1+n)^{1+s}} \therefore I = \sum_{n=0}^{\infty} \frac{\Gamma(1+s)}{(1+n)^{2(1+s)}}$$

$$\therefore I = \sum_{n=0}^{\infty} \frac{\Gamma^2(1+s)}{n^{2(1+s)}} = \Gamma^2(1+s) \cdot \sum_{n=0}^{\infty} \frac{1}{n^{2+2s}} = \Gamma^2(1+s) \mathcal{G}(2+2s), s > -\frac{1}{2}$$

$$\therefore \int_0^1 \int_0^1 \frac{(\ln(x) \ln(y))^s}{1-xy} dx dy = \Gamma^2(1+s) \mathcal{G}(2+2s), s > -\frac{1}{2}$$

UP.085. Let k be positive integer. Calculate:

$$\lim_{x \rightarrow \infty} \left((\Gamma(x+2))^{\frac{k+1}{x+1}} - (\Gamma(x+1))^{\frac{k+1}{x}} \right) (\Gamma(x+1))^{-\frac{k}{x}},$$

where $\Gamma(x)$ is the Gamma function (or Euler's second integral).

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} & \lim_{n \rightarrow \infty} \left((\Gamma(n+2))^{\frac{k+1}{n+1}} - (\Gamma(n+1))^{\frac{k+1}{n}} \right) \cdot (\Gamma(n+1))^{-\frac{k}{n}} \\ &= \lim_{\substack{n \rightarrow \infty \\ n \in \mathbb{N}}} \left((\Gamma(n+2))^{\frac{k+1}{n+1}} - (\Gamma(n+1))^{\frac{k+1}{n}} \right) \cdot (\Gamma(n+1))^{-\frac{k}{n}} \\ &= \lim_{n \rightarrow \infty} \left(\left((n+1)\sqrt{(n+1)!} \right)^{k+1} - (n\sqrt{n!})^{k+1} \right) \cdot (n\sqrt{n!})^{-k} \\ &= \lim_{n \rightarrow \infty} \left(\frac{n\sqrt{n!}}{n} \cdot \frac{u_n - 1}{\ln u_n} \cdot \ln u_n^n \right) \text{ where } u_n = \left(\frac{(n+1)\sqrt{(n+1)!}}{n\sqrt{n!}} \right)^{k+1} \quad \forall n \in \mathbb{N} \\ \text{Now, } \lim_{n \rightarrow \infty} \frac{n\sqrt{n!}}{n} &= \frac{1}{e}, \lim_{n \rightarrow \infty} u_n = \lim_{n \rightarrow \infty} \left(\frac{(n+1)\sqrt{(n+1)!}}{n+1} \cdot \frac{n}{n\sqrt{n!}} \cdot \frac{n+1}{n} \right)^{k+1} = 1 \\ \therefore \lim_{n \rightarrow \infty} \frac{u_n - 1}{\ln u_n} &= 1 \cdot \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \cdot \frac{n+1}{n\sqrt{(n+1)!}} \right)^{k+1} = e^{k+1} \\ \therefore \lim_{n \rightarrow \infty} \left((\Gamma(n+2))^{\frac{k+1}{n+1}} - (\Gamma(n+1))^{\frac{k+1}{n}} \right) \cdot (\Gamma(n+1))^{-\frac{k}{n}} \\ &= \frac{\ln e^{k+1}}{e} = \frac{k+1}{e} \end{aligned}$$

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UP.086. Let $a > 0, b, c > 1$ and $f, g: \mathbb{R} \rightarrow \mathbb{R}$ be continuous and odd functions. Prove that:

$$\int_{-a}^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx = (\ln(bc)) \int_0^a f(x) g(x) dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Shivam Sharma-New Delhi-India

Let

$$I = \int_{-a}^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx$$

As we know, the following Lemma,

If $f(x)$ is a continuous function defined on $[-a, a]$, then,

$$\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx, & \text{if } f(x) \text{ is an even function} \\ 0, & \text{if } f(x) \text{ is an odd function} \end{cases}$$

Using the above lemma, we get,

$$\begin{aligned} \Rightarrow \int_{-a}^a f(-x) \ln(b^{g(-x)} + c^{g(-x)}) dx &\Rightarrow - \int_{-a}^a f(x) \ln(b^{-g(x)} + c^{-g(x)}) dx \\ &\Rightarrow - \int_{-a}^a f(x) \ln(b^{g(x)} + c^{g(x)}) + \int_{-a}^a f(x) \ln((bc)^{g(x)}) dx \end{aligned}$$

$$2I \Rightarrow \int_{-a}^a f(x) g(x) \ln(bc) dx \text{ (OR) } 2I = 2 \ln(bc) \int_0^a f(x) g(x) dx \text{ (OR)}$$

$$I = \ln(bc) \int_0^a f(x) g(x) dx \text{ (Proved)}$$

Solution 2 by Hasan Bostanlik-Sarkisla-Turkey

$$\begin{aligned} A &= \int_{-a}^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx \\ x = -u &\Rightarrow - \int_a^{-a} f(-u) \ln\left(\frac{1}{b^{g(u)}} + \frac{1}{c^{g(u)}}\right) du \end{aligned}$$

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$$A = - \int_{-a}^a f(u) (\ln(b^{g(u)} + c^{g(u)}) - g(x) \ln(b \cdot c)) du$$

$$2A = \int_{-a}^a f(u) g(u) \ln(bc) du \Rightarrow h(x) = g(x) f(x)$$

$$h(-x) = g(x) f(x) = h(x)$$

$$2A = 2 \int_0^a f(u) g(u) \ln(bc) du$$

$$A = \int_0^a f(u) g(u) h(bc) du$$

Solution 3 by Abdelhak Maoukouf-Casablanca-Morocco

$$\begin{aligned} I &= \int_{-a}^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx \\ &= \int_0^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx + \int_a^0 f(-t) \ln(b^{g(-t)} + c^{g(-t)}) d(-t) \\ &= \int_0^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx + \int_0^a -f(t) \ln(b^{-g(t)} + c^{-g(t)}) dt \\ &= \int_0^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx - \int_0^a f(x) \ln(b^{-g(x)} c^{-g(x)} (c^{g(x)} + b^{g(x)})) dx \\ &= \int_0^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx - \int_0^a f(x) (\ln(b^{-g(x)} c^{-g(x)}) + (c^{g(x)} + b^{g(x)})) dx \\ &= - \int_0^a f(x) \ln((bc)^{-g(x)}) dx = \ln(bc) \int_0^a f(x) g(x) dx \end{aligned}$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

$$f(x) + f(-x) = 0, g(x) + g(-x) = 0$$

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$$\begin{aligned}
 \Omega &= \int_{-a}^a f(x) \ln(b^{g(x)} + c^{g(x)}) dx = \int_{-a}^a f(a-a-x) \ln(b^{g(a-a-x)} + c^{g(a-a-x)}) dx \\
 &= \int_{-a}^a f(-x) \ln(b^{g(-x)} + c^{g(-x)}) dx = - \int_{-a}^a f(x) \ln(b^{-g(x)} + c^{-g(x)}) dx \\
 &= - \int_{-a}^a f(x) \ln(b^{g(x)} + c^{g(x)}) + \ln(bc) \int_{-a}^a f(x) g(x) dx \\
 &\Rightarrow 2\Omega = \ln(bc) \int_{-a}^a f(x) g(x) dx = 2 \ln(bc) \int_0^a f(x) g(x) dx \\
 \Omega &= \ln(bc) \int_0^a f(x) g(x) dx \quad (\text{Proved})
 \end{aligned}$$

UP.087. Let $a, b \in \mathbb{R}, a < b$ and continuous functions

$f, g, h: \mathbb{R} \rightarrow \mathbb{R}$ such that $f(a+b-x) = -f(x), g(a+b-x) = g(x),$
 $h(a+b-x) = -h(x), \forall x \in \mathbb{R}$. Prove that

$$\int_a^b f(x) (\arctan(x)) \ln(1 + e^{h(x)}) dx = \frac{1}{2} \int_a^b f(x) h(x) \arctan(x) dx$$

Proposed by D.M. Băţineţu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Shivam Sharma-New Delhi-India

$$\int_a^b f(x) (\tan^{-1} g(x)) \ln(1 + e^{h(x)}) dx = \frac{1}{2} \int_a^b f(x) h(x) \tan^{-1}(x) dx$$

As we know, the following Lemma,

If $f(x)$ is a continuous function defined on $[a, b]$, then,

$$\int_a^b f(x) dx = \int_a^b f(a+b-x) dx$$

Using the above Lemma, we get. Let,

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$$I \Rightarrow \int_a^b f(x) \tan^{-1}(g(x)) \ln(1 + e^{h(x)}) dx$$

$$\Rightarrow \int_a^b f(a + b - x) \tan^{-1}(g(a + b - x)) \ln(1 + e^{h(a+b-x)}) dx$$

Given: $f(a + b - x) = -f(x)$; $g(a + b - x) = g(x)$; $h(a + b - x) = -h(x)$

Use the above given values, we get,

$$\Rightarrow \int_a^b (f(x)) \tan^{-1}(g(x)) \ln(1 + e^{-h(x)}) dx$$

$$\Rightarrow - \int_a^b f(x) \tan^{-1}(g(x)) \ln(1 + e^{h(x)}) dx + \int_a^b f(x) \tan^{-1}(g(x)) \ln(e^{h(x)}) dx$$

$$2I = \int_a^b f(x) \tan^{-1}(g(x)) \ln(e^{h(x)}) dx; 2I = \int_a^b f(x) h(x) \tan^{-1}(g(x)) dx$$

$$(OR) I = \frac{1}{2} \int_a^b f(x) h(x) \tan^{-1}(g(x)) dx \text{ (Proved)}$$

Solution 2 by Abdelhak Maoukouf-Casablanca-Morocco

$$I = \int_a^b f(x) \arctan(g(x)) \ln(1 + e^{h(x)}) dx =$$

$$= \int_a^b f(x) \arctan(g(x)) \ln(e^{h(x)}(e^{-h(x)} + 1)) dx$$

$$= \int_a^b f(x) \arctan(g(x)) (h(x) + \ln(e^{-h(x)} + 1)) dx$$

$$= \int_a^b f(x) h(x) \arctan(g(x)) dx +$$

$$+ \int_a^b f(a + b - t) \arctan(g(a + b - t)) \ln(e^{-h(a+b-t)} + 1) d(a + b - t)$$

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$$\begin{aligned}
 &= \int_a^b f(x) h(x) \arctan(g(x)) dx - \int_a^b f(x) \arctan(g(x)) \ln(1 + e^{h(x)}) dx \\
 \Leftrightarrow I &= \int_a^b f(x) h(x) \arctan(g(x)) dx - I \Leftrightarrow I = \frac{1}{2} \int_a^b f(x) h(x) \arctan(g(x)) dx
 \end{aligned}$$

Solution 3 by Anas Adlany-El Zemamra-Morocco

$$\begin{aligned}
 B &= \int_a^b f(x) \arctan(g(x)) \ln(1 + e^{h(x)}) dx \quad (u = a + b - x \Rightarrow du = -dx) \\
 &= - \int_a^b f(a + b - u) \arctan(g(a + b - x)) \ln(1 + e^{h(a+b-u)}) du \\
 &= \int_a^b -f(x) \arctan(g(x)) \ln\left(\frac{1 + e^{h(x)}}{e^{h(x)}}\right) dx \\
 &= - \int_a^b f(x) \arctan(g(x)) [\ln(1 + e^{h(x)}) - h(x)] dx \\
 &= -B + \int_a^b f(x) h(x) \arctan(g(x)) dx \Rightarrow \\
 \int_a^b f(x) \arctan(g(x)) \ln(1 + e^{h(x)}) dx &= \frac{1}{2} \int_a^b f(x) h(x) \arctan(g(x)) dx. !
 \end{aligned}$$

UP.088. Let $f: R \rightarrow R$ be a continuous function such that

$$f(x) = f(1 - x), \forall x \in R. \text{ Prove that}$$

$$\int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{1 + \sqrt{2x}} f(x) dx = \sqrt{2} \cdot \int_0^1 f(x) dx$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

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Solution 1 by Thanasis Xenos-Greece

$$I = \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{1 + \sqrt{2x}} f(x) dx \quad (1)$$

$$t = 1 - x$$

$$I = \int_1^0 \frac{\sqrt{t} + \sqrt{1-t}}{1 + \sqrt{2(1-t)}} f(1-t) (-dt)$$

$$I = \int_0^1 \frac{\sqrt{x} + \sqrt{1-x}}{1 + \sqrt{2(1-x)}} f(x) dx \quad (2)$$

$$(1) + (2) \Rightarrow 2I = \int_0^1 (\sqrt{x} + \sqrt{1-x}) \cdot \left(\frac{1}{1 + \sqrt{2x}} + \frac{1}{1 + \sqrt{2(1-x)}} \right) f(x) dx$$

$$2I = \int_0^1 (\sqrt{x} + \sqrt{1-x}) \cdot \frac{2 + \sqrt{2x} + \sqrt{2(1-x)}}{(1 + \sqrt{2x}) \cdot (1 + \sqrt{2(1-x)})} f(x) dx$$

$$2I = \sqrt{2} \int_0^1 \frac{(\sqrt{x} + \sqrt{1-x})(\sqrt{2} + \sqrt{x} + \sqrt{1-x})}{(1 + \sqrt{2x}) \cdot (1 + \sqrt{2(1-x)})} f(x) dx \quad (3)$$

$$(\sqrt{x} + \sqrt{1-x}) \cdot (\sqrt{2} + \sqrt{x} + \sqrt{1-x})$$

$$= \sqrt{2x} + x + \sqrt{x(1-x)} + \sqrt{2(1-x)} + \sqrt{x(1-x)} + 1 - x$$

$$= 1 + \sqrt{2x} + \sqrt{2(1-x)} + 2\sqrt{x(1-x)}$$

$$= (1 + \sqrt{2x})(1 + \sqrt{2(1-x)})$$

$$(3) \Rightarrow 2I = \sqrt{2} \cdot \int_0^1 f(x) dx \Rightarrow I = \frac{\sqrt{2}}{2} \cdot \int_0^1 f(x) dx$$

Solution 2 by Shivam Sharma-New Delhi-India

As we know the following Lemma,

If $f(x)$ is a continuous function defined on $[0, a]$, then,

$$\int_0^a f(x) dx = \int_0^a f(a-x) dx \quad (I)$$

Using the above lemma, we get $\Rightarrow \int_0^1 \frac{\sqrt{x} + \sqrt{1-x}}{1 + \sqrt{2(1-x)}} f(1-x) dx$. As it is given,

$$f(x) = f(1-x) \quad (II)$$

Using (II), we get, $\Rightarrow \int_0^1 \frac{\sqrt{x} + \sqrt{1-x}}{1 + \sqrt{2(1-x)}} f(x) dx$. Now,

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$$2I = \int_0^1 f(x) \left(\frac{1}{1 + \sqrt{2x}} + \frac{1}{1 + \sqrt{2(1-x)}} \right) (\sqrt{1-x} + \sqrt{x}) dx$$

$$2I = \sqrt{2} \int_0^1 f(x) \left(\frac{1 + \sqrt{2}\sqrt{1-x} + 1 + \sqrt{2}\sqrt{x}}{2x(1-x) + 1 + \sqrt{2}\sqrt{1-x} + \sqrt{2}\sqrt{x}} \right) (\sqrt{1-x} + \sqrt{x}) dx$$

$$2I = \sqrt{2} \int_0^1 f(x) \left(\frac{1 + 2x(1-x) + \sqrt{2}\sqrt{1-x} + \sqrt{2}\sqrt{x}}{1 + 2x(1-x) + \sqrt{2}\sqrt{1-x} + \sqrt{2}\sqrt{x}} \right) dx$$

$$2I = \sqrt{2} \int_0^1 f(x) dx \quad (\text{OR}) \quad 2I = \sqrt{2} \int_0^1 f(x) dx \quad (\text{OR}) \quad I = \frac{1}{\sqrt{2}} \int_0^1 f(x) dx \quad (\text{Proved})$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$\Omega = \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{1 + \sqrt{2x}} f(x) dx = \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{1 + \sqrt{2(1-x)}} f(1-x) dx$$

Applying $g(a+b-x) = g(x)$

$$\therefore 2\Omega = \int_0^1 (\sqrt{x} + \sqrt{1-x}) \left(\frac{1}{1 + \sqrt{2x}} + \frac{1}{1 + \sqrt{2(1-x)}} \right) f(x) dx$$

$$= \int_0^1 (\sqrt{x} + \sqrt{1-x}) \cdot \frac{2 + \sqrt{2x} + \sqrt{2(1-x)}}{(1 + \sqrt{2x})(1 + \sqrt{2(1-x)})} f(x) dx$$

$$= \int_0^1 \frac{\sqrt{2}f(x)(\sqrt{x} + \sqrt{1-x})(\sqrt{2} + \sqrt{x} + \sqrt{1-x})}{\sqrt{2}(\sqrt{x} + \sqrt{1-x}) + (\sqrt{x} + \sqrt{1-x})^2} dx = \sqrt{2} \int_0^1 f(x) dx$$

$$\therefore \int_0^1 \frac{\sqrt{x} + \sqrt{1-x}}{1 + \sqrt{2x}} f(x) dx = \frac{1}{\sqrt{2}} \int_0^1 f(x) dx$$

Solution 4 by Hasan Bostanlik-Sarkisla-Turkey

$$I = \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{1 + \sqrt{2}\sqrt{x}} f(x) dx$$

$$x = 1 - u \Rightarrow - \int_1^0 \frac{\sqrt{u} + \sqrt{1-u}}{1 + \sqrt{2}\sqrt{1-u}} f(1-u) (-du) = \int_0^1 \frac{\sqrt{u} + \sqrt{1-u}}{1 + \sqrt{2}\sqrt{1-u}} f(u) du$$

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$$\begin{aligned}
 2I &= \int_0^1 (\sqrt{u} + \sqrt{1-u}) f(u) \left(\frac{1}{1 + \sqrt{2}\sqrt{1-u}} + \frac{1}{1 + \sqrt{2}\sqrt{u}} \right) du \\
 2I &= \sqrt{2} \int (\sqrt{u} + \sqrt{1-u}) (f(u)) \left(\frac{\sqrt{2} + \sqrt{1-u} + \sqrt{u}}{(1 + \sqrt{2}\sqrt{1-u})(1 + \sqrt{2}\sqrt{u})} \right) du \\
 2I &= \sqrt{2} \int_0^1 f(u) \left[\frac{\sqrt{2}\sqrt{u} + \sqrt{2}\sqrt{1-u} + \sqrt{u}\sqrt{1-u} + 1 - u + u + \sqrt{u}\sqrt{1-u}}{1 + \sqrt{2}\sqrt{u} + \sqrt{2}\sqrt{1-u} + 2\sqrt{1-u}\sqrt{u}} \right] du \\
 2I &= \sqrt{2} \int_0^1 f(u) \left[\frac{1 + \sqrt{2}\sqrt{u} + \sqrt{2}\sqrt{1-u} + 2\sqrt{u}\sqrt{1-u}}{1 + 2\sqrt{u} + \sqrt{2}\sqrt{1-u} + 2\sqrt{u}\sqrt{u-1}} \right] du \\
 2I &= \sqrt{2} \int_0^1 f(u) du; \quad I = \frac{1}{\sqrt{2}} \int_0^1 f(u) du
 \end{aligned}$$

Solution 5 by Nirapada Pal-Jhargram-India

$$\begin{aligned}
 \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{\sqrt{2}(1 + \sqrt{2x})} f(x) dx &= \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{\sqrt{2} + 2\sqrt{x}} f(x) dx \\
 &= \frac{1}{2} \int_0^1 \frac{2\sqrt{1-x} + 2\sqrt{x} + \sqrt{2} - \sqrt{2}}{\sqrt{2} + 2\sqrt{x}} f(x) dx = \frac{1}{2} \int_0^1 f(x) dx + \frac{1}{2} \int_0^1 \frac{\sqrt{2(1-x)} - 1}{1 + \sqrt{2x}} f(x) dx
 \end{aligned}$$

$$\text{Let } I = \int_0^1 \frac{\sqrt{2(1-x)} - 1}{1 + \sqrt{2x}} f(x) dx. \text{ Or,}$$

$$I = \int_0^1 \frac{\sqrt{2x} - 1}{1 + \sqrt{2(1-x)}} f(1-x) dx = \int_0^1 \frac{\sqrt{2x} - 1}{1 + \sqrt{2(1-x)}} f(x) dx$$

$$\text{Or, } 2I = \int_0^1 \left[\frac{2(1-x) - 1 + 2x - 1}{(1 + \sqrt{2x})(1 + \sqrt{2(1-x)})} \right] f(x) dx = 0. \text{ Or, } I = 0$$

$$\text{So, } \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{\sqrt{2}(1 + \sqrt{2x})} f(x) dx = \frac{1}{2} \int_0^1 f(x) dx. \text{ Or, } \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{1 + \sqrt{2x}} f(x) dx = \frac{1}{\sqrt{2}} \int_0^1 f(x) dx$$

Solution 6 by Abdelhak Maoukuf-Casablanca-Morocco

$$I = \int_0^1 \frac{\sqrt{1-x} + \sqrt{x}}{1 + \sqrt{2x}} f(x) dx = \int_0^1 \frac{1-2x}{1-2x} \cdot \frac{1-\sqrt{2x}}{\sqrt{1-x}-\sqrt{x}} f(x) dx$$

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$$\begin{aligned}
 I &= \int_0^1 \frac{1 - \sqrt{2x}}{\sqrt{1-x} - \sqrt{x}} f(x) dx = \int_0^1 \frac{1 - \sqrt{2(1-x)}}{\sqrt{x} - \sqrt{1-x}} f(x) dx \\
 &\quad \text{"} f(x) = f(1-x) \text{"} \\
 &= \int_0^1 \frac{1 - \sqrt{2x} + \sqrt{2x} - \sqrt{2(1-x)}}{\sqrt{x} - \sqrt{1-x}} f(x) dx = -I + \int_0^1 \frac{\sqrt{2x} - \sqrt{2(1-x)}}{\sqrt{x} - \sqrt{1-x}} f(x) dx \\
 &\rightarrow 2I = \sqrt{2} \int_0^1 f(x) dx \Leftrightarrow I = \frac{\sqrt{2}}{2} \int_0^1 f(x) dx
 \end{aligned}$$

UP.089. Evaluate:

$$\int_0^1 [\ln(x) \ln(1-x) + Li_2(x)] \left(\frac{Li_2(x)}{x(1-x)} - \frac{\zeta(2)}{1-x} \right) dx$$

Proposed by Shivam Sharma-New Delhi-India

Solution by Ali Shather-Iraq

$$\begin{aligned}
 I &= \int_0^1 (\ln(x) \ln(1-x) + Li_2(x)) \left(\frac{Li_2(x)}{x(1-x)} - \frac{\zeta(2)}{1-x} \right) dx \\
 I &= \int_0^1 \frac{\ln(x) \ln(1-x) Li_2(x)}{x} dx + \int_0^1 \frac{\ln(x) \ln(1-x) Li_2(x)}{1-x} dx - \\
 &\quad - \zeta(2) \int_0^1 \frac{\ln(x) \ln(1-x)}{1-x} dx + \int_0^1 \frac{Li_2^2(x)}{x} dx + \int_0^1 \frac{Li_2^2(x) - \zeta(2) Li_2(x)}{1-x} dx
 \end{aligned}$$

combining the second and the last term, we get

$$\begin{aligned}
 \int_0^1 \frac{Li_2^2(x) - Li_2(x) [\zeta(2) - \ln(x) \ln(1-x)]}{1-x} dx &= \int_0^1 \frac{Li_2^2(x) - Li_2(x) [Li_2(x) + Li_2(1-x)]}{1-x} dx = \\
 &= - \int_0^1 \frac{Li_2(x) Li_2(1-x)}{1-x} dx = - \int_0^1 \frac{Li_2(1-x) Li_2(x)}{x} dx =
 \end{aligned}$$

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$$\begin{aligned}
 &= - \int_0^1 \frac{Li_2(x)}{x} [\zeta(2) - \ln(x) \ln(1-x) - Li_2(x)] dx = -\zeta(2) \int_0^1 \frac{Li_2(x)}{x} dx + \\
 &\quad + \int_0^1 \frac{\ln(x) \ln(1-x) Li_2(x)}{x} dx + \int_0^1 \frac{Li_2^2(x)}{x} dx \\
 \therefore I &= \int_0^1 \frac{\ln(x) \ln(1-x) Li_2(x)}{x} dx - \zeta(2) \int_0^1 \frac{\ln(x) \ln(1-x)}{1-x} dx + \int_0^1 \frac{Li_2^2(x)}{x} dx - \\
 &\quad - \zeta(2) \int_0^1 \frac{Li_2(x)}{x} dx + \int_0^1 \frac{\ln(x) \ln(1-x) Li_2(x)}{x} dx + \int_0^1 \frac{Li_2^2(x)}{x} dx \\
 I &= 2 \int_0^1 \frac{\ln(x) \ln(1-x) Li_2(x)}{x} dx - \zeta(2) \int_0^1 \frac{\ln(x) \ln(1-x)}{1-x} dx + 2 \int_0^1 \frac{Li_2^2(x)}{x} dx - \zeta(2) \int_0^1 \frac{Li_2(x)}{x} dx \\
 I &= [-Li_2^2(x) \ln(x)]_0^1 + \int_0^1 \frac{Li_2^2(x)}{x} - \zeta(2) \int_0^1 \frac{\ln(1-x) \ln(x)}{x} dx + 2 \int_0^1 \frac{Li_2^2(x)}{x} dx - \zeta(2) \zeta(3) \\
 I &= 3 \int_0^1 \frac{Li_2^2(x)}{x} dx + \zeta(2) \sum_{n=1}^{\infty} \frac{1}{n} \int_0^1 x^{n-1} \ln(x) dx - \zeta(2) \zeta(3) = \\
 &= 3 \int_0^1 \frac{Li_2^2(x)}{x} dx - \zeta(2) \sum_{n=1}^{\infty} \frac{1}{n^3} - \zeta(2) \zeta(3) = 3 \int_0^1 \frac{Li_2^2(x)}{x} dx - 2\zeta(2) \zeta(3) \\
 \int_0^1 \frac{Li_2^2(x)}{x} dx &= \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(nk)^2} \int_0^1 x^{n+k-1} dx = \sum_{n=1}^{\infty} \sum_{k=1}^{\infty} \frac{1}{(nk)^2(n+k)} = \\
 &= \sum_{n=1}^{\infty} \frac{1}{k^3} \sum_{n=1}^{\infty} \frac{k}{n^2(n+k)} = \sum_{n=1}^{\infty} \frac{1}{k^3} \left[\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{n(n+k)} \right] \\
 &= \sum_{k=1}^{\infty} \frac{1}{k^3} \left[\sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{H_k}{k} \right] = \zeta(3) \zeta(2) - \sum_{n=1}^{\infty} \frac{H_k}{k^4} = \zeta(3) \zeta(2) - (3\zeta(5) - \zeta(3) \zeta(2)) = \\
 &= 2\zeta(3) \zeta(2) - 3\zeta(5) \\
 \therefore I &= 3(2\zeta(2) \zeta(3) - 3\zeta(5)) - 2\zeta(2) \zeta(3) = 4\zeta(2) \zeta(3) - 9\zeta(5)
 \end{aligned}$$

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UP.090. Evaluate:

$$\int_0^1 (\ln(\Gamma(x))) (\sin(2k\pi x)) dx, \quad k \geq 1$$

Proposed by Shivam Sharma – New Delhi – India

Solution by Khalef Ruhemi-Jarash-Jordan

$$I = \int_0^1 \ln(\Gamma(x)) \sin(2\pi kx) dx \dots \#$$

$$\text{Notice that } \int_0^1 \frac{1-t^{x-1}}{1-t} \cdot dt = \frac{\Gamma'(x)}{\Gamma(x)} - \Gamma'(1)$$

$$\begin{aligned} \therefore \int_0^1 \left(\frac{1}{1-t} \right) \left(\int_1^x (1-t^{v-1}) dv \right) \cdot dt &= \int_1^x \left(\frac{\Gamma'(v)}{\Gamma(v)} - \Gamma'(1) \right) dv \\ &= \ln(\Gamma(v)) - v\Gamma'(1) \Big|_1^x = \ln(\Gamma(x)) - x\Gamma'(1) + \Gamma'(1) \end{aligned}$$

$$= \int_0^1 \left(\frac{1}{1-t} \right) \cdot \left(v - \frac{t^{v-1}}{\ln(t)} \right) \cdot dt = \int_0^1 \left(\frac{1}{1-t} \right) \cdot \left(x - \frac{t^{x-1}}{\ln(t)} - 1 + \frac{1}{\ln(t)} \right) dt$$

$$= \int_0^1 \left(\frac{1-t^{x-1} - (1-x)\ln(t)}{(1-t)\ln(t)} \right) \cdot dt = \ln(\Gamma(x)) + \Gamma'(1)(1-x)$$

$$\therefore \ln(\Gamma(x)) = \Gamma'(1)(x-1) + \int_0^1 \frac{1-t^{x-1} - (1-x)\ln(t)}{(1-t)\ln(t)} \cdot dt \quad (1)$$

$$\therefore \ln(\Gamma(x)) \sin(2\pi kx) = \Gamma'(1)(x-1) \sin(2\pi kx)$$

$$+ \int_0^1 \frac{\sin(2\pi kx) - \ln(t) \sin(2\pi kx) + \ln(t)x \sin(2\pi kx) - t^{x-1} \cdot \sin(2\pi kx)}{(1-t)\ln(t)} \cdot dt \quad (2)$$

$$\therefore \int_0^1 \ln(\Gamma(x)) \sin(2\pi kx) dx = \Gamma'(1) \int_0^1 (x-1) \sin(2\pi kx) dx$$

$$+ \int_0^1 \frac{dt}{(1-t)\ln(t)} \cdot \left[\int_0^1 \sin(2\pi kx) dx - \ln(t) \int_0^1 \sin(2\pi kx) dx + \ln(t) \int_0^1 x \sin(2\pi kx) dx - \frac{1}{t} \int_0^1 t^x \sin(2\pi kx) dx \right] \quad (3)$$

$$\int_0^1 \sin(2\pi kx) dx = \cos \frac{(2\pi kx)}{2\pi k} \Big|_0^1 = \frac{1-1}{2\pi k} = 0$$

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$$\begin{aligned}
 \int_0^1 x \sin(2\pi kx) dx &= x \cos \frac{(2\pi kx)}{2\pi k} \Big|_0^1 + \frac{1}{2\pi k} \cdot \int_0^1 \cos(2\pi kx) dx \\
 &= -\frac{\cos(2\pi k)}{2\pi k} + \left(\frac{1}{4\pi^2 k^2} \right) \sin(2\pi kx) \Big|_0^1 = -\frac{1}{2\pi k} \\
 \int_0^1 t^x \sin(2\pi kx) dx &= \int_0^1 e^{(\ln(t))x} \cdot \sin(2\pi kx) dx \\
 &= \frac{e^{\ln(t)x}}{\ln(t)} \cdot \sin(2\pi kx) \Big|_0^1 - \frac{2\pi k}{\ln(t)} \int_0^1 e^{\ln(t)x} \cdot \cos(2\pi kx) \\
 &= -\frac{2\pi k}{\ln(t)} \cdot \left(\frac{e^{\ln(t)x}}{\ln(t)} \cdot \cos(2\pi kx) \Big|_0^1 + \frac{2\pi k}{\ln(t)} \int_0^1 e^{\ln(t)x} \cdot \sin(2\pi kx) dx \right) \\
 &= -\frac{2\pi k}{\ln(t)} \cdot \left(\frac{(t-1)}{\ln(t)} + \frac{2\pi k}{\ln(t)} \int_0^1 e^{\ln(t)x} \sin(2\pi kx) dx \right) \\
 \therefore \int_0^1 t^x \sin(2\pi kx) dx &= \frac{2\pi k(1-t)}{\ln^2(t)} - \frac{4\pi^2 k^2}{\ln^2(t)} \cdot \int_0^1 t^x \sin(2\pi kx) dx \\
 \therefore \int_0^1 t^x \sin(2\pi kx) dx &= \frac{2\pi k(1-t)}{1 + \frac{4\pi^2 k^2}{\ln^2(t)}} = \frac{2\pi k(1-t)}{4\pi^2 k^2 + \ln^2(t)} \\
 \therefore \int_0^1 t^x \sin(2\pi kx) dx &= \frac{2\pi k(1-t)}{4\pi^2 k^2 + \ln^2(t)} \\
 \therefore I &= \int_0^1 \left(-\frac{\ln(t)}{2\pi k} - \frac{2\pi k(1-t)}{t(4\pi^2 k^2 + \ln^2(t))} \right) \cdot \frac{dt}{(1-t)\ln(t)} - \frac{\Gamma'(1)}{2\pi k} \quad (4) \\
 \therefore I &= -\frac{\Gamma'(1)}{B} - \int_0^1 \left(\frac{\ln(x)}{B} + \frac{B(1-x)}{x(B^2 + \ln^2 x)} \right) \cdot \frac{dx}{(1-x)\ln(x)}, B := 2\pi k \\
 \therefore I &= -\frac{\Gamma'(1)}{B} - \int_0^1 \left(\frac{1}{B(1-x)} + \frac{1}{B \ln(x)} + \frac{B}{x \ln x (B^2 + \ln^2 x)} - \frac{1}{B \ln(x)} \right) dx
 \end{aligned}$$

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$$\begin{aligned}
 &= -\frac{\Gamma'(1)}{B} - \frac{1}{B} \int_0^1 \left(\frac{1}{1-x} + \frac{1}{\ln x} \right) dx + \int_0^1 \frac{1}{\ln(x)} \left(\frac{1}{B} - \frac{B}{x(B^2 + \ln^2 x)} \right) dx \\
 &= -\frac{\Gamma'(1)}{B} + \frac{\Gamma'(1)}{B} + \int_0^1 \frac{1}{\ln(x)} \left(\frac{1}{B} - \frac{B}{x(B^2 + \ln^2 x)} \right) dx \\
 \therefore I &= \int_0^1 \frac{1}{\ln(x)} \left(\frac{1}{B} - \frac{B}{x(B^2 + \ln^2 x)} \right) dx \quad (6)
 \end{aligned}$$

Let $\ln\left(\frac{1}{x}\right) = y \Rightarrow x = e^{-y} \Rightarrow dx = -e^{-y} \cdot dy$

$$\begin{aligned}
 \therefore I &= \int_0^\infty -\frac{1}{y} \left(\frac{1}{B} - \frac{B}{e^{-y}(B^2 + y^2)} \right) e^{-y} \cdot dy \\
 &= \int_0^\infty \left(\frac{-e^{-y}}{By} + \frac{B}{y(B^2 + y^2)} \right) dy = \int_0^1 \left(\frac{1}{By} - \frac{e^{-y}}{By} - \frac{y}{B(B^2 + y^2)} \right) dy \\
 &= \frac{1}{B} \int_0^\infty \left(\frac{1-e^{-x}}{x} - \frac{x}{B^2+x^2} \right) dx = I \quad (7)
 \end{aligned}$$

Let $x = By \Rightarrow dx = Bdy$

$$\begin{aligned}
 \therefore I &= \left(\frac{1}{B} \right) \int_0^\infty \left(\frac{1 - e^{-Bx}}{x} - \frac{x}{1+x^2} \right) dx \\
 \therefore I &= \left(\frac{1}{B} \right) \int_0^\infty \left(\frac{1 - e^{-Bx}}{x} + \frac{1}{1+x} - \frac{x}{1+x^2} - \frac{1}{1+x} \right) dx \\
 &= \left(\frac{1}{B} \right) \int_0^\infty \left(\frac{1 - e^{-Bx}}{x} - \frac{x}{1+x^2} \right) dx + \left(\frac{1}{B} \right) \int_0^\infty \left(\frac{1 - e^{-Bx}}{x} - \frac{1}{1+x} \right) dx
 \end{aligned}$$

Since $\int_0^\infty \left(\frac{1}{1+x} - \frac{x}{1+x^2} \right) dx = \ln\left(\frac{1+x}{\sqrt{1+x^2}}\right) \Big|_0^\infty$

$$= \lim_{n \rightarrow \infty} \ln \left(\frac{\frac{1}{x} + 1}{\sqrt{\frac{1}{x^2} + 1}} \right) = \ln(1) = 0$$

$$\therefore I = \left(\frac{1}{B} \right) \int_0^\infty \left(\frac{1 - e^{-Bx}}{x} - \frac{1}{1+x} \right) dx$$

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$$\begin{aligned} \therefore I &= \left(\frac{1}{B}\right) \int_0^{\infty} \left(\frac{1}{x} - \frac{1}{1+x} - \frac{e^{-x}}{x} + \frac{e^{-x}}{x} - \frac{e^{-Bx}}{x}\right) dx \\ &= \left(\frac{1}{B}\right) \int_0^{\infty} \left(\frac{1}{1+x} - e^{-x}\right) \frac{1}{x} dx + \left(\frac{1}{B}\right) \int_0^{\infty} \frac{e^{-x} - e^{-Bx}}{x} dx \\ &= -\frac{\Gamma'(1)}{B} + \left(\frac{1}{B}\right) \int_0^{\infty} \frac{e^{-x} - e^{-Bx}}{x} dx = I \quad (8) \end{aligned}$$

$$\text{Let } F(A) := \int_0^{\infty} \frac{e^{-x} - e^{-Ax}}{x} dx, A > 0$$

$$\Rightarrow F(1) = 0 \Rightarrow F'(A) = \int_0^{\infty} e^{-Ax} dx = \frac{1}{A}$$

$$\therefore F(A) = \int_0^A \frac{dx}{x} = \ln(A) \Rightarrow I = \frac{\gamma}{B} + \frac{\ln(B)}{B}$$

$$\therefore I = \int_0^1 \ln(\Gamma(x)) \sin(2\pi kx) dx = \frac{\gamma + \ln(2\pi k)}{2\pi k}$$