SEVERAL WAYS TO DEVELOP AN INEQUALITY

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ABSTRACT. In this paper its presented a way to develop a given inequality. After six different solutions we show how to build new refinements and generalizations.

Main result:

Let $a, b > 0$. Prove that:

$$\mathbf{9 \leq \left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right) \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)}$$

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Solution 1 by Soumava Chakraborty - Kolkata - India.

$$\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right) = 3+\frac{4\sqrt{a}}{a+b}+\frac{a+b}{\sqrt{ab}}+\frac{4ab}{(a+b)^2}+\frac{(a+b)^2}{4ab} \leq$$

$$\leq 3 + 2 + \frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} = 6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab}.$$

It suffice to prove that: $6 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \leq 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right)$

(1) *which is equivalent to:* $1 + \frac{a+b}{\sqrt{ab}} + \frac{(a+b)^2}{4ab} \leq \frac{2(a^2+b^2)}{ab}$

Now, by the ***GM-HM inequality,*** $\sqrt{ab} \geq \frac{2ab}{a+b}$, *hence* $\frac{a+b}{\sqrt{ab}} \leq \frac{(a+b)^2}{2ab}$

It follows that:

$$\frac{a+b}{\sqrt{ab}} + 1 + \frac{(a+b)^2}{4ab} \leq 1 + \frac{3(a+b)^2}{4ab} \tag{2}$$

1 and 1 shows that the problem will be solved if we manage to prove

$$\frac{3(a+b)^2 + 4ab}{4ab} \leq \frac{2(a^2+b^2)}{ab}.$$

This is equivalent to $3a^2 + 3b^2 + 10ab \leq 8a^2 + 8b^2$, *or*

$5a^2 + 5b^2 - 10ab \geq 0$, *which is simply* $5(a-b)^2 \geq 0$

□

Key words and phrases. Inequalities, Schweitzer, Refinements, Generalizations.

Solution 2 by Kevin Soto Palacios - Huarmey - Peru.

We employ the obvious $\dfrac{2\sqrt{ab}}{a+b} \leq 1$ *and* $\dfrac{4ab}{(a+b)^2} \leq 1$ *to obtain:*

$$\left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right) \leq 6 + \frac{3(a+b)^2}{4ab}.$$

Suffice to prove that $6 + \dfrac{3(a+b)^2}{4ab} \leq 5 + 2\left(\dfrac{a}{b} + \dfrac{b}{a}\right)$.

Now note that: $1 + \dfrac{3(a+b)^2}{4ab} \leq \dfrac{5}{2} + \dfrac{3}{4}\left(\dfrac{a}{b} + \dfrac{b}{a}\right)$, *because* $\dfrac{5}{2} \leq \dfrac{5}{4}\left(\dfrac{a}{b} + \dfrac{b}{a}\right)$.

□

Solution 3 by Soumava Chakraborty - Kolkata - India.

Using the AM-GM inequality, $\dfrac{a+b}{2} \geq \sqrt{ab}$ *and* $\dfrac{2}{a+b} \leq \dfrac{1}{\sqrt{ab}}$

$$\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right) \leq \left(\sqrt{ab}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{ab}}\right)$$

$$= \left(2\sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{2}{\sqrt{ab}}\right) = \frac{a+b}{\sqrt{ab}} + 4 + \frac{(a+b)^2}{4ab} + \frac{a+b}{\sqrt{ab}} =$$

$$= 4 + \frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}}$$

Thus, suffice it to prove that: $4 + \dfrac{(a+b)^2}{4ab} + \dfrac{2(a+b)}{\sqrt{ab}} \leq 5 + 2\left(\dfrac{a}{b} + \dfrac{b}{a}\right)$.

which is equivalent to:

$$\frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \leq 1 + 2\frac{a^2+b^2}{ab} = \frac{2a^2+2b^2+ab}{ab}.$$

Now, from $\dfrac{1}{\sqrt{ab}} \leq \dfrac{a+b}{2ab}$ *we get* $\dfrac{2(a+b)}{\sqrt{ab}} \leq \dfrac{(a+b)^2}{ab}$, *implying*

$$\frac{(a+b)^2}{4ab} + \frac{2(a+b)}{\sqrt{ab}} \leq \frac{(a+b)^2}{4ab} + \frac{(a+b)^2}{a} = \frac{5(a+b)^2}{4ab}.$$

Thus, suffice it to prove $\dfrac{5}{4}\cdot\dfrac{(a+b)^2}{ab} \leq \dfrac{2a^2+2b^2+ab}{ab}$, *i.e.,* $5a^2+5b^2+10ab \leq 8a^2+8b^2+4ab$

which reduces to $3(a-b)^2 \geq 0$.

□

Solution 4 by Abdallah El Farisi - Bechar - Algerie.

$$\textit{Let: } A = \left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{a+b}{2ab} + \frac{1}{\sqrt{ab}} + \frac{2}{a+b}\right) =$$

$$= \left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)\left(\frac{2}{a+b} + \frac{1}{\sqrt{ab}} + \frac{a+b}{2ab}\right) =$$

$$= \frac{1}{ab}\left(\frac{2ab}{a+b} + \sqrt{ab} + \frac{a+b}{2}\right)^2 \leq \frac{1}{ab}(\sqrt{ab} + a + b)^2 =$$

$$= \frac{1}{ab}\left(ab + 2\sqrt{ab}(a+b) + (a+b)^2\right) \leq \frac{1}{ab}\left(ab + 2(a+b)^2\right) =$$

$$= \frac{1}{ab}\left(5ab + 2(a^2+b^2)\right) = 5 + 2\left(\frac{a}{b} + \frac{b}{a}\right).$$

□

Solution 5 by Soumava Chakrabory - Kolkata - India.

Define $=\dfrac{2ab}{a+b}, y=\sqrt{ab}, z=\dfrac{a+b}{2}$. *We have* $x\le y\le z$ *and* $y^2=xz$.

$$\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right)=\left(\sum_{cycl}x\right)\left(\sum_{cycl}\frac{1}{x}\right)=$$

$$=\frac{(\sum_{cycl}x(xy+yz+zx)}{xyz}=\frac{(\sum_{cycl}x)(xy+yz+y^2)}{xyz}=\frac{(x+y+z)^2}{xz}=\frac{(x+y+z)^2}{y^2}$$

$$5+2\left(\frac{a}{b}+\frac{b}{a}\right)=\frac{2a^2+2b^2+5ab}{ab}=\frac{2(a+b)^2+ab}{ab}=1+\frac{8z^2}{y^2}.$$

The required inequality is equivalent to $\dfrac{(x+y+z)^2}{y^2}\le 1+\dfrac{8z^2}{y^2}$ *which is*

$$(x+y+z)^2-y^2\le 8z^2,\ or,\ (x+2y+z)(x+z)\le 8z^2.$$

Since $x\le y\le z, x+2y+z\le 4z$ *and* $x+z\le 2z$, *which shows that, indeed,*

$$(x+2y+z)(x+z)\le 8z^2.$$

□

Solution 6 by Daniel Sitaru - Romania.

We know that, for positive $a,b,c:\dfrac{2ab}{a+b}\le\sqrt{ab}\le\dfrac{a+b}{2}$.

We'll use Schweitzer's inequality:

$$\left(\sum_{k=1}^{n}x_k\right)\left(\sum_{k=1}^{n}\frac{1}{x_k}\right)\le\frac{(m+M)^2n^2}{4mM},$$

where $x_1,\dots,x_n\in[m,M], m>0$.

with $n=3, m=x_1=\dfrac{2ab}{a+b}, x_2=\sqrt{ab}$, *and* $x_3=\dfrac{a+b}{2}=M$, *we directly get*

$$A=\left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right)\le 9\frac{\left(\frac{2ab}{a+b}+\frac{a+b}{2}\right)^2}{4ab}=$$

$$=\frac{9}{4ab}\left[\left(\frac{2ab}{a+b}\right)^2+2ab+\left(\frac{a+b}{2}\right)^2\right]\le\frac{9}{4ab}\left[(\sqrt{ab})^2+2ab+\left(\frac{a+b}{2}\right)^2\right]=$$

$$=\frac{9}{4ab}\left[3ab+\left(\frac{a+b}{2}\right)^2\right]=\frac{9}{16ab}\left[14ab+(a^2+b^2)\right]=\frac{63}{8}+\frac{9}{16}\left(\frac{a}{b}+\frac{b}{a}\right).$$

Now, $1\le\dfrac{1}{2}\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$. *It then follows that* $\dfrac{23}{8}\le\dfrac{23}{16}\left(\dfrac{a}{b}+\dfrac{b}{a}\right)$ *and, subsequently,*

$$\frac{63}{8}+\frac{9}{16}\left(\frac{a}{b}+\frac{b}{a}\right)\le\frac{40}{8}+\frac{32}{16}\left(\frac{a}{b}+\frac{b}{a}\right)=5+2\left(\frac{a}{b}+\frac{b}{a}\right).$$

In all solutions first inequality its proved by AM-GM: $E=(x+y+z)\left(\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z}\right)\ge 9$

This double inequality can be developed.

□

Extension 1 (Marin Chirciu):
Let $a, b > 0$. Prove that:

$$\mathbf{9 \leq \left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right) \leq 9-2n+n\left(\frac{a}{b}+\frac{b}{a}\right), n \geq \frac{3}{4}.}$$

Proof.

We denote $\frac{2ab}{a+b} = x, \sqrt{ab} = y, \frac{a+b}{2} = z$ *and* $E = \left(\frac{2ab}{a+b}+\sqrt{ab}+\frac{a+b}{2}\right)\left(\frac{a+b}{2ab}+\frac{1}{\sqrt{ab}}+\frac{2}{a+b}\right)$.

(1) *We have* $E = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) = 3+\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)+\left(\frac{z}{x}+\frac{x}{z}\right)$

(2)

We prove that: $\left(\frac{x}{y}+\frac{y}{x}\right) \leq 1+\frac{(a+b)^2}{4ab}, \left(\frac{y}{z}+\frac{z}{y}\right) \leq 1+\frac{(a+b)^2}{4ab}, \left(\frac{z}{x}+\frac{x}{z}\right) = \frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2}$

Indeed:

$$\left(\frac{x}{y}+\frac{y}{x}\right) = \frac{x^2+y^2}{xy} = \frac{(\frac{2ab}{a+b})^2+ab}{\frac{2ab}{a+b}\cdot\sqrt{ab}} \leq \frac{(\frac{2ab}{a+b})^2+ab}{\frac{2ab}{a+b}\cdot\frac{2ab}{a+b}} = 1+\frac{(a+b)^2}{4ab}$$

where the last inequality follows from means inequality $\sqrt{ab} \geq \frac{2ab}{a+b}$.

$$\left(\frac{y}{z}+\frac{z}{y}\right) = \frac{y^2+z^2}{yz} = \frac{ab+(\frac{a+b}{2})^2}{\sqrt{ab}\cdot\frac{a+b}{2}} \leq \frac{ab+(\frac{a+b}{2})^2}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}} = 1+\frac{(a+b)^2}{4ab}$$

where the last inequality follows from means inequality $\sqrt{ab} \geq \frac{2ab}{a+b}$.

$$\left(\frac{z}{x}+\frac{x}{z}\right) = \frac{x^2+z^2}{xz} = \frac{(\frac{2ab}{a+b})^2+(\frac{a+b}{2})^2}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}} = \frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2}.$$

Next, we look for an inequality having the form $E \leq k+n\left(\frac{a}{b}+\frac{b}{a}\right)$.

Taking into account 1 and 2 it remains to impose

$$3+\left[1+\frac{(a+b)^2}{4ab}\right]+\left[1+\frac{(a+b)^2}{4ab}\right]+\frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2} \leq k+n\left(\frac{a}{b}+\frac{b}{a}\right) \Leftrightarrow$$

$$2\cdot\frac{(a+b)^2}{4ab}+\frac{(a+b)^4+16a^2b^2}{4ab(a+b)^2} \leq x-5+\frac{y(a^2+b^2)}{ab} \Leftrightarrow$$

$$3(a+b)^4+16a^2b^2 \leq 4(x-5)ab(a+b)^2+4y(a^2+b^2)(a+b)^2 \Leftrightarrow$$

(3)

$$(4n-3)a^4+(4k+8n-32)a^3b+(8k+8n-74)a^2b^2+(4k+8n-32)a^3b+(4n-3)b^4 \geq 0$$

In Horner's scheme we put the condition that 1 to be double root and we obtain:

$16k+32n-144 = 0, 32k+64n-288 = 0,$ *wherefrom* $k+2n = 9$

It follows that 3 can be written

$$(a-b)^2\left[(4n-3)a^2+(4k+16n-38)ab+(16k+36n-143)b^2\right] \geq 0 \Leftrightarrow$$

$$(a-b)^2\left[(4n-3)a^2+(8n-2)ab+(4n-3)b^2\right] \geq 0.$$

Putting the condition that the right parenthesis to be positive we obtain $4n-3 \geq 0$.

Equality holds if and only if $a = b$.

$$\textit{Obviously } E=(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)\geq 9.$$

Remark.

We denote the arithmetic mean, the geometric mean, the harmonic mean, respectively the squared mean with

$$M_a=\frac{a+b}{2}, M_g=\sqrt{ab}, M_h=\frac{2ab}{a+b}, M_p=\sqrt{\frac{a^2+b^2}{2}}.$$

Inequality 2a) is:

$$4\leq (M_a+M_g+M_h)\Big(\frac{1}{M_a}+\frac{1}{M_g}+\frac{1}{M_h}\Big)\leq 9-2n+n\Big(\frac{a}{b}+\frac{b}{a}\Big),\ \textit{where } n\geq\frac{3}{4}$$

We will prove that $9\leq (x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)\leq 9-2n+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$, *where* $n\geq\frac{3}{4}$ *or* $n\geq\frac{5}{4}$, *for* $\{x,y,z\}\subset\{M_a,M_g,M_h,M_p\}$.

□

Extension 2 (Marin Chirciu):

$$\boldsymbol{9\leq\left(\frac{a+b}{2}+\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{2}{a+b}+\frac{1}{\sqrt{ab}}+\sqrt{\frac{2}{a^2+b^2}}\right)\leq 9-2n+n\Big(\frac{a}{b}+\frac{b}{a}\Big)}$$

with $\boldsymbol{n\geq\frac{3}{4}}$**.**

Proof.

$$\{x,y,z\}=\{M_a,M_g,M_p\}.$$

We denote $\frac{a+b}{2}=x, \sqrt{ab}=y, \sqrt{\frac{a^2+b^2}{2}}=z$ *and*

$$E=\left(\frac{a+b}{2}+\sqrt{ab}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{2}{a+b}+\frac{1}{\sqrt{ab}}+\sqrt{\frac{2}{a^2+b^2}}\right).$$

(1) *We have* $E=(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)=3+\Big(\frac{x}{y}+\frac{y}{x}\Big)+\Big(\frac{y}{z}+\frac{z}{y}\Big)+\Big(\frac{z}{x}+\frac{x}{z}\Big)$

(2)

We prove that $\Big(\frac{x}{y}+\frac{y}{x}\Big)\leq 1+\frac{(a+b)^2}{4ab}, \Big(\frac{y}{z}+\frac{z}{y}\Big)\leq 1+\frac{a^2+b^2}{2ab}, \Big(\frac{z}{x}+\frac{x}{z}\Big)\leq 1+\frac{2(a^2+b^2)}{(a+b)^2}$

Indeed:

$$\Big(\frac{x}{y}+\frac{y}{x}\Big)=\frac{x^2+y^2}{xy}=\frac{(\frac{a+b}{2})^2+ab}{\frac{a+b}{2}\cdot\sqrt{ab}}\leq\frac{(\frac{a+b}{2})+ab}{\frac{a+b}{2}\cdot\frac{2ab}{a+b}}=\frac{(\frac{a+b}{2})+ab}{ab}=1+\frac{(a+b)^2}{4ab}$$

where the last inequality follows from the means inequality $\sqrt{ab}\geq\frac{2ab}{a+b}$.

$$\Big(\frac{y}{z}+\frac{z}{y}\Big)=\frac{y^2+z^2}{yz}=\frac{ab+\frac{a^2+b^2}{2}}{\sqrt{ab}\cdot\sqrt{\frac{a^2+b^2}{2}}}\leq\frac{ab+\frac{a^2+b^2}{2}}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}}=\frac{ab+\frac{a^2+b^2}{2}}{ab}=1+\frac{a^2+b^2}{2ab}$$

where the last inequality follows from $\sqrt{ab}\geq\frac{2ab}{a+b}$ *and* $\sqrt{\frac{a^2+b^2}{2}}\geq\frac{a+b}{2}$.

$$\Big(\frac{z}{x}+\frac{x}{z}\Big)=\frac{x^2+z^2}{xz}=\frac{(\frac{a+b}{2})^2+\frac{a^2+b^2}{2}}{\frac{a+b}{2}\cdot\sqrt{\frac{a^2+b^2}{2}}}\le\frac{(\frac{a+b}{2})^2+\frac{a^2+b^2}{2}}{\frac{a+b}{2}\cdot\frac{a+b}{2}}=1+\frac{2(a^2+b^2)}{(a+b)^2}$$

Next, we look for an inequality having the form $E\le k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$.

Taking into account 1 and 2 it remains to impose

$$3+\left[1+\frac{(a+b)^2}{4ab}\right]+\left[1+\frac{a^2+b^2}{2ab}\right]+\left[1+\frac{2(a^2+b^2)}{(a+b)^2}\right]\le k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)\Leftrightarrow$$

$$\frac{(a+b)^2}{4ab}+\frac{a^2+b^2}{2ab}+\frac{2(a^2+b^2)}{(a+b)^2}\le(k-6)+\frac{n(a^2+b^2)}{ab}\Leftrightarrow$$

$$(a+b)^4+2(a^2+b^2)(a+b)^2+8ab(a^2+b^2)\le4(k-6)ab(a+b)^2+4n(a^2+b^2)(a+b)^2\Leftrightarrow$$

(3)

$$(4n-3)a^4+(4k+8n-40)a^3b+(8k+8n-58)a^2b^2+(4k+8n-40)a^3b+(4n-3)b^4\ge0$$

In Horner's scheme we put the condition that 1 to be double root and we obtain: $16k+32n-144=0, 32k+64n-288=0$, *wherefrom* $k+2n=9$. *It follows that 3 can be written*

$$(a-b)^2\Big[(4n-3)a^2+(4k+16n-46)ab+(16k+36n-147)b^2\Big]\ge0\Leftrightarrow$$

$$(a-b)^2\Big[(4n-3)a^2+(8n-10)ab+(4n-3)b^2\Big]\ge0$$

Putting the condition that the right parenthesis to be positive we obtain $4n-3\ge0$

The equality holds if and only if $a=b$.

Obviously $E=(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)\ge9$

□

Extension 3 (Marin Chirciu):

$$9\le\Big(\frac{a+b}{2}+\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\Big)\Big(\frac{2}{a+b}+\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\Big)\le9-2n+n\Big(\frac{a}{b}+\frac{b}{a}\Big),\ \textit{with } n\ge\frac{3}{4}$$

Proof.

$$\{x,y,z\}=\{M_a,M_h,M_p\}.$$

We denote $\frac{a+b}{2}=x$, $\frac{2ab}{a+b}=y$, $\sqrt{\frac{a^2+b^2}{2}}=z$ *and*

$$E=\Big(\frac{a+b}{2}+\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\Big)\Big(\frac{2}{a+b}+\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\Big)$$

(1) *We have* $E=(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)=3+\Big(\frac{x}{y}+\frac{y}{x}\Big)+\Big(\frac{y}{z}+\frac{z}{y}\Big)+\Big(\frac{z}{x}+\frac{x}{z}\Big)$

We prove that $\Big(\frac{x}{y}+\frac{y}{x}\Big)=\frac{(a+b)^2}{4ab}+\frac{4ab}{(a+b)^2}$, $\Big(\frac{y}{z}+\frac{z}{y}\Big)\le\frac{(a+b)^2}{2ab}+\frac{4ab}{(a+b)^2}$,

$$\Big(\frac{z}{x}+\frac{x}{z}\Big)\le1+\frac{2(a^2+b^2)}{(a+b)^2}\tag{2}$$

Indeed:

$$\Big(\frac{x}{y}+\frac{y}{x}\Big)=\frac{x^2+y^2}{xy}=\frac{(\frac{a+b}{2})^2+(\frac{2ab}{a+b})^2}{\frac{a+b}{2}\cdot\frac{2ab}{a+b}}=\frac{(a+b)^2}{4ab}+\frac{4ab}{(a+b)^2}$$

$$\left(\frac{y}{z}+\frac{z}{y}\right)=\frac{y^2+z^2}{yz}=\frac{(\frac{2ab}{a+b})^2+\frac{a^2+b^2}{2}}{\frac{2ab}{a+b}\cdot\sqrt{\frac{a^2+b^2}{2}}}\leq\frac{(\frac{2ab}{a+b})^2+\frac{a^2+b^2}{2}}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}}=$$

$$=\frac{(\frac{2ab}{a+b})^2+\frac{a^2+b^2}{2}}{ab}=\frac{4ab}{(a+b)^2}+\frac{a^2+b^2}{2ab}$$

where the last inequality follows from $\sqrt{\frac{a^2+b^2}{2}}\geq\frac{a+b}{2}$.

$$\left(\frac{z}{x}+\frac{x}{z}\right)=\frac{x^2+z^2}{xz}=\frac{(\frac{a+b}{2})^2+\frac{a^2+b^2}{2}}{\frac{a+b}{2}\cdot\sqrt{\frac{a^2+b^2}{2}}}\leq\frac{(\frac{a+b}{2})^2+\frac{a^2+b^2}{2}}{\frac{a+b}{2}\cdot\frac{a+b}{2}}=1+\frac{2(a^2+b^2)}{(a+b)^2}$$

Next, we are looking for an inequality having the form $E\leq k+n\left(\frac{a}{b}+\frac{b}{a}\right)$

Taking into account 1 and 2 it remains to impose

$$3+\left[\frac{(a+b)^2}{4ab}+\frac{4ab}{(a+b)^2}\right]+\left[\frac{4ab}{(a+b)^2}+\frac{a^2+b^2}{2ab}\right]+\left[1+\frac{2(a^2+b^2)}{(a+b)^2}\right]\leq k+n\left(\frac{a}{b}+\frac{b}{a}\right)\Leftrightarrow$$

$$\frac{(a+b)^2}{4ab}+\frac{8ab}{(a+b)^2}+\frac{a^2+b^2}{2ab}+\frac{2(a^2+b^2)}{(a+b)^2}\leq(k-4)+\frac{n(a^2+b^2)}{ab}\Leftrightarrow$$

$$(a+b)^4+32a^2b^2+2(a^2+b^2)(a+b)^2+8ab(a^2+b^2)\leq 4(k-4)ab(a+b)^2+4n(a^2+b^2)(a+b)^2$$

(3)

$$(4n-3)a^4+(4k+8n-32)a^3b+(8k+8n-74)a^2b^2+(4k+8n-32)a^3b+(4n-3)b^4\geq 0$$

In Horner's scheme we put the condition that 1 to be double root and we obtain:

$16k+32n-144=0, 32k+64n-288=0,$ *wherefrom* $k+2n=9$

It follows that 3 can be written

$$(a-b)^2\left[(4n-3)a^2+(4k+16n-38)ab+(16k+36n-147)b^2\right]\geq 0\Leftrightarrow$$

$$(a-b)^2\left[(4n-3)a^2+(8n-2)ab+(4n-3)b^2\right]\geq 0.$$

Putting the condition that the right parenthesis to positive we obtain $4n-3\geq 0$

The equality holds if and only if $a=b$.

Obviously $E=(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)\geq 9$

□

Extension 4 (Marin Chirciu):

$$\mathbf{9\leq\left(\sqrt{ab}+\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\right)\left(\frac{1}{\sqrt{ab}}+\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\right)\leq 9-2n+n\left(\frac{a}{b}+\frac{b}{a}\right)}$$

with $\mathbf{n\geq\frac{5}{4}}$

Proof.

$$\{x,y,z\}=\{M_g,M_h,M_p\}.$$

We denote $\sqrt{ab}=x$, $\frac{2ab}{a+b}=y$, $\sqrt{\frac{a^2+b^2}{2}}=z$ *and*

$$E=\Big(\sqrt{ab}+\frac{2ab}{a+b}+\sqrt{\frac{a^2+b^2}{2}}\Big)\Big(\frac{1}{\sqrt{ab}}+\frac{a+b}{2ab}+\sqrt{\frac{2}{a^2+b^2}}\Big)$$

(1) *We have* $E=(x+y+z)\Big(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\Big)=3+\Big(\frac{x}{y}+\frac{y}{x}\Big)+\Big(\frac{y}{z}+\frac{z}{y}\Big)+\Big(\frac{z}{x}+\frac{x}{z}\Big)$

(2)

We prove that $\Big(\frac{x}{y}+\frac{y}{x}\Big)\le 1+\frac{(a+b)^2}{4ab}$, $\Big(\frac{y}{z}+\frac{z}{y}\Big)\le\frac{(a+b)^2}{2ab}+\frac{4ab}{(a+b)^2}$, $\Big(\frac{z}{x}+\frac{x}{z}\Big)\le 1+\frac{a^2+b^2}{2ab}$

Indeed:

$$\Big(\frac{x}{y}+\frac{y}{x}\Big)=\frac{x^2+y^2}{xy}=\frac{ab+(\frac{2ab}{a+b})^2}{\sqrt{ab}\cdot\frac{2ab}{a+b}}\le\frac{ab+(\frac{2ab}{a+b})^2}{\frac{2ab}{a+b}\cdot\frac{2ab}{a+b}}=1+\frac{(a+b)^2}{4ab}$$

where the inequality follows from $\sqrt{ab}\ge\frac{2ab}{a+b}$.

$$\Big(\frac{y}{z}+\frac{z}{y}\Big)=\frac{y^2+z^2}{yz}=\frac{(\frac{2ab}{a+b})+\frac{a^2+b^2}{2}}{\frac{2ab}{a+b}\cdot\sqrt{\frac{a^2+b^2}{2}}}\le\frac{(\frac{2ab}{a+b})^2+\frac{a^2+b^2}{2}}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}}=\frac{(\frac{2ab}{a+b})^2+\frac{a^2+b^2}{2}}{ab}=\frac{4ab}{(a+b)^2}+\frac{a^2+b^2}{2ab}$$

where the inequality follows from $\sqrt{\frac{a^2+b^2}{2}}\ge\frac{a+b}{2}$

$$\Big(\frac{z}{x}+\frac{x}{z}\Big)=\frac{x^2+z^2}{xz}=\frac{ab+\frac{a^2+b^2}{2}}{\sqrt{ab}\cdot\frac{a^2+b^2}{2}}\le\frac{ab+\frac{a^2+b^2}{2}}{\frac{2ab}{a+b}\cdot\frac{a+b}{2}}=1+\frac{a^2+b^2}{2ab}$$

where the inequality follows form $\sqrt{\frac{a^2+b^2}{2}}\ge\frac{a+b}{2}$ *and* $\sqrt{ab}\ge\frac{2ab}{a+b}$.

Next, we are looking for an inequality having the form $E\le k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)$.

Taking into account 1 and 2 it remains to impose

$$3+\left[1+\frac{(a+b)^2}{4ab}\right]+\left[\frac{(a+b)^2}{2ab}+\frac{4ab}{(a+b)^2}\right]+\left[1+\frac{a^2+b^2}{2ab}\right]\le k+n\Big(\frac{a}{b}+\frac{b}{a}\Big)\Leftrightarrow$$

$$\frac{(a+b)^2}{4ab}+\frac{4ab}{(a+b)^2}+\frac{a^2+b^2}{ab}\le(k-5)+\frac{n(a^2+b^2)}{ab}\Leftrightarrow$$

$$(a+b)^4+16a^2b^2+4(a^2+b^2)(a+b)^2\le 4(k-5)ab(a+b)^5+4n(a^2+b^2)(a+b)^2$$

(3)

$$(4n-5)a^4+(4k+8n-32)a^3b+(8k+8n-70)a^2b^2+(4k+8n-32)a^3b+(4n-5)b^4\ge 0$$

In Horner's scheme we put the condition that 1 to be double root and we obtain: $16k+32n-144=0, 32k+64n-288=0$, *wherefrom* $k+2n=9$. *It follows that 3 can be written*

$$(a-b)^2\Big[(4n-5)a^2+(4k+16n-42)ab+(16k+36n-149)b^2\Big]\ge 0\Leftrightarrow$$

$$(a-b)^2\Big[(4n-5)a^2+(8n-6)ab+(4n-5)b^2\Big]\ge 0.$$

Putting the condition the the right parenthesis to be positive we obtain $4n-5\ge 0$

The equality holds if and only if $a=b$.

$$\textit{Obviously } E = (x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9.$$

$\square$

References

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