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If $a, b, c \geq 0$ then:

$$\begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & a^2 + b^2 & a^2 + c^2 & 1 \\ b^2 & a^2 + b^2 & 0 & b^2 + c^2 & 1 \\ c^2 & a^2 + c^2 & b^2 + c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} \leq \frac{1}{8} \prod (a + b)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

Solution 2 by Nirapada Pal-Jhargram-India

Solution 1 by Serban George Florin-Romania

$$\begin{aligned} & \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & -a^2 & b^2 & c^2 & 1 \\ b^2 & a^2 & -b^2 & c^2 & 1 \\ c^2 & a^2 & b^2 & -c^2 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} = \\ & = a^2 b^2 c^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{vmatrix} = a^2 b^2 c^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ 0 & 2 & 2 & 2 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{vmatrix} = \\ & = a^2 b^2 c^2 \cdot \begin{vmatrix} 2 & 2 & 2 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 8a^2 b^2 c^2 \leq \frac{1}{8} (a + b)^2 (b + c)^2 (a + c)^2 \\ & \Rightarrow 8abc \leq (a + b)(b + c)(a + c) \\ & \quad a + b \geq 2\sqrt{ab} \\ & \quad b + c \geq 2\sqrt{bc} \\ & \quad a + c \geq 2\sqrt{ac} \\ & \Rightarrow (a + b)(b + c)(a + c) \geq 8abc \end{aligned}$$

Solution 2 by Nirapada Pal-Jhargram-India

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$$\begin{aligned}
 & \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & 0 & a^2 + b^2 & a^2 + c^2 & 1 \\ b^2 & a^2 + b^2 & 0 & b^2 + c^2 & 1 \\ c^2 & a^2 + c^2 & b^2 + c^2 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & a^2 & b^2 & c^2 & 1 \\ a^2 & -a^2 & b^2 & c^2 & 1 \\ b^2 & a^2 & -b^2 & c^2 & 1 \\ c^2 & a^2 & b^2 & -c^2 & 1 \\ 1 & 0 & 0 & 0 & 0 \end{vmatrix} C_i - C_1 \rightarrow C_i, i = 2, 3, 4 \\
 & = \begin{vmatrix} a^2 & b^2 & c^2 & 1 \\ -a^2 & b^2 & c^2 & 1 \\ a^2 & -b^2 & c^2 & 1 \\ a^2 & b^2 & -c^2 & 1 \end{vmatrix} = a^2 b^2 c^2 \begin{vmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \end{vmatrix} \\
 & = a^2 b^2 c^2 \begin{vmatrix} 1 & 0 & 0 & 0 \\ -1 & 2 & 2 & 2 \\ 1 & -2 & 2 & 2 \\ 1 & 0 & -2 & 0 \end{vmatrix} C_i - C_1 \rightarrow C_i, i = 2, 3, 4 \\
 & = a^2 b^2 c^2 \begin{vmatrix} 2 & 2 & 2 \\ -2 & 0 & 0 \\ 0 & -2 & 0 \end{vmatrix} = 8a^2 b^2 c^2 = (2ab)(2bc)(2ca) \\
 & \leq \frac{1}{2}(a+b)^2 \frac{1}{2}(b+c)^2 \frac{1}{2}(c+a)^2, (a+b)^2 - (a-b)^2 = 4ab \Rightarrow (a+b)^2 \geq 4ab \\
 & = \frac{1}{8} \prod (a+b)^2
 \end{aligned}$$