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If $a,b,c>0, a+b+c=\pi$ then:

$$2\sum\int_0^a \frac{\arctan^2 x}{x}dx+\log(1+a^2)\log(1+b^2)\log(1+c^2)<\pi^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Saptak Bhattacharya-Kolkata-India,

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Saptak Bhattacharya-Kolkata-India

Let $f(x)=x-\tan^{-1}x$

$$f'(x)=1-\frac{1}{1+x^2}>0$$

So, $\forall x>0$

$$f(x)>f(0)=0$$

thus: $(\tan^{-1}x)^2<x^2\Rightarrow \frac{(\tan^{-1}x)^2}{x}<x;$

$$\int_0^a \frac{(\tan^{-1}x)^2dx}{x}<\frac{a^2}{2}$$

Thus,

$$2\sum\int_0^a \frac{(\tan^{-1}x)^2dx}{x}<\sum a^2 \quad (I)$$

Now, consider $\phi(x)=x-\ln(1+x^2)$

$$\phi(0)=0;\ \phi'(x)=1-\frac{2x}{1+x^2}=\frac{(x-1)^2}{1+x^2}>0$$

So, $\phi(x)>0\ \ \forall x>0\Rightarrow \ln(1+x^2)<x$

Thus, $\prod\ln(1+a^2)<abc \quad (ii)$

Now, by $AM \geq HM$

$$\sum \frac{1}{a} \geq \frac{9}{\pi^2}$$

$$\Rightarrow \sum \frac{2}{a} \geq \frac{18}{\pi^2} > 1 \quad [\because \pi < 4;\ \pi^2 < 16 < 18]$$

Thus, $abc < \sum 2ab = 2\sum ab$ ***(iii)***

Combining (i) & (iii)

$$LHS < \sum a^2 + 2\sum ab = (a+b+c)^2 = \pi^2$$

(Proved)

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $t = a\tan\theta,\ dt = a\sec^2\theta\, d\theta$

when $t = 0, \theta = 0$***, when*** $t = x, \theta = \tan^{-1} x$

$$\Omega(a) = \lim_{x\to\infty}\int_0^x \frac{\log t}{t^2+a^2}dt = \frac{1}{a}\lim_{x\to\infty}\int_a^{\tan^{-1}x} \log(a\tan\theta)\, d\theta$$

$$= \frac{1}{a}\lim_{x\to\infty}\int_0^{\tan^{-1}x} \log(a\tan(\tan^{-1}x-\theta))\, d\theta = \frac{1}{a}\lim_{x\to\infty}\int_0^{\tan^{-1}x} \log\left(a\cdot\frac{x-\tan\theta}{1+x\tan\theta}\right)d\theta$$

$$= \frac{1}{a}\lim_{x\to\infty}\int_0^{\tan^{-1}x} \log\left(a\cdot\frac{1-\frac{\tan\theta}{x}}{\frac{1}{x}+\tan\theta}\right)d\theta = \frac{1}{a}\int_0^{\frac{\pi}{2}} \log\left(\frac{a^2}{a\tan\theta}\right) = \pi\log a^{\frac{1}{a}} - \Omega(a)$$

$$\Rightarrow 2\Omega(a) = \pi\log a^{\frac{1}{a}} \Rightarrow \Omega(a) = \frac{\pi}{2}\log a^{\frac{1}{a}}. \text{ So,} \sum_{cyc}\Omega^2(a) = \frac{\pi^2}{4}\log^2\left(a^{\frac{1}{a}}\right)$$

$$\geq \frac{\pi^2}{12}\left(\sum_{cyc}\log\left(a^{\frac{1}{a}}\right)\right)^2 = \frac{\pi^2}{12}\log^2\left(a^{\frac{1}{a}}\cdot b^{\frac{1}{b}}\cdot c^{\frac{1}{c}}\right)$$

(Proved)