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In $\Delta\ ABC$:

$$2\sqrt{3} \le \frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} \le \frac{R\sqrt{3}}{r}$$

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Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam, Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia, Solution 3 by Soumitra Mandal-Chandar Nagore-India

Solution 1 by Khanh Hung Vu-Ho Chi Minh-Vietnam

We have $\frac{a}{r_a} + \frac{b}{r_b} + \frac{r}{r_c} = \frac{a(b+c-a)}{2S} + \frac{b(c+a-b)}{2S} + \frac{c(a+b-c)}{2S} =$

$$= \frac{a^2 + b^2 + c^2 - (a-b)^2 - (b-c)^2 - (c-a)^2}{2S} \ge \frac{4S\sqrt{3}}{2S} = 2\sqrt{3}$$

On the other hand, we have

$$\frac{a}{r_a} + \frac{b}{r_b} + \frac{c}{r_c} = \frac{a(p-a)}{S} + \frac{b(p-b)}{S} + \frac{c(p-c)}{S} = \frac{a(p-a)+b(p-b)+c(p-c)}{\sqrt{p(p-a)(p-b)(p-c)}} \quad \textbf{(1)}$$

And $\frac{R\sqrt{3}}{r} = \frac{abc\sqrt{3}}{4S\cdot\frac{S}{p}} = \frac{abc\sqrt{3}}{4(p-a)(p-b)(p-c)}$ ***(2)***

We need to prove that $\frac{a(p-a)+b(p-b)+c(p-c)}{\sqrt{p(p-a)(p-b)(p-c)}} \le \frac{abc\sqrt{3}}{4(p-a)(p-b)(p-c)}$ ***(3)***

Put $p-a = x, p-b = y, p-c = z \Rightarrow z = y+z, b = x+z$ ***and*** $c = y+z$

We have (3) $\Rightarrow \frac{x(y+z)+y(x+z)+z(x+y)}{\sqrt{xyz(x+y+z)}} \le \frac{(x+y)(y+z)(x+z)\sqrt{3}}{4xyz} \Rightarrow$

$$\Rightarrow \frac{8xyz(xy+yz+zx)}{\sqrt{3xyz(x+y+z)}} \le (x+y)(y+z)(x+z)$$

We have $(x+y)(y+z)(x+z) \ge \frac{8(x+y+z)(xy+yz+zx)}{9}$

On the other hand we have

$$(x+y+z)\sqrt{3(x+y+z)} \ge 9\sqrt{xyz} \Rightarrow \frac{x+y+z}{9\sqrt{xyz}} \ge \frac{1}{\sqrt{3(x+y+z)}} \Rightarrow$$

$$\Rightarrow \frac{x+y+z}{9xyz} \ge \frac{1}{\sqrt{3xyz(x+y+z)}} \Rightarrow$$

$$\Rightarrow \frac{8(x+y+z)(xy+yz+zx)}{9} \ge \frac{8xyz(xy+yz+zx)}{\sqrt{3xyz(x+y+z)}}$$

$\Rightarrow$ ***(3) true***

(1), (2) and (3) $\Rightarrow \frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c} \le \frac{R\sqrt{3}}{r} \Rightarrow$ ***Q.E.D.***

Solution 2 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$2\sqrt{3} \le \frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c} \le \frac{R\sqrt{3}}{r}$$

$$\sum \frac{a}{r_a} = \frac{1}{\Delta}\cdot\sum a\cdot(p-a) = \frac{1}{\Delta}\cdot\left(\sum ap\right)$$

$$\Delta\ ABC$$

$$2\sqrt{3} \le \frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c} \le \frac{R\sqrt{3}}{r}$$

$$\sum \frac{a}{r_a} = \frac{1}{\Delta}\cdot\sum a\,(p-a) = \frac{1}{\Delta}\cdot\left[p\cdot\sum a - \sum a^2\right]$$

$$= \frac{1}{\Delta}\cdot\left[2p^2 - \sum a^2\right] = \frac{1}{\Delta}\cdot(2p^2 - 2p^2 + 8Rr + 2r^2) =$$

$$= \frac{1}{\Delta}\cdot 2r(4R+r) = \frac{2\cdot(4R+r)}{p} \quad (*)$$

LHS $\Rightarrow \sqrt{3}p \le 4R + r \,|\cdot\frac{2}{p} \Leftrightarrow 2\sqrt{3} \le \underbrace{\frac{2\cdot(4R+r)}{p}}_{LHS}$

$$RHS \Rightarrow \frac{2\cdot(4R+r)}{p} \le \frac{2(4R+r)}{3\sqrt{3}r} \overset{Euler}{\le} \frac{2\cdot\left(4R+\frac{R}{r}\right)}{3\sqrt{3}r} =$$

$$= \frac{2\cdot\left(\frac{9R}{2}\right)}{3\sqrt{3}r} = \frac{\sqrt{3}R}{r} \text{ (RHS)}$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$r_a = \frac{\Delta}{p-a}, r_b = \frac{\Delta}{p-b}, r_c = \frac{\Delta}{p-c} \text{ and } \frac{R(r+4R)^2}{2(2R-r)} \geq p^2$$

$$\therefore \sum_{cyc} \frac{a}{r_a} = \sum_{cyc} \frac{a(p-a)}{\Delta} = \frac{p(a+b+c)-(a^2+b^2+c^2)}{\Delta}$$

$$= \frac{2p^2 - 2(p^2 - r^2 - 4Rr)}{\Delta} = \frac{2r(r+4R)}{\Delta} = \frac{2(r+4R)}{p} \leq \frac{2\left(\frac{R}{2}+4R\right)}{3\sqrt{3}r}$$

$$= \frac{R\sqrt{3}}{r}$$

we need to prove, $\frac{2(r+4R)}{p} \geq 2\sqrt{3} \geq \frac{(r+4R)^2}{3} \geq p^2$

if we can show that, $\frac{(r+4R)^2}{3} \geq \frac{R(r+4R)^2}{2(2R-r)} \Leftrightarrow (r+4R)^2(R-2r) \geq 0$

which is true,

$$\therefore 2\sqrt{3} \leq \sum_{cyc} \frac{a}{r_a} \leq \frac{\sqrt{3}R}{r}$$

(Proved)