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If $a,b,c>0, abc=1$ then:

$$a+b+c\geq\frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}$$

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Solution 1 by Kevin Soto Palacios-Huarmey-Peru, Solution 2 by Geanina Tudose-Romania, Solution 3 by Ngo Tuan-Vietnam, Solution 4 by Nguyen Thanh Nho-Tra Vinh-Vietnam, Solution 5 by Sanong Hauerai-Nakon Pathom-Thailand, Solution 6 by Nguyen Ngoc Tu-Ha Giang-Vietnam, Solution 7 by Seyran Ibrahimov-Maasilli-Azerbaidian, Solution 8 by Suvam Bhattacherjee-India

Solution 1 by Kevin Soto Palacios-Huarmey-Peru

Siendo $a,b,c>0$, de tal manera que $abc=1$. Probar que

$$a+b+c\geq\frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}$$

Es suficiente probar

$$\frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}\leq 3\Leftrightarrow$$

$$(b+2)(c+2)+(c+2)(a+2)+(a+2)(b+2)\leq$$

$$\leq(a+2)(b+2)(c+2)$$

$$\Leftrightarrow 4(a+b+c)+ab+bc+ca+12\leq 8+4(a+b+c)+2(ab+bc+ca)+abc$$

$$\Leftrightarrow 12\leq 8+ab+bc+ca+abc=9+ab+bc+ca\Leftrightarrow 3\leq ab+bc+ca$$

(Válido por $MA\geq MG$)

Por lo tanto

$$\frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}\leq 3\leq a+b+c$$

(LQQD)

Solution 2 by Geanina Tudose-Romania

We have $\frac{a+b+c}{3}\geq\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}$

$$\text{From } AM \geq GM \Rightarrow \frac{a+b+c}{3} \geq \sqrt[3]{abc} = 1$$

$$\text{We will show } \frac{1}{a+2} + \frac{1}{b+2} + \frac{1}{c+2} \leq 1 \text{ provided } abc = 1$$

$$\text{Let } \left.\begin{array}{l} a = \frac{x}{y} \\ b = \frac{y}{z} \\ c = \frac{z}{x} \end{array}\right\} \Rightarrow \text{The inequality becomes,}$$

$$\frac{y}{x+2y} + \frac{z}{y+2z} + \frac{x}{z+2x} \leq 1 \Big| \cdot (-1) \Leftrightarrow -\frac{y}{x+2y} - \frac{z}{y+2z} - \frac{x}{z+2x} \geq -1 \Big| + \frac{3}{2}$$

$$\Leftrightarrow \frac{x}{x+2y} + \frac{y}{y+2z} + \frac{z}{z+2x} \geq 1$$

$$\Leftrightarrow \frac{x^2}{x^2+2yx} + \frac{y^2}{y^2+2yz} + \frac{z^2}{z^2+2xz} \geq 1$$

But from CBS (Engel form)

$$\frac{x^2}{x^2+2xy} + \frac{y^2}{y^2+2yz} + \frac{z^2}{z^2+2xz} \geq \frac{(x+y+z)^2}{x^2+y^2+z^2+2xy+2xz+2yz} = 1$$

Solution 3 by Ngo Tuan-Vietnam

$$\text{If } a; b; c > 0, abc = 1 \text{ then}$$

$$a+b+c \geq \frac{3}{a+2} + \frac{3}{b+2} + \frac{3}{c+2}$$

$$\Leftrightarrow (a+2)(b+2)(c+2)(a+b+c) \geq$$

$$\geq 3(b+2)(c+2) + 3(c+2)(a+2) + 3(a+2)(b+2)$$

$$\Leftrightarrow [9 + 2(ab+bc+ca) + 4(a+b+c)](a+b+c) \geq$$

$$3(bc+ca+ab) + 12(a+b+c) + 36$$

$$\Leftrightarrow 4(a+b+c)^2 + 2(a+b+c)(ab+bc+ca) - 3(a+b+c) \geq$$

$$\geq 3(ab+bc+ca) + 36$$

$$\Leftrightarrow 4[(a+b+c)^2 - 9] + (a+b+c)(ab+bc+ca-3) + (ab+bc+ca)(a+b+c-3) \geq 0 \quad (1)$$

By AM-GM, we have:

$$a+b+c \ge 3\sqrt[3]{abc} = 3$$

$$ab+bc+ca \ge 3\sqrt[3]{a^2b^2c^2} = 3$$

$$\Leftrightarrow (a+b+c)^2 \ge 9 \Rightarrow \textbf{(1) true} \Rightarrow \textbf{Q.E.D.}$$

Solution 4 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$a,b,c>0, abc=1$$

$$(a+b+c)^2 \ge 3(ab+bc+ca) \ge 3\cdot\frac{\left(\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}\right)^3}{9}$$

$$\ge \left(\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}\right)^2\cdot\left(\frac{\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}}{3}\right)$$

$$\ge \left(\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}\right)^2\cdot\left(\frac{3\sqrt[3]{\sqrt[3]{a^2b^2c^2}}}{3}\right)$$

$$\Rightarrow (a+b+c)^2 \ge \left(\sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca}\right)^2, (abc=1)$$

$$\Rightarrow a+b+c \ge \sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca} \quad \textbf{(1)}$$

$$a+1+1 \ge 3\sqrt[3]{a} \Leftrightarrow a+2 \ge \frac{3}{\sqrt[3]{bc}} \Leftrightarrow \sqrt[3]{bc} \ge \frac{3}{a+2}$$

$$\Rightarrow \sqrt[3]{ab}+\sqrt[3]{bc}+\sqrt[3]{ca} \ge \frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2} \quad \textbf{(2)}$$

$$\textbf{(1)} \,\&\, \textbf{(2)} \Rightarrow a+b+c \ge \frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}$$

Solution 5 by Sanong Hauerai-Nakon Pathom-Thailand

$$\textbf{Give } a=\frac{x}{y}, b=\frac{y}{z}, d=\frac{z}{x}, \frac{x}{y}=\frac{m^3}{n^3}; \frac{y}{z}=\frac{n^3}{p^3}, \frac{z}{x}=\frac{p^3}{m^3}$$

$$\frac{3}{a+2}+\frac{3}{b+z}+\frac{3}{c+z} \le a+b+d$$

$$\textbf{Iff } \frac{1}{a\left(1+\frac{2}{a}\right)}+\frac{1}{a\left(1+\frac{2}{b}\right)}+\frac{1}{c\left(1+\frac{2}{c}\right)} \le \frac{a+b+c}{3}$$

$$Iff\ \frac{\frac{1}{a}}{\left(1+\frac{1}{a}+\frac{1}{a}\right)}+\frac{\frac{1}{b}}{\left(1+\frac{1}{b}+\frac{1}{b}\right)}+\frac{\frac{1}{c}}{\left(1+\frac{1}{c}+\frac{1}{c}\right)}\le\frac{a+b+c}{3}$$

$$Iff\ \frac{\frac{y}{x}}{\left(1+\frac{y}{x}+\frac{y}{x}\right)}+\frac{\frac{z}{y}}{\left(1+\frac{z}{y}+\frac{z}{y}\right)}+\frac{\frac{x}{z}}{\left(1+\frac{x}{z}+\frac{x}{z}\right)}\le\frac{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}{3}$$

$$Iff\ \frac{\frac{y}{x}}{3\sqrt[3]{\frac{y^2}{x^2}}}+\frac{\frac{z}{y}}{3\sqrt[3]{\frac{z^2}{y^2}}}+\frac{\frac{x}{z}}{3\sqrt[3]{\frac{x^2}{z^2}}}\le\frac{\frac{x}{y}+\frac{y}{z}+\frac{z}{x}}{3}$$

$$Iff\ \sqrt[3]{\frac{y}{x}}+\sqrt[3]{\frac{z}{y}}+\sqrt[3]{\frac{x}{z}}\le\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$$

$$Iff\ \frac{n}{m}+\frac{p}{n}+\frac{m}{p}\le\frac{m^3}{n^3}+\frac{n^3}{p^3}+\frac{p^3}{m^3}$$

and it is to be true

$$Because\ \frac{m^3}{n^3}+\frac{n^3}{p^3}+\frac{p^3}{m^3}\ge\frac{m}{p}\sqrt{\frac{m}{p}}+\frac{n}{m}\sqrt{\frac{n}{m}}+\frac{p}{n}\sqrt{\frac{p}{n}}$$

Therefore this is to be true

Solution 6 by Nguyen Ngoc Tu-Ha Giang-Vietnam

With $abc=1$ **we have** $a+b+c\ge 3, ab+bc+ca\ge 3,$

$$a^2+b^2+c^2\ge a+b+c$$

We have

$$a+b+c\ge\frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}$$

$$\Leftrightarrow (a+b+c)(a+2)(b+2)(c+2)\ge 12(a+b+c)+3(ab+bc+ca)+36$$

$$\Leftrightarrow 2(a+b+c)(ab+bc+ca)+9(a+b+c)+4(a+b+c)^2\ge$$

$$\ge 12(a+b+c)+3(ab+bc+ca)+36$$

We have

$$2(a+b+c)(ab+bc+ca)\ge 6(ab+bc+ca)\ge 3(ab+bc+ca)+9$$

$$4(a+b+c)^2=4(a^2+b^2+c^2)+8(ab+bc+ca)\ge 3(a+b+c)+27$$

$$\Rightarrow 2(a+b+c)(ab+bc+ca)+9(a+b+c)+4(a+b+c)^2 \geq$$
$$\geq 12(a+b+c)+3(ab+bc+ca)+36$$

Solution 7 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\begin{matrix} a+1+1 \geq 3\sqrt[3]{a} \\ b+1+1 \geq 3\sqrt[3]{b} \\ c+1+1 \geq 3\sqrt[3]{c} \end{matrix} \Rightarrow \sum a \geq \frac{27\left(\sum \sqrt[3]{ab}\right)}{27\sqrt[3]{abc}}$$

$$\sum a \geq \sum \sqrt[3]{ab}$$

$$\begin{matrix} a = x^3 \\ b = y^3 \\ z = z^3 \end{matrix} \Rightarrow \begin{matrix} x^3+y^3+z^3 \geq xy+yz+zx \\ x^3+y^3+z^3 \geq \frac{1}{3}(x+y+z)(xy+yz+xz) \end{matrix}$$

$$\sum x = \sqrt[3]{a}+\sqrt[3]{b}+\sqrt[3]{c} \geq 3\sqrt[9]{abc} = 3$$

Solution 8 by Suvam Bhattacherjee-India

$$a, b, c > 0 \quad abc = 1$$

Then $a+b+c \geq \frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}$

Let $f(x) = \frac{1}{x+2}$ **and** $f(x) < 0$ **Hence** $f(x)$

decreases

now $\frac{a+b+c}{3} \geq \sqrt[3]{abc} = 1$

Hence $\frac{a+b+c}{3}$ **increases from one**

And $f\left(\frac{a+b+c}{3}\right)$ **decreases**

$x = 3y$ **and** $y = \frac{1}{x+2}$ **intersect at** $\left(1, \frac{1}{3}\right)$

after that $f\left(\frac{a+b+c}{3}\right) = \frac{3}{a+b+c+6}$ **decreases**

Hence $\frac{3}{(a+2)+(b+2)+(c+2)}$ **decreases**

$$\Rightarrow \frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+3} \text{ decreases}$$

Hence abscissa of the triangle having vertices

$(a,f(a)),(b,f(b)),(c,f(c))$ *increases and its ordinate decreases and*

$$\left(\frac{a+b+c}{3},\frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}\right)$$

x *increases from* 1 *and* y *decreases from* $\frac{1}{3}$

$$\text{Hence } \frac{a+b+c}{3} \geq 3\underbrace{\left(\frac{1}{a+2}+\frac{1}{b+2}+\frac{1}{c+2}\right)}_{3}$$

$$\Rightarrow a+b+c \geq \frac{3}{a+2}+\frac{3}{b+2}+\frac{3}{c+2}$$

A unique way to solve.