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1. In ΔABC

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 4r(4R+r)$$

Proposed by Kevin Soto Palacios - Huarmey - Peru

Proof.

Using the means inequality we obtain:

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 3\sqrt[3]{(abc)^2}=3\sqrt[3]{(4Rrp)^2}\geq 4r(4R+r),$$

where the last inequality is equivalent with:

$$27\cdot(4Rrp)^2\geq 64r^3(4Rr+r)^3\Leftrightarrow 27R^2p^2\geq 4r(4R+r)^3$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$27R^2(16Rr-5r^2)\geq 4r(4R+r)^3\Leftrightarrow 176R^3-327R^2r-48Rr^2-4r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)(176R^2+25Rr+2r^2)\geq 0,\ \textit{obviously from Euler's inequality } R\geq 2r.$$

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1.** can be strengthened:*

2. In ABC

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 18Rr.$$

Proof.

Using means inequality we obtain:

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 3\sqrt[3]{(abc)^2}=3\sqrt[3]{(4Rrp)^2}\geq 18Rr,$$

where the last inequality is equivalent with:

$$(4Rrp)^2\geq (6Rr)^3\Leftrightarrow 16R^2r^2p^2\geq 216R^3r^3\Leftrightarrow 2p^2\geq 27Rr$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$2(16Rr-5r^2)\geq 27Rr\Leftrightarrow R\geq 2r\ \textit{(Euler's inequality).}$$

The equality holds if and only if the triangle is equilateral.

□

Remark 2.

*Inequality **1.** is stronger then inequality **2.**:*

3. In ΔABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 18Rr\geq 4r(4R+r)}$$

Proof.

*See inequality **2.** and Euler's inequality $R\geq 2r$.*

The equality holds if and only if the triangle is equilateral.

□

Remark 3.

*Inequality **1.** can be developed:*

4. In ΔABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq r\Big[nR+(36-2n)r\Big]}\textbf{, where } \boldsymbol{n\leq 19}$$

Proposed by Marin Chirciu - Romania

Proof.

Using the means inequality we obtain:

$$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\geq 3\sqrt[3]{(abc)^2}=3\sqrt[3]{(4Rrp)^2}\geq r\Big[nR+(36-2n)r\Big],$$

where the last inequality is equivalent with:

$$27\cdot(4Rrp)^2\geq r^3\Big[nR+(36-2n)r\Big]^3\Leftrightarrow 27\cdot 16R^2r^2p^2\geq 64r^3\Big[nR+(36-2n)r\Big]^3\Leftrightarrow$$

$$432R^2p^2\geq r\Big[nR+(36-2n)r\Big]^3$$

which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$. It remains to prove that:

$$432R^2(16Rr-5r^2)\geq 4\Big[nR+(36-2n)r\Big]^3\Leftrightarrow 432R^2(16R-5r)\geq\Big[nR+(36-2n)r\Big]^3\Leftrightarrow$$

$$6912R^3-2160R^2r\geq (nR)^3+3\cdot(nR)^2\cdot(36-2n)r+3(nR)\Big[(36-2n)r\Big]^2+\Big[(36-2n)r\Big]^3\Leftrightarrow$$

$$(6912-n^3)R^3+(6n^3-108n^2-2160)R^2r+(-12n^3+432n^2-3888n)Rr^2+$$
$$+(8n^3-432n^2+7776n-46656)r^3\geq 0\Leftrightarrow$$

$$\Leftrightarrow (R-2r)\Big[(6912-n^3)R^2+(4n^3-108n^2+11664)Rr+(-4n^3+216n^2+23328)r^2\Big]\geq 0$$

obviously from Euler's inequality $R\geq 2r$ and the condition $n\geq 19$

which assures the positivity of the right parenthesis.

The equality holds if and only if the triangle is equilateral.

□

Note.

*For $n=16$ we obtain inequality **1.**, and for $n=18$ we obtain inequality **2.***

Let's find an inequality having an opposite sense.

5. In ΔABC

$$\boldsymbol{a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab}\leq 4(R+r)^2.}$$

Proof.

Using inequality $xy+yz+zx \leq x^2+y^2+z^2$ *for* $x=\sqrt{bc}, y=\sqrt{ca}, z=\sqrt{ab}$ *we obtain:*
$a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \leq ab+bc+ca = p^2+r^2+4Rr \leq 4R^2+4Rr+3r^2+r^2+4Rr = 4(R+r)^2.$
where the last inequality follows from Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$.

The equality holds if and only if the triangle is equilateral.

□

We can write the double inequality:

6. In $\boldsymbol{\Delta ABC}$

$$\boldsymbol{18Rr \leq a\sqrt{bc}+b\sqrt{ca}+c\sqrt{ab} \leq 4(R+r)^2}$$

Proof.

See inequalities ***2.*** *and* ***5.***

The equality holds if and only if the triangle is equilateral.

□

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