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$$f: \mathbb{R} \rightarrow [a, b], a < b$$

Find:

$$\Omega = \lim_{n \rightarrow \infty} \sum_{k=1}^n \frac{(n-k+1)^2 f(k)}{k(1^2 + 2^2 + \dots + n^2)}$$

Proposed by Daniel Sitaru – Romania

Solution by Saptak Bhattacharya-Kolkata-India

$$\begin{aligned} & \lim_{n \rightarrow \infty} 6 \sum_{k=1}^n \frac{(n-k+1)^2 (k)}{kn(n+1)(2n+1)} \\ = & \lim_{n \rightarrow \infty} \frac{6}{n(n+1)(2n+1)} \sum_{k=1}^n \frac{(n-k+1)^2 f(k)}{k} \end{aligned}$$

Now,

$$a \leq f(k) \leq b,$$

And,

$$\begin{aligned} & \lim_{n \rightarrow \infty} \frac{6}{n(n+1)(2n+1)} \sum_{k=1}^n \frac{(n-k+1)^2 a}{k} \\ = & \lim_{n \rightarrow \infty} \frac{6a}{n(n+1)(2n+1)} \cdot \sum_{k=1}^n \frac{n^2 + k^2 + 1 - 2k - 2nk + 2n}{k} \\ = & \lim_{n \rightarrow \infty} \frac{6a}{n(n+1)(2n+1)} \cdot \left[(n+1)^2 H_n + \frac{n(n+1)}{2} - 2n(n+1) \right] \\ = & \lim_{n \rightarrow \infty} \frac{6a}{n(n+1)(2n+1)} \left((n+1)^2 H_n - \frac{3n(n+1)}{2} \right) \\ = & a \lim_{n \rightarrow \infty} 6 \left(\frac{(n+1)H_n}{n(2n+1)} - \frac{3}{2(2n+1)} \right) \end{aligned}$$

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$$= 6a \left(\frac{1}{2} \lim_{n \rightarrow \infty} \frac{H_n}{n} \right)$$
$$= 3a \lim_{n \rightarrow \infty} \frac{H_n}{n} = 0 \text{ (Cauchy first theorem)}$$

Similarly,

$$\lim_{n \rightarrow \infty} \frac{6b}{n(n+1)(2n+1)} \sum_{k=1}^n \frac{(n-k+1)^2}{k} = 0$$

Thus by squeeze theorem, the given limit is 0