If $a, b, c > 0$ then:

$$\frac{1}{(2a^2+bc)^2}+\frac{1}{(2b^2+ca)^2}+\frac{1}{(2c^2+ab)^2}\ge\frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

Proposed by Nguyen Ngoc Tu-Ha Giang-Vietnam

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam, Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand, Solution 3 by Nguyen Thanh Nho-Tra Vinh-Vietnam, Solution 4 by SK Rejuan-West Bengal-India

Solution 1 by Hoang Le Nhat Tung-Hanoi-Vietnam

We have:

$$3(a^6+b^6+c^6)\ge(a^4+b^4+c^4)(a^2+b^2+c^2)\ge(a^4+b^4+c^4)\cdot\frac{(a+b+c)^2}{3}$$

$$\Rightarrow\frac{(a+b+c)^2}{9(a^6+b^6+c^6)}\le\frac{1}{a^4+b^4+c^4}\quad\textbf{(1)}$$

By Cauchy – Scwarz:

$$\sum\frac{1}{(2a^2+bc)^2}\ge\frac{\left(\sum\frac{1}{2a^2+bc}\right)^2}{3}\ge\frac{\left(\frac{9}{\sum(2a^2+bc)}\right)^2}{3}$$

$$\Rightarrow\sum\frac{1}{(2a^2+bc)^2}\ge\frac{27}{(2\sum a^2+\sum bc)^2}\ge\frac{27}{(2\sum a^2+\sum a^2)^2}=\frac{27}{9(\sum a^2)^2}$$

$$\Rightarrow\sum\frac{1}{(2a^2+bc)^2}\ge\frac{3}{(\sum a^2)^2}\ge\frac{3}{3\sum a^4}=\frac{1}{\sum a^4}\quad\textbf{(2)}$$

Then (1), (2) $\Rightarrow\sum\frac{1}{(2a^2+bc)^2}\ge\frac{(\sum a)^2}{9\cdot\sum a^6}$

$$\Rightarrow Q.E.D$$

Solution 2 by Sanong Hauerai-Nakon Pathom-Thailand

Because $(a^3+b^3+c^3)(a+b+c)\ge2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^2a$ ***is to be true***

Imply

$$(a^3+b^3+c^3)^2(a+b+c)^2 \geq (2(a^2b^2)+(b^2c^2)+(c^2a^2)+a^3b+b^3c+c^3a)^2$$

Imply

$$9(a^3+b^3+c^3)^2(a+b+c)^2 \geq 9(2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^3a)^2$$

Imply

$$9(a^6+b^6+c^6)(a+b+c)^2 \geq 3(2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^3a)^2$$

$$\textit{Imply}\ \frac{\frac{(a+b+c)^2}{\left(2(a^2b^2+b^2c^2+c^2a^2)+a^3b+b^3c+c^3a\right)^2}}{3} \geq \frac{1}{9(a^6+b^6+c^6)}$$

$$\textit{Imply}\ \frac{\left(\frac{(a+b+c)^2}{2(a^2b^2+b^2c^2+c^2a^2+a^3b+b^3c+c^3a)}\right)^2}{3} \geq \frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

$$\textit{Imply}\ \frac{\left(\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\right)^2}{3} \geq \frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

$$\textit{Imply}\ \frac{1}{(2a^2+bc)^2}+\frac{1}{(2b^2+ca)^2}+\frac{1}{(2c^2+ab)^2} \geq \frac{(a+b+c)^2}{9(a^6+b^6+c^6)}$$

There for it is to be true

Solution 3 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$LHS = \frac{1}{(2a^2+bc)^2}+\frac{1}{(2b^2+ca)^2}+\frac{1}{(2c^2+ab)^2} \geq \frac{1}{3}\left(\frac{1}{2a^2+bc}+\frac{1}{2b^2+ca}+\frac{1}{2c^2+ab}\right)^2$$

$$\geq \frac{1}{3}\left(\frac{9}{2(a^2+b^2+c^2)+ab+bc+ca}\right)^2 \geq \frac{1}{3}\left(\frac{9}{3(a^2+b^2+c^2)}\right)^2 = \frac{3}{(a^2+b^2+c^2)^2} \quad (*)$$

$$a^6+b^6+c^6 \geq 3\left(\frac{a^2+b^2+c^2}{3}\right)^3 = \frac{(a^2+b^2+c^2)^2}{9}\cdot(a^2+b^2+c^2)$$

$$\geq \frac{(a^2+b^2+c^2)}{9}\cdot\frac{1}{3}(a+b+c)^2$$

$$\Rightarrow \frac{3}{(a^2+b^2+c^2)^2} \geq \frac{(a+b+c)^2}{9(a^6+b^6+c^6)} = RHS \quad (**)$$

$$(*)\ \&\ (**) \Rightarrow LHS = \frac{3}{(a^2+b^2+c^2)} \geq RHS \Rightarrow LHS \geq RHS$$

$$\text{“ = “} \Leftrightarrow a = b = c$$

Solution 4 by SK Rejuan-West Bengal-India

$$a, b, c > 0$$

$$\textit{by AM} \geq \textit{GM we get, } 2a^2 + \frac{b^2+c^2}{2} \geq 2a^2 + bc$$

$$\Rightarrow \frac{4a^2 + b^2 + c^2}{2} \geq 2a^2 + bc \Rightarrow \sum \frac{4a^2 + b^2 + c^2}{2} \geq \sum (2a^2 + bc)$$

$$\Rightarrow 3\sum a^2 \geq \sum(2a^2 + bc) \Rightarrow \frac{1}{\sum(2a^2+bc)} \geq \frac{1}{3\sum a^2} \quad (1)$$

$$LHS = \sum\left(\frac{1}{2a^2+bc}\right)^2 = P \quad (\text{say})$$

$$\Rightarrow P \geq \frac{3}{9}\left(\sum\frac{1}{2a^2+bc}\right)^2 \quad [\text{by mth power theorem}]$$

$$\geq \frac{1}{3}\left\{\frac{9}{\sum(2a^2+bc)}\right\}^2 \quad [\text{by AM} \geq \text{HM}] \geq \frac{1}{3}\left\{\frac{9}{3\sum a^2}\right\}^2 \quad [\text{from (1)}]$$

$$\Rightarrow P \geq \frac{3}{(\sum a^2)^2} \geq \frac{1}{\sum a^4} \quad [\text{by Cauchy inequality}] \Rightarrow P \geq \frac{1}{\sum a^4} \quad (2)$$

$$RHS = \frac{1}{9}\,\frac{(\sum a)^2}{\sum a^6} = 9 \quad (\text{say})$$

$$\Rightarrow 9 = \frac{1}{9}\frac{(\sum a)^2}{\sum a^6} \leq \frac{1}{9}\cdot\frac{3\sum a^2}{\sum a^6} \quad [\text{by Cauchy inequality}] \Rightarrow 9 \leq \frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6} \quad (3)$$

We now have to prove that,

$$\frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6} \leq \frac{1}{\sum a^4} \Leftrightarrow \frac{\sum a^2}{3}\cdot\frac{\sum a^4}{3} \leq \frac{\sum a^6}{3} = \frac{\sum a^{2+9}}{3}$$

$$\Leftrightarrow \frac{\sum a^2}{3}\cdot\frac{\sum a^4}{3} \leq \frac{\sum a^6}{3} = \frac{\sum a^{2+4}}{3}$$

which is true

i.e., $\frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6}\leq\frac{1}{\sum a^4}$ ***(4)***

Combining (2), (3) & (4) we get

$$9\leq\frac{1}{3}\cdot\frac{\sum a^2}{\sum a^6}\leq\frac{1}{\sum a^4}\leq P\Rightarrow P\geq 9$$

$$\Rightarrow\sum\left(\frac{1}{2a^2+bc}\right)^2\geq\frac{1}{9}\cdot\frac{(\sum a)^2}{\sum a^6}$$

[Proved]