THEOREMS OF CONCAVITY AND CONVEXITY OF COMPOSITE FUNCTIONS-APPLICATION TO CLASSICAL INEQUALITIES.

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Abstract

The paper presents the theorems of concavity and convexity of composite functions. It goes further to present an inequality lemma. It ends with demonstrative examples.

1.0 INTRODUCTION

General form of a composite function of 'x'

$$\boldsymbol{f(x) = f_1 o f_2 o f_3 o \dots o f_i o \dots o f_n(x)}$$

Equivalently, we can write

$$\boldsymbol{f(x) = f_1(f_2\left(f_3(\dots(f_i(\dots f_n(x)\dots))\dots)\right))}$$

where f_i is the " ith-sub function"

and f_n is the " nth-sub function"

DEFINITIONS

1. Concavity of a function : If a function $f(x)$ is continuous in an interval $\boldsymbol{I}$ and $f"(x) < 0 \;\forall\; x \; in \; I$ then $f(x)$ is concave in I.
2. Convexity of a function : If a function $f(x)$ is continuous in an interval $\boldsymbol{I}$ and $f"(x) > 0 \;\forall\; x \; in \; I$ then $f(x)$ is convex in I.

2.0 THEOREMS.

Two theorems will be stated and proven.

Theorem 1: Theorem of Concavity of composite functions

If a composite function $\boldsymbol{f(x) = f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n(x)}$ is continuous in an interval $I \in (a, b)$ where $b > a$ such that

- $f_i > 0 \;\forall\; x \; in \; I \; and \;\; i \in [2, n]$,
- $f_i is \; increasing \;\; \forall\; x \; in \; I \; and \;\; i \in [1, n-1]$,
- $f_i \; is \; concave \;\; \forall\; x \; in \; I \; and \;\; i \in [1, n]$

  Then $\boldsymbol{f(x) = f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n(x)}$ $is \; \boldsymbol{concave} \; in \; I.$

Theorem 2: Theorem of Convexity of composite functions

If a composite function $\boldsymbol{f(x) = f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n(x)}$ is continuous in an $I \in (a, b)$ where $b > a$ such that

- $f_i > 0 \;\forall\; x \; in \; I \; and \;\; i \in [2, n]$,
- $f_i is \; increasing \;\; \forall\; x \; in \; I \; and \;\; i \in [1, n-1]$,
- $f_i \; is \; convex \;\; \forall\; x \; in \; I \; and \;\; i \in [1, n]$

  then $\boldsymbol{f(x) = f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n(x)}$ $is \; \boldsymbol{convex} \; in \; I.$

2.1 PROOF OF THEOREMS

The necessary and sufficient conditions will be shown in the proof of theorems.

Proof of Theorem1: Theorem of Concavity of a composite function

Consider $\boldsymbol{f(x) = f_1 o f_2 o f_3 o \dots o f_i o \dots o f_n(x)}$ (1)

Requirements:

$f_i(x) > 0 \ \forall\, i \ \in [2, n]\ , \forall\, x \ \in \ I$ $(1a)$

$f_i'(x) > 0 \ \forall\, i \ \in [1, n-1], \forall\, x \ \in I$ $(1b)$

$f_i''(x) < 0 \ \forall\, i \ \in [1, n], \forall\, x \ \in I$ $(1c)$

Aim: To prove $\boldsymbol{f''(x) < 0} \ \forall\, x \ \in I$

Procedure:

$\boldsymbol{f(x) = f_1 o f_2 o f_3 o \dots o f_i o \dots o f_n(x)}$

$f'(x) = \prod_{i=1}^{n} f_i'(x)$ (1d)

$f''(x) = f_1'' \times \prod_{i=2}^{n} (f_i')^2 + f_1' f_2'' \times \prod_{i=3}^{n} (f_i')^2 + \cdots + f_n' \times (\prod_{i=1}^{n-1} f_1')$ (1e)

By (1a)-(1c), $\boldsymbol{f''(x) < 0}$ **(**proof complete)

Proof of Theorem2: Theorem of Convexity of a composite function

Consider $\boldsymbol{f(x) = f_1 o f_2 o f_3 o \dots o f_i o \dots o f_n(x)}$ (2)

Requirements:

$$f_i(x) > 0 \; \forall \; i \; \in [2, n] \,, \forall \; x \; \in \; I \qquad (2a)$$
$$f_i'(x) > 0 \; \forall \; i \; \in [1, n-1], \forall \; x \; \in I \qquad (2b)$$
$$f_i''(x) > 0 \; \forall \; i \; \in [1, n], \forall \; x \; \in I \qquad (2c)$$

Aim: To prove $\boldsymbol{f''(x) > 0} \; \forall \; x \; \in I$

Procedure:

$$\boldsymbol{f(x) = f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n(x)}$$

$$f'(x) = \prod_{i=1}^{n} f_i'(x) \qquad \text{(2d)}$$

$$f''(x) = f_1'' \times \prod_{i=2}^{n} (f_i')^2 + f_1' f_2'' \times \prod_{i=3}^{n} (f_i')^2 + \cdots + f_n' \times (\prod_{i=1}^{n-1} f_1') \qquad \text{(2e)}$$

By (2a)-(2c), $\boldsymbol{f''(x) > 0}$ **(**proof complete)

3.0 <u>Application</u>

We shall present an application of the concept of composite functions to classical inequalities.

<u>3.1 Case1: Inequality theorem for Concave composite function</u>

Consider the real valued composite function

$$\boldsymbol{f(x) = f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n(x)}$$

If *the concavity of* $\boldsymbol{f(x)}$ ***is ascertained in I*** $(\boldsymbol{i.e} \;\; \boldsymbol{f''(x_i)} \; \boldsymbol{< 0} \; \forall \; \boldsymbol{x_i} \; \in \boldsymbol{I})$ then the following holds

$$\frac{\sum_{i=1}^{n} w_i \, (\boldsymbol{f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n}(x_i)\,)}{\sum_{i=1}^{n} w_i} \leq \boldsymbol{f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n}\Big(\frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}\Big) \quad \text{(3a)}$$

It is easy to see that (3a) is similar to the famous Jensen inequality for weighted concave functions.

3.2 Case2: Inequality theorem for Convex composite function

Consider the composite function

$\boldsymbol{f(x) = f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n(x)}$

If $\textit{the convexity of } \boldsymbol{f(x)}$ $\boldsymbol{\textit{is ascertained in I}}$ $(\boldsymbol{i.e\ f''(x_i)} > \boldsymbol{0}\ \forall\ \boldsymbol{x_i} \in \boldsymbol{I})$ then the following holds

$$\frac{\sum_{i=1}^{n} w_i\,(\boldsymbol{f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n}(x_i))}{\sum_{i=1}^{n} w_i} \geq \boldsymbol{f_1 o f_2 o f_3 o \ldots o f_i o \ldots o f_n}\left(\frac{\sum_{i=1}^{n} w_i x_i}{\sum_{i=1}^{n} w_i}\right) \quad (3b)$$

It is easy to see that (3b) is similar to the famous Jensen inequality for weighted convex functions.

4.0 Demonstrations

We shall now use simple inequality problems to demonstrate the use of these inequality theorems.

Problem 1.

Given that a and b are positive real numbers such that $a + b = 1$.

Find the minimum value of the expression: $(a + 1/a)^2$ +$(b + 1/b)^2$

Solution.

$(a + 1/a)^2$ +$(b + 1/b)^2$=$\sum_{cyl}(a + 1/a)^2$

Consider the composite function $f(x) = \boldsymbol{f_1 o f_2(x)}$ **where** $\boldsymbol{f_2(x)} = x + 1/x$ and $\boldsymbol{f_1(x) = x^2}$ **.** We can easily establish the convexity of $f(x) = \boldsymbol{f_1 o f_2(x)}$ following theorem 2.

For $\boldsymbol{f_1(x) = x^2}$, we have

$$\boldsymbol{f_1(x) = x^2 > 0 \; \forall \; x > 0 \; in \; reals.}$$

$$\boldsymbol{f_1'(x) = 2x > 0 \; \forall \; x > 0 \; in \; reals.}$$

$$\boldsymbol{f_1''(x) = 2 > 0 \; \forall \; x > 0 \; in \; reals.}$$

For $\boldsymbol{f_2(x)} = x + 1/x$, we have

$$\boldsymbol{f_1(x) = x + \frac{1}{x} > 0 \; \forall \; x \; \in \; (0, \infty)}$$

$$\boldsymbol{f_1'(x) = 1 - \frac{1}{x^2} < 0 \; \forall \; x \; (0, 1) \; and \; f_1'(x) = 1 - \frac{1}{x^2} > 0 \; \forall \; x \; \in \; (1, \infty)}$$

$$\boldsymbol{f_1''(x) = \frac{2}{x^3} > 0 \; \forall \; x \; \in \; (0, \infty)}$$

Clearly all sub-functions are convex and all sub-functions except the “n-th” sub-function are increasing monotone.

Clearly $f(x) = \boldsymbol{f_1 o f_2(x)}$ is convex $\boldsymbol{x \in (0, \infty)}$ by theorem 2. We proceed to apply Inequality Theorem for Convex Composite function (Case 1)

$$\sum_{cyl}\left(a + \frac{1}{a}\right)^2 \geq 2\left(\sum_{cyl} a + \frac{1}{\sum_{cyl} a}\right)^2 = 2\left(\frac{a+b}{2} + \frac{1}{\frac{a+b}{2}}\right)^2 = 2\left(\frac{1}{2} + \frac{1}{\frac{1}{2}}\right)^2 = \frac{25}{2}$$

Min$\{(a + 1/a)^2 + (b + 1/b)^2\} = \frac{25}{2}$

Equality at $a = b = 1/2$

Problem 2.

In any triangle ABC, prove the generalization (author's generalization of Daniel Sitaru's inequality published in RMM-2017)

$$\prod_{i=1}^{n}\left(\sum_{Cyl}\cos\left(\frac{A}{2}\right)\right)^{\frac{1}{2^i}} \geq 3^n\left(\frac{3}{4}\right)^{\frac{\sum_{i=1}^{n}2^{n-1}}{2^{n+1}}}$$

Solution

We identify $f(x) = \boldsymbol{f_1 o f_2(x)}$,where $\boldsymbol{f_1(x) = x^{\frac{1}{2^i}}}$ and $\boldsymbol{f_2(x)} = \cos\left(\frac{x}{2}\right)$

We can easily verify the concavity of $\boldsymbol{f_1(x)\ and\ f_2(x)}$ **.** Also we see that all sub-functions functions except the "n-th sub-functions" are increasing, hence $f(x) = \boldsymbol{f_1 o f_2(x)}$ is concave (by theorem 1)

We proceed to apply Inequality Theorem for Concave Composite function (Case 2)

$$\left(\sum_{Cyl}\cos\left(\frac{A}{2}\right)\right)^{\frac{1}{2^i}} \geq 3cos\left(\frac{1}{2}\frac{\sum_{cyl}A}{3}\right)^{\frac{1}{2^i}}=3\left(\frac{\sqrt{3}}{2}\right)^{\frac{1}{2^i}}=3\left(\frac{3}{4}\right)^{\frac{1}{2^{i+1}}}$$

$$\prod_{i=1}^{n}\left(\sum_{Cyl}\cos\left(\frac{A}{2}\right)\right)^{\frac{1}{2^i}} \geq \prod_{i=1}^{n}3\left(\frac{3}{4}\right)^{\frac{1}{2^{i+1}}} = 3^n\left(\frac{3}{4}\right)^{\sum_{i=1}^{n}\left(\frac{1}{2^{i+1}}\right)}=3^n\left(\frac{3}{4}\right)^{\frac{\sum_{i=1}^{n}2^{i-1}}{2^{n+1}}}$$ (proof complete)

Equality holds for equilateral triangle.