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COMMENTED PROBLEM 30

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1. Let be $a, b, c \in (0, \infty)$. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{30abc}{(a+b)(b+c)(c+a)} \geq \frac{27}{4}.$$

Proposed by Costel Anghel – Romania

Proof.

Denoting $\frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z$, *we have* $x, y, z > 0$ *and* $xyz = 1$.

We write $\frac{abc}{(a+b)(b+c)(c+a)} = \frac{1}{(\frac{a+b}{b})(\frac{b+c}{c})(\frac{c+a}{a})} = \frac{1}{(x+1)(y+1)(z+1)}$

We reformulate the problem:

If $x, y, z > 0$ *and* $xyz = 1$, *prove that* $x^2+y^2+z^2+\frac{30}{(x+1)(y+1)(z+1)} \geq \frac{27}{4}$.

Denoting $x+y+z=t$, *we have* $t \geq 3\sqrt[3]{xyz}=3; x^2+y^2+z^2 \geq \frac{(x+y+z)^2}{3}=\frac{t^2}{3}$

$(x+y+z)^2 \geq 3(xy+yz+zx) \Rightarrow xy+yz+zx \leq \frac{t^2}{3}; (x+y+z)^3 \geq 27xyz \Rightarrow xyz \leq \frac{t^3}{27};$

We have $(x+1)(y+1)(z+1) = xyz+xy+yz+zx+x+y+z+1 \leq \frac{t^3}{27}+\frac{t^2}{3}+t+1.$

It suffices to prove that:

$$\frac{t^2}{3}+\frac{30}{\frac{t^3}{27}+\frac{t^2}{3}+t+1} \geq \frac{27}{4} \Leftrightarrow \frac{t^2}{3}+\frac{810}{t^3+9t^2+27t+27} \geq \frac{27}{4} \Leftrightarrow$$

$$4t^2(t^3+9t^2+27t+27)+9720 \geq 81(t^3+9t^2+27t+27) \Leftrightarrow$$

$$4t^5+36t^4+27t^3-621t^2-2187t+7533 \geq 0 \Leftrightarrow (t-3)(4t^2+48t^3+171t^2-108t-2511) \geq 0,$$

Which follows from $t-3 \geq 0$ *and* $4t^2+48t^3+171t^2-108t-2511 \geq 324 > 0$.

Equality holds for $x=y=z=1$, *namely for* $a=b=c$.

□

Remark.

The inequality can be devoloped:

2. Let be $a, b, c \in (0, \infty)$ and $n \leq 32$. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+n \cdot \frac{abc}{(a+b)(b+c)(c+a)} \geq 3+\frac{n}{8}$$

Proposed by Marin Chirciu - Romania

Proof.

$$\text{Denoting } \frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z, \text{ we have } x,y,z>0 \text{ and } xyz=1.$$

$$\text{We write } \frac{abc}{(a+b)(b+c)(c+a)}=\frac{1}{(\frac{a+b}{b})(\frac{b+c}{c})(\frac{c+a}{a})}=\frac{1}{(x+1)(y+1)(z+1)}.$$

The problem can be reformulated:

If $x,y,z>0$ *and* $xyz=1$, *prove that* $x^2+y^2+z^2+\frac{n}{(x+1)(y+1)(z+1)}\geq 3+\frac{n}{8}$.

Denoting $x+y+z=t$, *we have* $t\geq 3\sqrt[3]{xyz}=3; x^2+y^2+z^2\geq\frac{(x+y+z)^2}{3}=\frac{t^2}{3}$

$(x+y+z)^2\geq 3(xy+yz+zx)\Rightarrow xy+yz+zx\leq\frac{t^2}{3}; (x+y+z)^3\geq 27xyz\Rightarrow xyz\leq\frac{t^3}{27};$

We have $(x+1)(y+1)(z+1)=xyz+xy+yz+zx+x+y+z+1\leq\frac{t^3}{27}+\frac{t^2}{3}+t+1.$

It suffices to prove that:

$$\frac{t^2}{3}+\frac{n}{\frac{t^3}{27}+\frac{t^2}{3}+t+1}\geq 3+\frac{n}{8}\Leftrightarrow\frac{t^2}{3}+\frac{27n}{t^3+9t^2+27t+27}\geq\frac{n+24}{8}\Leftrightarrow$$

$$8t^2(t^3+9t^2+27t+27)+648n\geq(3n+72)(t^3+9t^2+27t+27)\Leftrightarrow$$

$$8t^5+72t^4+(144-3n)t^3-(27n+432)t^2-(81n+1944)t+567n-1944\geq 0$$

$$\Leftrightarrow(t-3)(8t^4+96t^3+(432-3n)t^2+(864-36n)t+648-189)\geq 0,$$

Which follows from $t-3\geq 0$ *and* $8t^4+96t^3+(432-3n)t^2+(864-36n)t+648-189n\geq 0$,

for $n\leq 32$.

The equality holds for $x=y=z=1$, *namely for* $a=b=c$.

□

We can formulate the following problem:

3. Let be $a,b,c\in(0,\infty)$. Prove that

$$\frac{a^2}{b^2}+\frac{b^2}{c^2}+\frac{c^2}{a^2}+\frac{32abc}{(a+b)(b+c)(c+a)}\geq 7.$$

Proof 1.

$$\text{Denoting } \frac{a}{b}=x, \frac{b}{c}=y, \frac{c}{a}=z, \text{ we have } x,y,z>0 \text{ and } xyz=1.$$

$$\text{We write } \frac{abc}{(a+b)(b+c)(c+a)}=\frac{1}{(\frac{a+b}{b})(\frac{b+c}{c})(\frac{c+a}{a})}=\frac{1}{(x+1)(y+1)(z+1)}.$$

The problem can be reformulated:

If $x,y,z>0$ *and* $xyz=1$, *prove that* $x^2+y^2+z^2+\frac{30}{(x+1)(y+1)(z+1)}\geq\frac{27}{4}$.

Denoting $x+y+z=t$, *we have* $t\geq 3\sqrt[3]{xyz}=3; x^2+y^2+z^2\geq\frac{(x+y+z)^2}{3}=\frac{t^2}{3}$

$(x+y+z)^2\geq 3(xy+yz+zx)\Rightarrow xy+yz+zx\leq\frac{t^2}{3}; (x+y+z)^3\geq 27xyz\Rightarrow xyz\leq\frac{t^3}{27};$

We have $(x+1)(y+1)(z+1)=xyz+xy+yz+zx+x+y+z+1\leq\frac{t^3}{27}+\frac{t^2}{3}+t+1$

It suffices to prove that:

$$\frac{t^2}{3} + \frac{32}{\frac{t^3}{27} + \frac{t^2}{3} + t + 1} \geq 7 \Leftrightarrow \frac{t^2}{3} + \frac{864}{t^3 + 9t^2 + 27t + 27} \geq 7 \Leftrightarrow$$

$$t^2(t^3 + 9t^2 + 27t + 27) + 2592 \geq 21(t^3 + 9t^2 + 27t + 27) \Leftrightarrow$$

$$t^5 + 9t^4 + 6t^3 - 162t^2 - 567t + 2025 \geq 0 \Leftrightarrow (t-3)(t^4 + 12t^3 + 42t^2 - 36t - 675) \geq 0,$$

Which follows from $t - 3 \geq 0$ *and* $t^4 + 12t^3 + 42t^2 - 36t - 675 > 0$.

The equality holds for $x = y = z = 1$, *namely for* $a = b = c$.

□

Proof 2.

We put $n = 32$ *in* ***2.***

□

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