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PROBLEM 354 INEQUALITY IN TRIANGLE ROMANIAN MATHEMATICAL MAGAZINE 2017

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1. In ΔABC

$$\sum \frac{a^3}{b+c-a} \geq 4S\sqrt{3}$$

Proposed by D.M. Bătinețu - Giurgiu, Neculai Stanciu - Romania

Proof.

We prove that following lemma:

Lemma.
2. In ΔABC

$$\sum \frac{a^3}{b+c-a} = \frac{p^2(2R-3r)+r^2(4R+r)}{r}.$$

Proof.

$$\sum \frac{a^3}{b+c-a} = \frac{1}{2}\sum \frac{a^3}{p-a} = \frac{1}{2}\cdot\frac{2p^2(2R-3r)+2r^2(4R+r)}{r} = \frac{p^2(2R-3r)+r^2(4R+r)}{r}.$$

□

Let's pass to solving the problem from enunciation.

*Base on the **Lemma** we write the following inequality:*

$$\frac{p^2(2R-3r)+r^2(4R+r)}{r} \geq 4rp\sqrt{3} \Leftrightarrow p^2(2R-3r)+r^2(4R+r) \geq 4r\cdot p\sqrt{3}$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ and Doucet's inequality $4R+r \geq p\sqrt{3}$

It remains to prove that:

$$(16Rr-5r^2)(2R-3r)+r^2(4R+r) \geq 4r\cdot(4R+r) \Leftrightarrow 16R^2-35Rr+6r^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(16R-3r) \geq 0, \text{ obviously from Euler's inequality } R \geq 2r.$$

Equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **1.** can be strengthened:*

3. In ΔABC

$$\sum \frac{a^3}{b+c-a} \geq \frac{4p^2}{3}.$$

Proof.

Base on the ***Lemma*** *the inequality can be written:*

$$\frac{p^2(2R-3r)+r^2(4R+r)}{r} \geq \frac{4p^2}{3} \Leftrightarrow p^2(6R-13r)+3r^2(4R+r) \geq 0$$

We distinguish the cases:

1. If $6R-13r \geq 0$, *the inequality is obvious.*
2. If $6R-13r < 0$, *inequality can be rewritten:*

$$p^2(13r-6R) \leq 3r^2(4R+r)$$

which follows from Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$.

It remains to prove that:

$$(4R^2+4Rr+3r^2)(13r-6R) \leq 3r^2(4R+r) \Leftrightarrow 12R^3-14R^2r-11Rr^2-18r^3 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(12R^2+10Rr+9r^2) \geq 0,$$ *obviously from Euler's inequality* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3.** is stronger than inequality **1.***

4. In ΔABC

$$\sum \frac{a^3}{b+c-a} \geq \frac{4p^2}{3} \geq 4S\sqrt{3}$$

Proof.

*See inequality **3.** and Mitrinović's inequality* $p \geq 3r\sqrt{3}$.

The equality holds if and only if the triangle is equilateral.

□

*Inequality **3.** can be also strengthened:*

5. In ΔABC

$$\sum \frac{a^3}{b+c-a} \geq 2Rp\sqrt{3}$$

Proposed by Marin Chirciu - Romania

Proof.

Base on the ***Lemma*** *we write the inequality:*

$$\frac{p^2(2R-3r)+r^2(4R+r)}{r} \geq 2Rp\sqrt{3} \Leftrightarrow p^2(2R-3r)+r^2(4R+r) \geq 2R \cdot p\sqrt{3}$$

which follows from Gerretsen's inequality $p^2 \geq 16Rr-5r^2$ *and Doucet's inequality* $4R+r \geq p\sqrt{3}$.

It remains to prove that:

$$(16Rr-5r^2)(2R-3r)+r^2(4R+r) \geq 2R \cdot (4R+r) \Leftrightarrow 3R^2-7Rr+2r^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow (R-2r)(3R-r) \geq 0,$$ *obviously from Euler's inequality* $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **5.** is stronger than inequality **3.**:*

6. In ΔABC

$$\boldsymbol{\sum \frac{a^3}{b+c-a} \geq 2Rp\sqrt{3} \geq \frac{4p^2}{3}}$$

Proof.

*See inequality **5.** and Mitrinović's inequality* $p \leq \dfrac{3R\sqrt{3}}{2}$

The equality holds if and only if the triangle is equilateral.

□

Remark.

We write the following inequalities:

7. In ΔABC

$$\boldsymbol{\sum \frac{a^3}{b+c-a} \geq 2Rp\sqrt{3} \geq \frac{4p^2}{3} \geq 4S\sqrt{3}}$$

Proof.

*See inequality **5.** and Mitrinović's inequality* $3r\sqrt{3} \leq p \leq \dfrac{3R\sqrt{3}}{2}$

The equality holds if and only if the triangle is equilateral.

□

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