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INEQUALITY IN TRIANGLE 301
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1. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{18r^2}{abc}$$

Proposed by Panagiote Ligouras - Florence - Italy

Proof.

We prove the following lemma:

Lemma

2. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} = \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)}$$

We have

$$\sum \frac{a}{(b+c)(b+c-a)} = \sum\Big(\frac{1}{b+c-a} - \frac{1}{b+c}\Big) = \sum \frac{1}{b+c-a} - \sum \frac{1}{b+c} =$$

$$= \frac{4R+r}{2pr} - \frac{5p^2+r^2+4Rr}{2p(p^2+r^2+2Rr)} =$$

$$= \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)}$$

Let's pass to solving the problem from the enunciation.

*Using the **Lemma**, the inequality that we have to prove can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{18r^2}{abc} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{18r^2}{4pRr} \Leftrightarrow$$

$$p^2(4R^2-4Rr-9r^2)+r(8R^3+2R^2r-18Rr^2-9r^3) \geq 0$$

We distinguish the following cases:

1. If $4R^2-4Rr-9r^2 \geq 0$, the inequality is obvious.
2. If $4R^2-4Rr-9r^2 < 0$, the inequality can be rewritten:

$$p^2(9r^2+4Rr-4R^2) \leq r(8R^3+2R^2r-18Rr^2-9r^3).$$

Using Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$ *it is enough to prove that:*

$$(4R^2+4Rr+3r^2)(9r^2+4Rr-4R^2) \leq r(8R^3+2R^2r-18Rr^2-9r^3) \Leftrightarrow$$

$$\Leftrightarrow 8R^4+4R^3r-19R^2r^2-33Rr^3-18r^4 \geq 0 \Leftrightarrow (R-2r)(8R^3+20R^2r+21Rr^2+9r^3) \geq 0$$

which is obviously from Euler's inequality $R \geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

Remark

*Inequality **1.** can be strengthened:*

3. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9Rr}{abc}$$

Proof. *Using the **Lemma** the inequality can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9Rr}{abc} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9Rr}{4pRr} \Leftrightarrow$$

$$p^2(8R-17r)+r(16R^2-14Rr-9r^2) \geq 0.$$

We distinguish the cases:

1. If $8R-17r \geq 0$, the inequality is obvious.
2. If $8R-17r < 0$, the inequality can be rewritten:

$$p^2(17r-8R) \leq r(16R^2-14Rr-9r^2).$$

Using Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$ it suffices to prove that:

$$(4R^2+4Rr+3r^2)(17r-8R) \leq r(16R^2-14Rr-9r^2) \Leftrightarrow$$

$$\Leftrightarrow 16R^3-10R^2r-29Rr^2-30r^3 \geq 0 \Leftrightarrow (R-2r)(16R^2+22Rr+15r^2) \geq 0,$$

which is obvious from Euler's inequality $R \geq 2r$.

The inequality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **3.** is stronger than inequality **1.***

4. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9Rr}{abc} \geq \frac{18r^2}{abc}.$$

Proof.

*See inequality **3.** and Euler's inequality $R \geq 2r$.*

The inequality holds if and only if the triangle is equilateral.

□

Remark.

*Also inequality **3.** can be developed:*

5. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9R^2}{2abc}$$

Proof.

*Using the **Lemma**, the inequality that we have to prove can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9R^2}{2abc} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \geq \frac{9R^2}{9pRr} \Leftrightarrow$$

$$p^2(7R-16r)+Rr(14R-r) \geq 0.$$

We distinguish the cases:

1. If $7R-16r \geq 0$, the inequality is obvious.
2. If $7R-16r < 0$, the inequality can be rewritten:

$$\boldsymbol{p^2(16r-7R) \leq Rr(14R-r).}$$

Using Gerretsen's inequality $p^2 \leq 4R^2+4Rr+3r^2$ it suffices to prove that:

$$(4R^2+4Rr+3r^2)(16r-7R) \leq Rr(14R-r) \Leftrightarrow$$

$$\Leftrightarrow 14R^3-11R^2r-22Rr^2-24r^3 \geq 0 \Leftrightarrow (R-2r)(14R^2+17Rr+12r^2) \geq 0,$$

which is obvious from Euler's inequality $R \geq 2r$.

The inequality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **5.** is stronger than inequality **3.***

6. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9R^2}{2abc} \geq \frac{9Rr}{abc}$$

Proof.

*See inequality **5.** and Euler's inequality $R \geq 2r$.*

The inequality holds if and only if the triangle is equilateral.

□

Remark.

The inequalities can be written:

7. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \geq \frac{9R^2}{2abc} \geq \frac{9Rr}{abc} \geq \frac{18r^2}{abc}$$

Remark.

Let's obtain an inequality having an opposite sense.

8. In ΔABC

$$\sum \frac{a}{(b+c)(b+c-a)} \leq \frac{9R^2}{16Sr}.$$

Proof.

Using the ***Lemma****, the inequality that we have to prove can be written:*

$$\frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \le \frac{9R^2}{16Sr} \Leftrightarrow \frac{2p^2(R-r)+Rr(4R+r)}{pr(p^2+r^2+2Rr)} \le \frac{9R^2}{16r^2p} \Leftrightarrow$$

$$p^2(9R^2-32Rr+32r^2)+Rr(18R^2-55Rr-16r^2) \ge 0.$$

Because $9R^2-32Rr+32r^2>0(\Delta<0$, *for the trinomial* $9x^2-32x+32)$, *using Gerretsen's inequality* $p^2 \ge 16Rr-5r^2$ *is suffices to prove that:*

$$(16Rr-5r^2)(9R^2-32Rr+32r^2)+Rr(18R^2-55Rr-16r^2) \ge 0 \Leftrightarrow$$

$$\Leftrightarrow 81R^3-306R^2r+328Rr^2-80r^3 \ge 0 \Leftrightarrow (R-2r)(80R^2-144Rr+40r^2) \ge 0,$$

which is obviously from Euler's $R \ge 2r$.

The inequality holds if and only if the triangle is equilateral.

□

Remark.

We can write the double inequality:

9. In ΔABC

$$\frac{9R}{8S} \le \sum \frac{a}{(b+c)(b+c-a)} \le \frac{9R^2}{16Sr}.$$

Proposed by Marin Chirciu - Romania

Proof.

See inequalities ***2.*** *and 8.*

The inequality holds if and only if the triangle is equilateral.

We've obtained a refinement of Euler's inequality.

□

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