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TRIGONOMETRIC SUBSTITUTIONS IN PROBLEM SOLVING PART 2

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Abstract. In this paper are indicated a few useful trigonometric substitutions for solving problems. Solved problems are also a part of this article.

Problem 23. If $x, y, z > 0, xy + yz + zx = 1$ then:

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{3z}{1+z^2} \le \sqrt{10}$$

Proof.

$$x, y, z > 0 \Rightarrow (\exists) A, B, C \in \left(0, \frac{\pi}{2}\right);$$

$$x = \tan\frac{A}{2}; y = \tan\frac{B}{2}; z = \tan\frac{C}{2}$$

$$\frac{x}{1+x^2} = \frac{\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} = \frac{\sin\frac{A}{2}}{\cos\frac{A}{2}} \cdot \cos^2\frac{A}{2} = \sin\frac{A}{2}\cos\frac{A}{2}$$

$$\frac{y}{1+y^2} = \sin\frac{B}{2}\cos\frac{B}{2}; \frac{z}{1+z^2} = \sin\frac{C}{2}\cos\frac{C}{2}$$

$$\sin\frac{A}{2}\cos\frac{A}{2} + \sin\frac{B}{2}\cos\frac{B}{2} + 3\sin\frac{C}{2}\cos\frac{C}{2} \le \sqrt{10}$$

$$\sin A + \sin B + 3\sin C \le 2\sqrt{10}$$

$$\sin A + \sin B = 2\sin\frac{A+B}{2}\cos\frac{B-C}{2} =$$

$$= 2\cos\frac{C}{2}\cos\frac{B-C}{2} \le 2\cos\frac{C}{2}$$

$$2\cos\frac{C}{2} + 3\sin C = 2\cos\frac{C}{2} + 6\sin\frac{C}{2}\cos\frac{C}{2} \le$$

$$\le 2\cos\frac{C}{2} + 6\sin\frac{C}{2} =$$

$$= 2\left(\cos\frac{C}{2} + 3\sin\frac{C}{2}\right) \overbrace{\le}^{CBS} 2 \cdot \sqrt{1^2+3^2} \cdot \sqrt{\sin^2\frac{C}{2} + \cos^2\frac{C}{2}} =$$

$$= 2 \cdot \sqrt{10} \cdot 1 = 2\sqrt{10}$$

□

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Problem 24. If $x, y, z > 0; xy + yz + zx = 1$ then

$$\frac{x}{\sqrt{1+x^2}} + \frac{y}{\sqrt{1+y^2}} + \frac{z}{\sqrt{1+z^2}} \le \frac{3}{2}$$

Proof.

$$x, y, z > 0 \text{ then } (\exists)A, B, C \in \left(0, \frac{\pi}{2}\right);$$

$$x = \cot A; y = \cot B; z = \cot C$$

$$\frac{x}{\sqrt{1+x^2}} = \cos A; \frac{y}{\sqrt{1+y^2}} = \cos B; \frac{z}{\sqrt{1+z^2}} = \cos C$$

Inequality to prove become a known one:

$$\cos A + \cos B + \cos C \le \frac{3}{2}$$

□

Problem 25. If $x, y, z > 0; x + y + z = xyz$ then:

$$(x^2 - 1)(y^2 - 1)(z^2 - 1) \le \sqrt{(x^2+1)(y^2+1)(z^2+1)}$$

Proof.

$$\text{If } x, y, z > 0 \text{ then } (\exists)A, B, C \in (0, \pi)$$

$$x = \cot \frac{A}{2}; y = \cot \frac{B}{2}; z = \cot \frac{C}{2}$$

Inequality to prove become:

$$\frac{x^2-1}{x^2+1} \cdot \frac{y^2-1}{y^2+1} \cdot \frac{z^2-1}{z^2+1} \le \frac{1}{\sqrt{(x^2+1)(y^2+1)(z^2+1)}}$$

$$\frac{x^2-1}{x^2+1} = \cos A; \frac{y^2-1}{y^2+1} = \cos B; \frac{z^2-1}{z^2+1} = \cos C$$

$$\frac{1}{\sqrt{x^2+1}} = \sin \frac{A}{2}; \frac{1}{\sqrt{y^2+1}} = \sin \frac{B}{2}; \frac{1}{\sqrt{z^2+1}} = \sin \frac{C}{2}$$

Inequality to prove can be written:

$$\cos A \cos B \cos C \le \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}$$

$$\cos A \cos B = \frac{1}{2}\Big[\cos(A-B) + \cos(A+B)\Big] =$$

$$= \frac{1}{2}\Big[\cos(A-B) - \cos C\Big] \le$$

$$\le \frac{1}{2}(1 - \cos C) = \sin^2 \frac{C}{2}$$

Analogous:

$$\cos B \cos C \le \sin^2 \frac{A}{2}; \cos C \cos A \le \sin^2 \frac{B}{2}$$

and by multiplying its obtained the asked inequality.

□

Problem 26. If $x, y, z > 0; xy + yz + zx = 1$ then:

$$\frac{x}{1+x^2} + \frac{y}{1+y^2} + \frac{z}{1+z^2} \geq \frac{3\sqrt{3}}{4}$$

Proof.

$$x, y, z > 0 \text{ then } (\exists)A, B, C \in \left(0, \frac{\pi}{2}\right)$$
$$x = \tan A; y = \tan B; z = \tan C$$
$$\frac{x}{1+x^2} = \frac{\tan A}{1+\tan^2 A} = \frac{\sin A}{\cos A} \cdot \frac{\cos^2 A}{1} = \sin A \cos A$$
$$\frac{y}{1+y^2} = \frac{\tan B}{1+\tan^2 B} = \sin B \cos B; \frac{z}{1+z^2} = \sin C \cos C$$

Inequality to prove can be written:

$$\sin A \cos A + \sin B \cos B + \sin C \cos C \geq \frac{3\sqrt{3}}{4}$$
$$\sin A + \sin B + \sin C \geq \frac{3\sqrt{3}}{2}$$
$$f : (0, \pi) \to \mathbb{R}; f(x) = \sin x;$$
$$f'(x) = \cos x; f''(x) = -\sin x < 0$$

f concave. By Jensen:

$$f\left(\frac{A+B+C}{3}\right) \geq \frac{1}{3}\Big(f(A) + f(B) + f(C)\Big)$$
$$3 \sin \frac{\pi}{3} \geq \sin 2A + \sin 2B + \sin 2C$$
$$\sin 2A + \sin 2B + \sin 2C \leq \frac{3\sqrt{3}}{2}$$

□

Theorem 1. If $A, B, C \in (0, \pi); A + B + C = \pi$ then:

$$\tan\frac{A}{2}\tan\frac{B}{2} + \tan\frac{B}{2}\tan\frac{C}{2} + \tan\frac{C}{2}\tan\frac{A}{2} = 1$$

Theorem 2. If $A, B, C \in (0, \pi); A + B + C = \pi$

$$\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2} + 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} = 1$$

Problem 27. If $x, y, z \in (0, \infty); x^2 + y^2 + z^2 + 2xyz = 1$ then $xy + yz + zx \leq \frac{3}{4}$

Proof.

$$x, y, z \in (0, \infty) \Rightarrow (\exists)A, B, C \in (0, \pi)$$
$$x = \sin\frac{A}{2}; y = \sin\frac{B}{2}; z = \sin\frac{C}{2}$$

Inequality to prove becomes:

$$\sin\frac{A}{2}\sin\frac{B}{2} + \sin\frac{B}{2}\sin\frac{C}{2} + \sin\frac{C}{2}\sin\frac{A}{2} \leq \frac{3}{4}$$

By Jensen's inequality:

$$\frac{1}{2} = \sin\frac{\pi}{6} = \sin\frac{A+B+C}{6} \geq$$

$$\geq \frac{1}{3}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)$$
$$\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\leq\frac{3}{2}$$
On the other hand:
$$\sin\frac{A}{2}\sin\frac{B}{2}+\sin\frac{B}{2}\sin\frac{C}{2}+\sin\frac{C}{2}\sin\frac{A}{2}\leq$$
$$\leq\frac{1}{3}\left(\sin\frac{A}{2}+\sin\frac{B}{2}+\sin\frac{C}{2}\right)^2\leq$$
$$\leq\frac{1}{3}\cdot\left(\frac{3}{2}\right)^2=\frac{9}{12}=\frac{3}{4}$$

□

Problem 28. If $x, y, z > 0; x^2 + y^2 + z^2 + 2xyz = 1$ then:
$$x+y+z\geq 4xyz+1$$

Proof.
$$x,y,z>0\Rightarrow(\exists)A,B,C\in\left(0,\frac{\pi}{2}\right);$$
$$x=\cos A; y=\cos B; z=\cos C$$
Inequality to prove can be written:
$$\cos A+\cos B+\cos C\geq 4\cos A\cos B\cos C+1$$
$$2\cos\frac{A+B}{2}\cos\frac{A-B}{2}+1-2\sin^2\frac{C}{2}\geq 4\cos A\cos B\cos C+1$$
$$2\cos\frac{\pi-C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}\geq 4\cos A\cos B\cos C$$
$$2\sin\frac{C}{2}\cos\frac{A-B}{2}-2\sin^2\frac{C}{2}\geq 4\cos A\cos B\cos C$$
$$2\sin\frac{C}{2}\left(\cos\frac{A-B}{2}-\sin\frac{C}{2}\right)\geq 4\cos A\cos B\cos C$$
$$2\sin\frac{C}{2}\cdot\left(\cos\frac{A-B}{2}-\cos\frac{\pi-C}{2}\right)\geq 4\cos A\cos B\cos C$$
$$2\sin\frac{C}{2}\cdot 2\cdot\sin\frac{\frac{A-B}{2}+\frac{\pi-C}{2}}{2}\sin\frac{\frac{\pi-C}{2}-\frac{A-B}{2}}{2}\geq 4\cos A\cos B\cos C$$
$$4\sin\frac{C}{2}\sin\frac{A}{2}\sin\frac{B}{2}\geq 4\cos A\cos B\cos C$$
Remains to prove:
$$\cos A\cos B\cos C\leq\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$
$$\cos A\cos B\leq\frac{(\cos A+\cos B)^2}{4}=\sin^2\frac{C}{2}\cos^2\frac{A-B}{2}\leq\sin^2\frac{C}{2}$$
Analogous:
$$\cos B\cos C\leq\sin^2\frac{A}{2};\cos C\cos A\leq\sin^2\frac{B}{2}$$
By multiplying:
$$\cos^2 A\cos^2 B\cos^2 C\leq\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2}$$
$$\cos A\cos B\cos C\leq\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$$

$\square$

Proposed problems

29. If $a \in \mathbb{R}; |a| \geq 1$ then:

$$a + \frac{a}{\sqrt{a^2 - 1}} \geq 2\sqrt{2}$$

$$\left(\textit{Use: } a = \frac{1}{\cos p}; p \in \left[0, \frac{\pi}{2}\right) \cup \left[\pi, \frac{3\pi}{2}\right) \right)$$

30. If $a \in \mathbb{R}$ then:

$$\left| \frac{3a}{\sqrt{1 + a^2}} - \frac{4a^2}{\sqrt{(1 + a^2)^3}} \right| \leq 1$$

$$\left(\textit{Use: } a = \tan p; p \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right)$$

31. If $a \in \mathbb{R}$ then:

$$\frac{5}{2} \leq \frac{12a^4 + 8a^2 + 3}{(1 + 2a^2)^2} \leq 3$$

$$\left(\textit{Use: } a = \frac{1}{\sqrt{2}} \tan p; p \in \left(-\frac{\pi}{2}, \frac{\pi}{2} \right) \right)$$

32. If $a \in \mathbb{R}$ then:

$$-3 \leq \frac{6a + 4|a^2 - 1|}{a^2 + 1} \leq 5$$

33. If $a \in [1, 3]$ then:

$$|4a^3 - 24a^2 + 45a - 26| \leq 1$$

$$\left(\textit{Use: } a - 2 = \cos x \right)$$

34. If $x \in \mathbb{R}$ then:

$$\left| \frac{x(1 - x^2)(x^4 - 6x^2 + 1)}{(1 + x^2)^4} \right| \leq \frac{1}{8}$$

$$\left(\textit{Use: } x = \tan p \right)$$

35. If $a, b, c, d \in \mathbb{R}; a^2 + b^2 = c^2 + d^2 = 1$ then:

$$-\sqrt{2} \leq a(c + d) + b(c - d) \leq 2$$

$$\left(\textit{Use: } a = \sin x; b = \cos x; c = \sin y; d = \cos y \right)$$

36. If $a, b \in \mathbb{R}; a^2 + b^2 = 1$ then:

$$\left(a^2 + \frac{1}{a^2} \right)^2 + \left(b^2 + \frac{1}{b^2} \right)^2 \geq \frac{25}{2}$$

$$\left(\textit{Use: } a = \sin x; b = \cos x \right)$$

37. If $a, b \in \mathbb{R}; a^2 + b^2 - 2a - 4b + 4 = 0$ then:

$$|a^2 - b^2 + 2\sqrt{3}ab - 2(1 + 2\sqrt{3})a + (4 - 2\sqrt{3})b + 4\sqrt{3} - 3| \leq 2$$

$$\left(\textit{Use: } a - 1 = \sin x; b - 2 = \cos x\right)$$

38. If $a \in [0, 2]; b \in [0, 3]$ then:

$$4 \leq a^2 + b^2 + ab + \sqrt{4 - a^2} \cdot \sqrt{9 - b^2} \leq 19$$

$$\left(\textit{Use: } a = 2\cos x; b = 3\cos y; x, y \in \left[0, \frac{\pi}{2}\right]\right)$$

39. If $m, n \in [1, \infty)$ then:

$$n\sqrt{m - 1} + m\sqrt{n - 1} \leq mn$$

$$\left(\textit{Use: } m = \frac{1}{\cos^2 x}; n = \frac{1}{\cos^2 y}; x, y \in \left[0, \frac{\pi}{2}\right]\right)$$

40. If $m, n \in (0, \infty); m > n$ then:

$$\sqrt{m^2 - n^2} + \sqrt{2mn - n^2} \geq m$$

$$\left(\textit{Use: } \frac{n}{m} = \sin x; x \in \left[0, \frac{\pi}{2}\right]\right)$$

41. If $x, y \in \mathbb{R}; |x| \leq 1; |y| \leq 1$ then:

$$\sqrt{1 - x^2} + \sqrt{1 - y^2} \leq \sqrt{4 - (x + y)^2}$$

$$\left(\textit{Use: } x = \sin p; y = \sin q\right)$$

42. If $x_1, x_2, \ldots, x_n \in [-1, 1]; x_1^3 + x_2^3 + \ldots + x_n^3 = 0$ then:

$$x_1 + x_2 + \ldots + x_n \leq \frac{n}{3}$$

$$\left(\textit{Use: } x_i = \cos p_i; p_i \in [0, \pi]; i \in \overline{1, n}\right)$$

TO BE CONTINUED!

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