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PROBLEMS FOR JUNIORS

JP.061. Let a, b, c be positive real numbers such that $a + b + c = 3$. Prove that

$$\frac{1}{a^3 + b^3 + c^3} + \frac{8}{ab + bc + ca} \geq 3$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.062. Prove that if $a, b, c, d \in [1, \infty)$ then:

$$3a + 3b + 2c + d \leq 6 + ab(1 + c + cd)$$

Proposed by Daniel Sitaru - Romania

JP.063. Let x, y, z be non-negative real numbers satisfying $\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 3$. Find the minimum possible value of

$$\sqrt[3]{x + 2y + 5z} + \sqrt[3]{y + 2z + 5x} + \sqrt[3]{z + 2x + 5y}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.064. Let a, b, c be non-negative real numbers. Prove that:

$$\sqrt[3]{1 + a^3} + \sqrt[3]{1 + b^3} + \sqrt[3]{1 + c^3} \geq \frac{\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}}{\sqrt[6]{2}}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.065. Let ABC be an equilateral triangle inscribed in the circle (O) whose radius R . Prove that for an arbitrary point P lies on (O) ,

$$6\sqrt{2} < \frac{PA^3 + PB^3 + PC^3}{R^3} < 3\sqrt[4]{216}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

JP.066. Prove that in any triangle

$$\sum \frac{m_a^2}{h_b h_c} \geq n \cdot \frac{R}{r} + (3 - 2n) \cdot \frac{s}{3r\sqrt{3}}, n \leq \frac{3}{2}$$

Proposed by Marin Chirciu - Romania

JP.067. Prove that in any triangle

$$n \cdot \frac{s^2 + r^2}{Rr} + \sqrt[k]{\frac{2r}{R}} \geq 14n + 1, n \geq \frac{1}{2}, k \in \mathbb{N}, k \geq 2$$

Proposed by Marin Chirciu - Romania

JP.068. Let a, b and c be the side lengths of a triangle ABC , with circumradius R and inradius r . Prove that

$$\frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{a + b} + \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{b + c} + \frac{\tan \frac{C}{2} + \tan \frac{A}{2}}{c + a} \leq \frac{1}{r} - \frac{1}{R}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.069. Let a, b be positive real numbers such that $a^2 + ab + b^2 = 9$. Find the maximal value of expression:

$$(a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.070. Let a, b be positive real numbers with $a^2 + ab + b^2 = k^2, k > 0$. Prove that

$$\sqrt{a + b} + \sqrt[4]{ab} \leq \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \sqrt{k}.$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.071. Let ABC be a triangle with circumradius R and inradius r , and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c respectively. Prove that

$$\left(\frac{w_a}{a}\right)^2 \cdot \tan \frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \cdot \tan \frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan \frac{C}{2} \leq \frac{3\sqrt{3}}{8} \cdot \frac{R}{r}$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.072. Let a, b and c denote, as usual, the lengths of the sides BC, CA , and AB , respectively, in ΔABC . Let R be the circumradius, r the inradius of ΔABC , and r_a, r_b and r_c the exradii of A, B and C , respectively. Prove that

$$(a) \frac{r_a}{a^3} + \frac{r_b}{b^3} + \frac{r_c}{c^3} \leq \frac{\sqrt{3}}{8r^2},$$

$$(b) \frac{3R}{2r} \geq \sqrt{\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a}} + 6$$

Proposed by George Apostolopoulos - Messolonghi - Greece

JP.073. If $a, b, c > 0; n \geq 1$ then:

$$\frac{3n(a^4 + b^4 + c^4)}{(a^2 + b^2 + c^2)^2} + \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq n + 1$$

Proposed by Marin Chirciu - Romania

JP.074. If $a, b, c, n > 0; n(ab + bc + ca) + 2abc = n^3$ then:

$$\frac{1}{a + b + 2n} + \frac{1}{b + c + 2n} + \frac{1}{c + a + 2n} \leq \frac{1}{n}$$

Proposed by Marin Chirciu - Romania

JP.075. Let R and r be the circumradius and the inradius of a triangle ABC respectively. Prove that

$$\csc A + \csc B + \csc C \geq 3\sqrt{3} \frac{R}{R + r}$$

Proposed by Martin Lukarevski - Stip - Macedonia

PROBLEMS FOR SENIORS

SP.061. Let x_1, x_2, \dots, x_n be non-negative real numbers satisfying

$$\frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \dots + \frac{nx_n}{1+x_n} = 1.$$

Find the maximum possible value of

$$P = x_1 x_2^2 \dots x_n^n.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.062. Let ABC be an equilateral triangle inscribed in the circle with center O and radius R . Prove that for an arbitrary point P does not lie on (O) , we always have

$$PA \cdot PB + PB \cdot PC + PC \cdot PA \geq 3 \cdot \max \{PO^2, R^2\}$$

Determine position of the point P so that the equality holds.

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.063. Prove that for any triangle ABC ,

$$\begin{aligned} & \cos \frac{A}{2} \cos \frac{B}{2} + \cos \frac{B}{2} \cos \frac{C}{2} + \cos \frac{C}{2} \cos \frac{A}{2} \geq \\ & \geq \sqrt{3} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \frac{1}{2} \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right) \end{aligned}$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

SP.064. Prove that $abc + 2 \geq \frac{9}{a^2+b^2+c^2} \forall a, b, c \geq 0$ such that $a + b + c \geq 3$.

Proposed by Leonard Giugiuc - Romania

SP.065. Let a, b, c and d be real numbers such that $a+b+c+d = 0$ and $(a+b)^2 - ab - cd = 2$.

(i) Prove that $abcd \leq 1$. When does the equality holds?

(ii) Prove that $abcd \geq -\frac{1}{3}$.

Proposed by Leonard Giugiuc - Romania

SP.066. Let $t \in \mathbb{R}^+$ and $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ such that

$$\lim_{x \rightarrow \infty} \frac{f(x+1)}{x^t f(x)} \in \mathbb{R}_+$$

and

$$\lim_{x \rightarrow \infty} (f(x))^{\frac{1}{x}} \cdot \frac{1}{x^t} \in \mathbb{R}_+$$

Prove that

$$\lim_{x \rightarrow \infty} (f(x))^{\frac{1}{x}} \cdot \frac{1}{x^t} = \frac{1}{e^t} \cdot \lim_{x \rightarrow \infty} \frac{f(x+1)}{x^t f(x)}$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

SP.067. Let $t \in \mathbb{R}^+$ and $f : \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ be a function such that

$$\lim_{x \rightarrow \infty} \frac{x^t f(x+1)}{f(x)} \in \mathbb{R}_+$$

and

$$\lim_{n \rightarrow \infty} x^t (f(x))^{\frac{1}{x}} \in \mathbb{R}_+^*$$

Prove that

$$\lim_{n \rightarrow \infty} x^t (f(x))^{\frac{1}{x}} = e^t \cdot \lim_{x \rightarrow \infty} \frac{n^t f(x+1)}{f(x)}$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

SP.068. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left({}^{mn+n}\sqrt{(2n+1)!!} - {}^{mn}\sqrt{(2n-1)!!} \right) \cdot n^{\frac{m-1}{m}}$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

SP.069. If $a, b, c \in (0, 1)$; $x, y \in \mathbb{R}$ then:

$$(a+b+c)(\tan^{-1} x - \tan^{-1} y) \leq 3 \left(\tan^{-1}(\sqrt{x^2+y^2}) - \tan^{-1} y \right)$$

Proposed by Daniel Sitaru - Romania

SP.070. Prove that if $a, b, c \in \mathbb{R}$ then:

$$(2-a-b-c+abc)^2 \leq (a^2+2)(b^2+2)(c^2+2)$$

Proposed by Daniel Sitaru - Romania

SP.071. Prove that if $a, b, c \in \mathbb{R}$ then:

$$(3abc - a^3 - b^3 - c^3)^2 \leq (a^2 + b^2 + c^2)^3$$

Proposed by Daniel Sitaru - Romania

SP.072. If $a, b, c > 0; n \in \mathbb{N}^*$ then:

$$\left(\frac{2na}{b + (2n-1)c}\right)^{\frac{2}{3}} + \left(\frac{2nb}{c + (2n-1)a}\right)^{\frac{2}{3}} + \left(\frac{2nc}{a + (2n-1)b}\right)^{\frac{2}{3}} \geq 3$$

Proposed by Marin Chirciu - Romania

SP.073. If $x, y, z > 0$ then:

$$\log\left(1 + \frac{1}{x}\right) + \log\left(1 + \frac{1}{y}\right) + \log\left(1 + \frac{1}{z}\right) \geq 3 \log\left(1 + \frac{3}{x+y+z}\right)$$

Proposed by Marin Chirciu - Romania

SP.074. Let x, y, z be positive real numbers such that:

$x + y + z = 3$. Find the minimum of expression:

$$P = \frac{x^4}{y^4 \cdot \sqrt[3]{4z(x^5+1)}} + \frac{y^4}{z^4 \cdot \sqrt[3]{4x(y^5+1)}} + \frac{z^4}{x^4 \cdot \sqrt[3]{4y(z^5+1)}}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

SP.075. Let x, y, z be positive real numbers such that: $xyz = 1$.

Find the minimum of expression:

$$P = 2(x + y + z) + \frac{x}{y^3 + z^3 + 1} + \frac{y}{z^3 + x^3 + 1} + \frac{z}{x^3 + y^3 + 1}$$

Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam

UNDERGRADUATE PROBLEMS

UP.061. Prove that in all triangle ABC with usual notations holds the following inequalities:

$$(a) \frac{\tan^3 \frac{A}{2}}{m \cdot \tan \frac{B}{2} + n \cdot \tan \frac{C}{2}} + \frac{\tan^3 \frac{B}{2}}{m \cdot \tan \frac{C}{2} + n \cdot \tan \frac{A}{2}} + \frac{\tan^3 \frac{C}{2}}{m \cdot \tan \frac{A}{2} + n \cdot \tan \frac{B}{2}} \geq \frac{((4R+r)^2 - 2s^2)^2}{(m+n)s^4};$$

$$(b) \frac{\tan \frac{A}{2}}{m+n \cdot \tan \frac{B}{2} \cdot \tan \frac{C}{2}} + \frac{\tan \frac{B}{2}}{m+n \cdot \tan \frac{C}{2} \cdot \tan \frac{A}{2}} + \frac{\tan \frac{C}{2}}{m+n \cdot \tan \frac{A}{2} \cdot \tan \frac{B}{2}} \geq \frac{(4R+r)^2}{s(m(4R+r)+3nr)};$$

$$(c) \frac{\tan^3 \frac{A}{2}}{m \cdot \cot \frac{B}{2} + n \cdot \cot \frac{C}{2}} + \frac{\tan^3 \frac{B}{2}}{m \cdot \cot \frac{C}{2} + n \cdot \cot \frac{A}{2}} + \frac{\tan^3 \frac{C}{2}}{m \cdot \cot \frac{A}{2} + n \cdot \cot \frac{B}{2}} \geq \frac{(4R+r)r}{(m+n)s^2};$$

$$(d) \frac{\tan \frac{A}{2}}{(x+y \cdot \tan \frac{B}{2} \cdot \tan \frac{C}{2})^m} + \frac{\tan \frac{B}{2}}{(x+y \cdot \tan \frac{C}{2} \cdot \tan \frac{A}{2})^m} + \frac{\tan \frac{C}{2}}{(x+y \cdot \tan \frac{A}{2} \cdot \tan \frac{B}{2})^m} \geq \frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$$

for any positive real numbers m, n, x, y

Proposed by D.M Băţineţu-Giurgiu, Neculai Stanciu - Romania

UP.062. Given the equilateral triangle ABC and let P be any point in its plane. R, R_a, R_b, R_c denote respectively radii of the circumcircles of the triangles ABC, BPC, CPA, APB and x, y, z are respectively distances from P to the sides BC, CA, AB . Prove that

$$xR_a + yR_b + zR_c \geq \frac{3}{2}R^2.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.063. Let a, b, c, d be real numbers such that

$$a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 = 6.$$

Prove that

$$\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d} \geq 4.$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.064. Let a, b, c, d be non-negative real numbers such that

$$a + b + c + d = 4.$$

Prove that

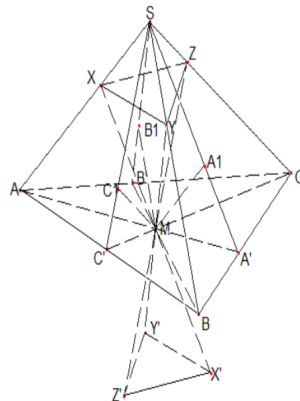
$$ab(a + b) + cd(c + d) + 4(a + b)(c + d) \leq \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d} + 16$$

Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.065. Let $SABC$ be a tetrahedron and let M be any point inside the triangle ABC . The lines through M parallel with the planes SBC, SCA, SAB intersect SA, SB, SC at X, Y, Z , respectively. Prove that

$$\text{Vol}(MXYZ) \leq \frac{2}{27} \text{Vol}(SABC).$$

Determine position of the point M such that the equality holds.



Proposed by Nguyen Viet Hung - Hanoi - Vietnam

UP.066. Evaluate:

$$\sum_{n=1}^{\infty} \sum_{k=1}^n \left(\frac{1}{n^3(2k-1)} \right)$$

Proposed by Shivam Sharma - New Delhi - India

UP.067. Evaluate:

$$S = \sum_{n=1}^{\infty} H_n \left[\zeta(8) - \frac{1}{1^8} - \frac{1}{2^8} - \frac{1}{3^8} \cdots \frac{1}{n^8} \right]$$

Proposed by Shivam Sharma - New Delhi - India

UP.068. Prove that if $a, b, c, d \in \mathbb{R}^*$ then:

$$(abc - ac - bc - ac)^2 \leq 4(1 + a^2)(1 + b^2)(1 + c^2)$$

Proposed by Daniel Sitaru - Romania

UP.069. Prove that if $n \in \mathbb{N}^*$; $a > 1$ then:

$$(n + a - 1)(a - 1)^{n-1} \leq a^n$$

Proposed by Daniel Sitaru - Romania

UP.070. If $a, b \in \mathbb{R}$; $a < b$; $f, g : [a, b] \rightarrow \mathbb{R}$ are continuous functions such that:

$$f(a + b - x) = f(x), g(a + b - x)g(x) = 1; (\forall)x \in [a, b]$$

then:

$$\int_a^b \frac{f(x) \cdot g(x)}{1 + g(x)} dx = \frac{1}{2} \int_a^b f(x) dx$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

UP.071. Evaluate:

$$\int_0^1 \ln \left[\left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \right] \frac{x dx}{1-x^2}$$

Proposed by Shivam Sharma - New Delhi - India

UP.072. Show that there exists $\gamma \in \left(0, \frac{\pi}{4}\right)$ such that

$$\frac{\pi}{4} \cdot \frac{e^{\tan(\gamma)}}{\cos^2(\gamma)} + 2e^{\tan(\frac{7\pi}{48})} + 2 \geq 3e^{\sqrt{2}-1} + e^{\frac{1}{\sqrt{3}}}$$

Proposed by Soumitra Mandal - Chandar Nagore - India

UP.073. Let be $a, b, c \in \mathbb{C}$. Solve the following equation:

$$x^3 - (a + b + c)x^2 + (ab + bc + ca - 1)x + b - abc = 0$$

Proposed by Daniel Sitaru - Romania

UP.074. Let $(a_n)_{n \geq 1}$ be positive real sequences with

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^* \text{ and } f \in \mathbb{R}[x], f(x) \in \mathbb{R}_+^*, \forall x \in \mathbb{R}_+^*; u, v \in \mathbb{R}$$

such that $u + v = 1$. Find the following limit

$$\lim_{n \rightarrow \infty} \left((n+1)^u \sqrt[n+1]{(a_{n+1}f(n+1))^v} - n^u \sqrt[n]{(a_n f(n))^v} \right).$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

UP.075. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$, be real positive sequences with

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a \in \mathbb{R}_+^*. \text{ If}$$

$$\lim_{n \rightarrow \infty} (n(a_n - a)) = b \in \mathbb{R} \text{ and } \lim_{n \rightarrow \infty} (n(b_n - a)) = c \in \mathbb{R},$$

$$\text{evaluate } \lim_{n \rightarrow \infty} \left(a_{n+1} \sqrt[n+1]{(n+1)!} - b_n \sqrt[n]{n!} \right)$$

Proposed by D.M. Bătinețu - Giurgiu; Neculai Stanciu - Romania

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