RMM - Calculus Marathon 1 - 100

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1. **Find:**

$$\Omega = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)(i+3)}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Hallar: $\Omega = \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)(i+3)}$

Trabajando la expresión el lado derecho:

$$\Rightarrow \frac{1}{3}\frac{(i+3)-(i)}{i(i+1)(i+2)(i+3)} = \frac{1}{3}\left(\frac{1}{i(i+1)(i+2)} - \frac{1}{(i+1)(i+2)(i+3)}\right)$$

$$\text{Si: } i = 1 \to \frac{1}{3}\left(\frac{1}{1\times2\times3} - \frac{1}{2\times3\times4}\right)$$

$$\Rightarrow i = 2 \to \frac{1}{3}\left(\frac{1}{2\times3\times4} - \frac{1}{3\times4\times5}\right)$$

$$\Rightarrow i = 3 \to \frac{1}{3}\left(\frac{1}{3\times4\times5} - \frac{1}{4\times5\times6}\right)$$

....

$$\Rightarrow i = n \to \frac{1}{3}\left(\frac{1}{k(k+1)(k+2)} - \frac{1}{(k+1)(k+2)(k+3)}\right)$$

$$\Rightarrow \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{i(i+1)(i+2)(i+3)} = \frac{1}{3}\left(\frac{1}{6} - \frac{1}{(n+1)(n+2)(n+3)}\right)$$

$$\Rightarrow \frac{1}{18} - \lim_{n\to\infty}\frac{1}{3}\frac{1}{(n+1)(n+2)(n+3)} = \frac{1}{18}$$

2. Find:

$$\Omega = \lim_{n\to\infty} \frac{n\int_1^{n^2}[\sqrt{x}]\,dx}{\int_1^{n^3}[\sqrt{x}]\,dx}, [*] - \textit{great integer function}$$

Proposed by Daniel Sitaru – Romania

Solution by Francis Fregeau – Quebec – Canada

Let β be a natural number. $\beta^2 < x \le (\beta+1)^2 \Rightarrow [\sqrt{x}] = \beta + 1$

Let n be a perfect square, and $1 \le x \le n^2$. Divide the interval $(1, n^2)$ into the partition: $(1,4] \cup (4,9] \cup \dots \cup (k^2,(k+1)^2] \cup \dots \cup ((n-1)^2, n^2)$*.Now:*

$$\int_{j^2}^{(j+1)^2} [\sqrt{x}]\,dx = \int_{j2}^{(j+1)^2} (j+1)\,dx = (2j+1)(j+1) = 2j^2 + 3j + 1$$

$$\int_1^{n^2} [\sqrt{x}]\,dx = \sum \int_{j^2}^{(j+1)^2} [\sqrt{x}]\,dx; 1 \le j \le n-1$$

Let $n - 1 = m$

$$\sum_1^m \int_{j^2}^{(j+1)^2} [\sqrt{x}]\,dx = \sum_1^m 2j^2 + 3j + 1 = \frac{2m^3 + 3m^2 + m}{3} + \frac{3m(m+1)}{2} + m$$

$$= \frac{4m^3 + 15m^2 + 17m}{6}$$

$$n\int_1^{n^2} [\sqrt{x}]\,dx = (m+1)\cdot\frac{4m^3 + 15m^2 + 17m}{6} = n\cdot\frac{4(n-1)^3 + 15(n-1)^2 + 17(n-1)}{6}$$

But n^3 is also a perfect square since n is a perfect square.

$$\Rightarrow \int_1^{n^3} [\sqrt{x}]\,dx = \sum 2j^2 + 3j + 1\,;\, 1 \le j \le n^{\frac{3}{2}} - 1$$

$$\int_1^{n^3} [\sqrt{x}]\,dx = \frac{4\left(n^{\frac{3}{2}} - 1\right)^3 + 15\left(n^{\frac{3}{2}} - 1\right)^2 + 17\left(n^{\frac{3}{2}} - 1\right)}{6}$$

$$\Rightarrow \lim_{n\to\infty} \frac{n \cdot \int_1^{n^2} [\sqrt{x}]\,dx}{\int_1^{n^3} [\sqrt{x}]\,dx} = \lim_{n\to\infty} \frac{n\left(4(n-1)^3 + 15(n-1)^2 + 17(n-1)\right)}{4\left(n^{\frac{3}{2}} - 1\right)^3 + 15\left(n^{\frac{3}{2}} - 1\right)^2 + 17\left(n^{\frac{3}{2}} - 1\right)}$$

Applying De l'Hôpital's rule four times yields: $\lim_{n\to\infty} \frac{n \cdot \int_1^{n^2} [\sqrt{x}]dx}{\int_1^{n^3} [\sqrt{x}]dx} = 0$

when n is a perfect square. But for every real number R, there exists a perfect square n such that $R < n$.

$$\lim_{R\to\infty} \frac{R \cdot \int_1^{R^2} [\sqrt{x}]\,dx}{\int_1^{R^3} [\sqrt{x}]\,dx} = \lim_{n\to\infty} \frac{n \cdot \int_1^{n^2} [\sqrt{x}]\,dx}{\int_1^{n^3} [\sqrt{x}]\,dx} = 0$$

3. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} e^{-\left(\frac{k}{n}\right)^2}}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren – Mongolia

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} e^{-\left(\frac{k}{n}\right)^2}} = \lim_{n\to\infty} \sqrt[n]{e^{-\left(\left(\frac{1}{n}\right)^2 + \cdots + \left(\frac{n}{n}\right)^2\right)}} =$$

$$=\lim_{n\to\infty}\frac{1}{e^{\left(\left(\frac{1}{n}\right)^2+\cdots+\left(\frac{n}{n}\right)^2\right)\cdot\frac{1}{n}}}=\frac{1}{e^{\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2}\cdot\frac{1}{n}}=$$

$$=\frac{1}{e^{\lim_{n\to\infty}\frac{1}{n}}\cdot\sum_{k=1}^{n}\left(\frac{k}{n}\right)^2}=\frac{1}{e^{\int_0^1 x^2dx}}=\frac{1}{\sqrt[3]{e}}$$

Solution 2 by Togrul Ehmedov-Baku-Azerbaidjian

$$\Omega=\lim_{n\to\infty}\sqrt[n]{\prod_{k=1}^{n}e^{-\left(\frac{k}{n}\right)^2}}=\lim_{n\to\infty}\sqrt[n]{e^{-\left[\frac{1^2+2^2+\cdots+n^2}{n^2}\right]}}=$$

$$=\lim_{n\to\infty}e^{-\left[\frac{1^2+2^2+\cdots+n^2}{n^3}\right]}=e^{-\lim_{n\to\infty}\left(\left[\frac{n(n+1)(2n+1)}{6n^3}\right]\right)}=e^{-\frac{1}{3}}$$

4. Calculate:

$$\Omega=\lim_{n\to\infty}\left\{\frac{\sqrt[n+1]{\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}}}{e^{\frac{(n+1)}{2}}(n+1)^{-\frac{3}{2}}}-\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{3}{2}}}\right\}$$

Proposed by Soumitra Moukherjee – Kolkata – India

Solution 1 by proposer

$$\Omega=\lim_{n\to\infty}\left\{\frac{\sqrt[n+1]{\binom{n+1}{1}\binom{n+1}{2}\cdots\binom{n+1}{n+1}}}{e^{\frac{(n+1)}{2}}(n+1)^{-\frac{3}{2}}}-\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{3}{2}}}\right\}$$

$$=\lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{1}{2}}}$$

Applying Reverse Caesaro – Stolz

$$\Rightarrow \ln\Omega = \lim_{n\to\infty}\frac{1}{n}\left\{\sum_{k=1}^{n}\ln n! - \sum_{k=1}^{n}\ln k! - \sum_{k=1}^{n}\ln(n-k)!\right\} - \frac{1}{2}\lim_{n\to\infty} n + \frac{1}{2}\lim_{n\to\infty}\ln n$$

$$= \lim_{n\to\infty}\frac{1}{n}\left\{(n+1)\ln n! - 2\sum_{k=1}^{n}\ln k!\right\} - \frac{1}{2}\lim_{n\to\infty} n + \frac{1}{2}\lim_{n\to\infty}\ln n$$

$$= \lim_{n\to\infty}\left\{(n+1)\ln n! - 2\sum_{k=1}^{n}(n+1-k)\ln k\right\} - \frac{1}{2}\lim_{n\to\infty} n + \frac{1}{2}\lim_{n\to\infty}\ln n$$

$$= \lim_{n\to\infty}\left\{2\sum_{k=1}^{n}k\ln k - (n+1)\ln n!\right\} - \frac{1}{2}\lim_{n\to\infty} n + \frac{1}{2}\lim_{n\to\infty}\ln n$$

$$= 2\lim_{n\to\infty}\frac{1}{n}\ln(1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n) - \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\ln n! - \frac{1}{2}\lim_{n\to\infty} n + \frac{1}{2}\lim_{n\to\infty}\ln n$$

Glaisher – Kenkelin constant: $A_n = \dfrac{1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n}{n^{\left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)}e^{-\frac{n^2}{4}}}$

$$\Rightarrow \ln A_n = \ln(1\cdot 2^2\cdot 3^3\cdot\ldots\cdot n^n) - \left(\frac{n^2}{2}+\frac{n}{2}+\frac{1}{12}\right)\ln n + \frac{n^2}{4}$$

again, $n! = \sqrt{2\pi n}\left(\frac{n}{e}\right)^n\left(1 + O\left(\frac{1}{n}\right)\right)$

$$\ln n! = \frac{1}{2}\ln 2\pi + \frac{1}{2}\ln n + n\ln n - n + \ln\left(1 + O\left(\frac{1}{n}\right)\right)$$

$$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\ln n! = \lim_{n\to\infty}\frac{1}{2}\left(1+\frac{1}{n}\right)\ln 2\pi + \lim_{n\to\infty}\left(1+\frac{1}{n}\right)\left(\frac{1}{2}+n\right)\ln - \lim_{n\to\infty}(n+1)$$

$$\therefore \ln\Omega = 2\lim_{n\to\infty}\frac{A_n}{n} + \lim_{n\to\infty}\left(n+\frac{3}{2}+\frac{1}{6n}\right)\ln n - \lim_{n\to\infty} n - \frac{1}{2}\lim_{n\to\infty}\left(1+\frac{1}{n}\right)\ln 2\pi$$

$$- \lim_{n\to\infty}\left(n+\frac{3}{2}+\frac{1}{2n}\right)\ln + \lim_{n\to\infty}(n+1)$$

$\Rightarrow \ln\Omega = 1 - \ln\sqrt{2\pi} = \ln\frac{e}{\sqrt{2\pi}} \Rightarrow \Omega = \frac{e}{\sqrt{2\pi}}$ ***(Ans :)***

Solution 2 by Yubian Andres Bedoya Henao – Medellin – Colombia

$$\textit{Let } P_n = \frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{3}{2}}} \Rightarrow \Omega = \lim_{n\to\infty}(P_{n+1}-P_n)$$

$$= \lim_{n\to\infty}\frac{P_{n+1}-P_n}{n+1-n} \overset{Stolz}{=} \lim_{n\to\infty}\frac{P_n}{n}$$

$$= \lim_{n\to\infty}\frac{\sqrt[n]{\binom{n}{1}\binom{n}{2}\cdots\binom{n}{n}}}{e^{\frac{n}{2}}n^{-\frac{1}{2}}} = \lim_{n\to\infty}\frac{\sqrt{n}\, n!\,(n!)^{\frac{1}{n}}}{e^{\frac{n}{2}}(1!\,2!\ldots n!)^{\frac{2}{n}}};$$

$$= \lim_{n\to\infty}\sqrt[n]{\frac{\left(\sqrt[n]{n}\right)^n n!\,(n!)^n}{e^{\frac{n^2}{2}}(1!\,2!\ldots n!)^2}} =$$

$$\overset{Cauchy-D'Alembert}{=} \lim_{n\to\infty}\frac{\dfrac{\left(\sqrt{n+1}\right)^{n+1}(n+1)!\,[(n+1)!]^{n+1}}{e^{\frac{(n+1)^2}{2}}(1!\,2!\ldots(n+1)!)^2}}{\dfrac{\sqrt{n}^{\,n}n!\,(n!)^n}{e^{\frac{n^2}{2}}(1!\,2!\ldots n!)^2}} =$$

$$= \lim_{n\to\infty}\frac{\sqrt{n+1}\left(\sqrt{1+\frac{1}{n}}\right)^n(n+1)(n+1)!\,(n+1)^n}{e^{\frac{n+1}{2}}(n+1)!^2} =$$

$$= \lim_{n\to\infty}\frac{e^{\frac{1}{2}}\sqrt{n+1}(n+1)^n}{e^{\frac{1}{2}}e^n n!}\cdot 1 \ \textit{(Stirling from } 1 = \lim_{n\to\infty}\frac{n!}{\sqrt{2n\pi}n n^n e^{-n}}\textit{)}$$

$$= \lim_{n\to\infty}\frac{\sqrt{n+1}(n+1)^n}{e^n n!}\cdot\lim_{n\to\infty}\frac{n!}{\sqrt{2\pi n}n^n e^{-n}} =$$

$$= \lim_{n\to\infty}\frac{1}{\sqrt{2\pi}}\sqrt{1+\frac{1}{n}}\left(1+\frac{1}{n}\right)^n = \frac{e}{\sqrt{2\pi}}$$

5. If $E_n := \left(1+\frac{1}{n}\right)^n$, $H_n := 1+\frac{1}{2}+\dots+\frac{1}{n}$, then:

$$\lim_{n\to\infty}\left(E_{n+1}^{H_{n+1}} - E_n^{H_n}\right) = e^{\gamma},$$

where e and γ are well known Euler's constants.

Proposed by Dorin Marghidanu – Romania

Solution by Soumitra Mandal – Kolkata – India

$$E_n = \left(1+\frac{1}{n}\right)^n \text{ and } H_n = 1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}$$

$$\lim_{n\to\infty}\left(E_{n+1}^{H_{n+1}} - E_n^{H_n}\right) = \lim_{n\to\infty}\frac{E_{n+1}^{H_{n+1}} - E_n^{H_n}}{n+1-n} = \lim_{n\to\infty}\frac{E_n^{H_n}}{n}$$

(Caesaro – Stolz Lemma)

$$= \lim_{n\to\infty}\frac{\left(1+\frac{1}{n}\right)^{n(\gamma_n+\ln n)}}{n} = \frac{\left\{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right\}^{\lim_{n\to\infty}(\ln n+\gamma_n)}}{\lim_{n\to\infty} n} =$$

$$= \lim_{n\to\infty}\frac{e^{\ln n+\gamma_n}}{n} = \lim_{n\to\infty} e^{\gamma_n} = e^{\gamma}$$

6. From the book: "Math Phenomenon"

$$\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{1}{1!\,2!\,3!\cdot\ldots\cdot k!}\right)^{\frac{1}{k}} \le e(e-1)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $a_k = \left(\frac{1}{1!2!3!\cdot\ldots\cdot k!}\right)^{\frac{1}{k}}$. We show that $a_k \le \frac{2^k-1}{k!}\ \forall\ k\ge 1$

For $k=1$, $a_k = 1 \le \frac{2^1-1}{1!}$. Assume that $a_k = \frac{1}{k!}\left(2^k-1\right)$ for some $k\in\mathbb{N}$

$$\Rightarrow \frac{1}{1!\,2!\cdot\ldots\cdot k!} \le \left(\frac{1}{k!}\right)^k \left(2^k-1\right)^k = \frac{1}{1!\,2!\cdot\ldots\cdot k!\,(k+1)!}$$

$$\le \left(\frac{1}{k!}\right)^k \left(2^k-1\right)^k \frac{1}{(k+1)!}$$

We now show that $\left(\frac{1}{k!}\right)^k \left(2^k-1\right)^k \frac{1}{(k+1)!} \le \left(\frac{1}{(k+1)!}\right)^{k+1} \left(2^{k+1}-1\right)^{k+1}$

$$\Leftrightarrow \left(\frac{(k+1)!}{k!}\right)^k \left(2^k-1\right)^k \le \left(2^{k+1}-1\right)^{k+1}$$

$$\Leftrightarrow (k+1)^k\left(2^k-1\right)^k \le \left(2^{k+1}-1\right)^{k+1} \quad \textit{(1)}$$

$$LHS$$

$$(k+1)^k\left(2^k-1\right) \le \left(2^{k+1}-1\right)^k\left(2^{k+1}-1\right) = \left(2^{k+1}-1\right)^{k+1}$$

$\left[\therefore k+1 \le 2^{k+1}-1 \ \forall k \in \mathbb{N}\right] \therefore (1)$ ***is true***

Thus, $a_k \le \frac{1}{k!}\left(2^k-1\right) \Rightarrow \sum_{k=1}^{n} a_k \le \sum_{k=1}^{n} \frac{1}{k!}\left(2^k-1\right)$

$$\Rightarrow \lim_{n\to\infty} \sum_{k=1}^{n} a_k \le e^2 - e = e(e-1)$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

Let $a_k = \frac{1}{k!}$ ***where*** $k = 1, 2, \ldots, n$***. Applying CARLEMANS Inequality***

$$\sum_{k=1}^{n} \sqrt[k]{a_1 a_2 \ldots a_k} < e(a_2 + a_3 + \cdots + a_n)$$

$$\therefore \sum_{k=1}^{n} \left(\frac{1}{1!\,2!\ldots k!}\right)^{\frac{1}{k}} < e\left(\frac{1}{2!} + \frac{1}{3!} + \cdots + \frac{1}{n!}\right)$$

$$\therefore \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{1}{1!\,2!\ldots k!}\right)^{\frac{1}{k}} < e(e-1)$$

7. **Solve for natural numbers:**

$$\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=3025$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

Resolver la ecuación: $\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=3025 \Leftrightarrow x>0$

$\Rightarrow \frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=(55)^2$. *Desde que:*

$$\sum_{i=1}^{n} i^3=\left(\frac{n(n+1)}{2}\right)^2$$

Si: $n=10 \Leftrightarrow 1^3+2^3+3^3+\cdots+10^3=(55)^2$

$$\frac{1^4}{x}+\frac{2^4}{x+1}+\frac{3^4}{x+2}+\cdots+\frac{10^4}{x+9}=1^3+2^3+3^3+\cdots+10^3$$

$$\left(\frac{1}{x}-1\right)+\left(\frac{16}{x+1}-8\right)+\left(\frac{81}{x+2}-27\right)+\cdots+\left(\frac{10000}{x+9}-1000\right)=0$$

$$\left(\frac{1-x}{x}\right)+\frac{8(1-x)}{x+1}+\frac{27(1-x)}{(x+2)}+\frac{1000(1-x)}{(x+9)}=0$$

$$(1-x)\left(\frac{1}{x}+\frac{8}{x+1}+\frac{27}{x+2}+\cdots+\frac{1000}{x+9}\right)=0 \Leftrightarrow$$

$\Leftrightarrow \left(\frac{1}{x}+\frac{8}{x+1}+\frac{27}{x+2}+\cdots+\frac{1000}{x+9}\right)>0$, *ya que:* $x>0$. *Por la tanto:* $x=1$

8. **Solve in natural numbers:**

$$1-\sum_{k=1}^{n}\frac{k}{\sqrt{k!}\left(k+1+\sqrt{k+1}\right)}\le\frac{1}{12\sqrt{5}}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru:

Resolver en números naturales.

$$1-\sum_{k=1}^{n}\frac{k}{\sqrt{k!}\left(k+1+\sqrt{k+1}\right)}\le\frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\frac{k}{\sqrt{k!}\left(\sqrt{k+1}\right)\left(\sqrt{k+1}+1\right)}\le\frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\frac{k\left(\sqrt{k+1}-1\right)}{\sqrt{(k+1)!}\left(\sqrt{k+1}+1\right)\left(\sqrt{k+1}-1\right)}\le\frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\frac{k\left(\sqrt{k+1}-1\right)}{\sqrt{(k+1)!}\,(k)}\le\frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\left(\frac{\sqrt{k+1}}{\sqrt{(k+1)k!}}-\frac{1}{\sqrt{(k+1)!}}\right)\le\frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\sum_{k=1}^{n}\frac{1}{\sqrt{k!}}+\sum_{k=1}^{n}\frac{1}{\sqrt{(k+1)!}}\le\frac{1}{12\sqrt{5}}$$

$$\Rightarrow 1-\frac{1}{\sqrt{1!}}+\frac{1}{\sqrt{2!}}-\frac{1}{\sqrt{2!}}+\frac{1}{\sqrt{3!}}-\cdots+\frac{1}{\sqrt{(k+1)!}}\le\frac{1}{12\sqrt{5}}$$

$$\Rightarrow\frac{1}{\sqrt{(k+1)!}}\le\frac{1}{12\sqrt{5}}\rightarrow\sqrt{(k+1)!}\ge 12\sqrt{5}\rightarrow(k+1)!\ge 720\rightarrow$$

$$\rightarrow(k+1)!\ge 6!\rightarrow k\ge 5$$

9. Find $x\in\mathbb{N}$ such that:

$$\sum_{k=1}^{n}\frac{x+k}{3k+2}=\sum_{k=1}^{n}\frac{k+1}{x+3k+1}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios –Huarmey- Peru

$$\sum_{k=1}^{n}\frac{x+k}{3k+2}=\sum_{k=1}^{n}\frac{k+1}{x+3k+1} \Leftrightarrow x\wedge n>0$$

$$\Rightarrow \frac{x+1}{5}+\frac{x+2}{8}+\frac{x+3}{11}+\dots+\frac{x+n}{3n+2}=\frac{2}{x+4}+\frac{3}{x+7}+\frac{4}{x+10}+\dots+\frac{n+1}{x+1+3n}$$

$$\Rightarrow \left(\frac{x+1}{5}-\frac{2}{x+4}\right)+\left(\frac{x+2}{8}-\frac{3}{x+7}\right)+\left(\frac{x+3}{11}-\frac{4}{x+10}\right)+\dots+\left(\frac{x+n}{3n+2}-\frac{n+1}{x+1+3n}\right)=0$$

$$\Rightarrow \frac{x^2+5x-6}{5(x+4)}+\frac{x^2+9x-10}{8(x+7)}+\frac{x^2+13x-14}{11(x+10)}+\dots+\frac{(x+n)(x+1+3n)-(n+1)(3n+2)}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow \frac{(x-1)(x+6)}{5(x+4)}+\frac{(x-1)(x+10)}{8(x+7)}+\frac{(x-1)(x+14)}{11(x+10)}+\dots+\frac{x^2+x+4nx+n+3n^2-3n^2-5n-2}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow \frac{(x-1)(x+6)}{5(x+4)}+\frac{(x-1)(x+10)}{8(x+7)}+\frac{(x-1)(x+14)}{11(x+10)}+\dots+\frac{x^2+x(4n+1)-4n-2}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow \frac{(x-1)(x+6)}{5(x+4)}+\frac{(x-1)(x+10)}{8(x+7)}+\frac{(x-1)(x+14)}{11(x+10)}+\dots+\frac{(x-1)(x+4n+2)}{(3n+2)(x+1+3n)}=0$$

$$\Rightarrow (x-1)\left(\frac{x+6}{5(x+4)}+\frac{x+10}{8(x+7)}+\frac{x+14}{11(x+10)}+\dots+\frac{x+4n+2}{(3n+2)(x+1+3n)}\right)=0$$

$\Rightarrow$ ***Por la tanto:*** $x=1$***, ya que:***

$$\left(\frac{x+6}{5(x+4)}+\frac{x+10}{8(x+7)}+\frac{x+14}{11(x+10)}+\dots+\frac{x+4n+2}{(3n+3)(x+1+3n)}\right)>0 \Leftrightarrow x\wedge n>0$$

10. Solve in natural numbers:

$$\sum_{k=3}^{x}\binom{\binom{k}{2}}{2}\leq 168$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\binom{k}{2}=\frac{1}{2}k(k-1)=m \quad \textbf{\textit{(say)}} \therefore \binom{\binom{k}{2}}{2}=\binom{m}{2}=\frac{1}{2}m(m-1)$$

$$=\frac{1}{8}k(k-1)[k^2-k-2]=\frac{1}{8}k(k-1)(k+1)(k-2)=$$

$$=\frac{3}{24}(k+1)k(k-1)(k-2)=3\binom{k+1}{4}$$

$$\sum_{k=3}^{x}\binom{\binom{k}{2}}{2}=3\sum_{k=3}^{x}\binom{k+1}{4}=3\left[\binom{4}{4}+\binom{5}{4}+\cdots+\binom{x+1}{4}\right]=$$

$$=3\left[\binom{5}{5}+\binom{5}{4}+\cdots+\binom{x+1}{4}\right]=3\left[\binom{6}{5}+\binom{6}{4}+\cdots+\binom{x+1}{4}\right]=$$

$$=3\left[\binom{7}{5}+\binom{7}{4}+\cdots+\binom{x+1}{4}\right]=\cdots=3\binom{x+2}{5}$$

Thus, $3\binom{x+2}{5}\le 168\Rightarrow\binom{x+2}{5}\le 56$. *As* $\binom{n}{r}=0$ *for* $n<r$,

$$x=1,2,3,4,5,6$$

Solution 2 by Soumava Pal-Kolkata-India

We know $\binom{\binom{k}{2}}{2}=3\binom{k}{3}+3\binom{k}{4},\ k\ge 3$ *(where* $\binom{3}{4}=0$ *defined)*

$$\sum_{k=3}^{x}\binom{\binom{k}{2}}{2}\le 168\Rightarrow 3\left(\sum_{k=3}^{x}\binom{k}{3}+\sum_{k=3}^{x}\binom{k}{4}\right)\le 168$$

$$\Rightarrow\left(\binom{3}{3}+\binom{4}{3}+\cdots+\binom{x}{3}+\binom{3}{4}+\binom{4}{4}+\cdots+\binom{x}{3}\right)\le 56$$

(where $\binom{3}{4}=0$*) we check that for* $x=3, LHS=1$

$x=4, LHS=6$; $x=5, LHS=21$; $x=6, LHS=56$

so possible values of x *are* $3,4,5,6$

Solution 3 by Saptak Bhattacharya – Kolkata – India

$\sum_{k=3}^{x}\left\{\binom{k}{3}+\binom{k}{4}\right\}\le 56\Rightarrow\sum_{k=4}^{x+1}\binom{n}{4}\le 56$ *(Pascal). Now,*

$$\sum_{k=4}^{x+1}\binom{n}{4}=\sum_{n-4=0}^{x-3}\binom{n-4+5-1}{5-1}=T(let)$$

$= \alpha_1 + \alpha_2 + \cdots + \alpha_5 \le x - 3$. *We introduce a 6th variable*

$\alpha_6 = x - 3 - \sum_{k=i}^{5} \alpha_i$. *So,* $\sum_{k=1}^{6} \alpha_i = x - 3$ *has* T *two integer solutions*

$$\Rightarrow T = \binom{x - 3 + 6 - 1}{6 - 1} = \binom{x + 2}{5}. \text{ Thus, } \binom{x + 2}{5} \le 56 \Rightarrow x = 3, 4, 5, 6$$

11. Solve for real numbers:

$$4^x + 25^{\frac{1}{x}} + 4^x \cdot 25^{\frac{1}{x}} = 101$$

Proposed by Marian Ursarescu-Romania

Solution by Radu Butelca-Romania

$E = 4^x + 25^{\frac{1}{x}} + 4^x \cdot 25^{\frac{1}{x}} = 101$; $x < 0 \Rightarrow 4^x + 25^{\frac{1}{x}} + 4^x \cdot 25^{\frac{1}{x}} < 3 < 101$*: false*

If $x > 0$*, we notice that* $x = 2$ *and* $x = \log_4 5$ *satisfies the equation.*

Let $f: \mathbb{R}_+^* \to (0, +\infty), f(x) = 4^x + 25^{\frac{1}{x}}$

We prove that f *is strictly decreasing on* $(0, \sqrt{\alpha})$ *and strictly increasing on* $(\sqrt{\alpha}, +\infty)$*,*

where $\alpha = \log_4 25$; $4^{\log_4 25} = 25 \Rightarrow f(x) = 4^x + 4^{\frac{\alpha}{x}}$

Suppose that $\sqrt{\alpha} \le x \le y \Rightarrow f(y) - f(x) = (4^y - 4^x) + \left(4^{\frac{\alpha}{y}} - 4^{\frac{\alpha}{x}}\right) =$

$= 4^x(4^{y-x} - 1) - 4^{\frac{\alpha}{y}}\left(4^{\frac{\alpha(y-x)}{y-1}}\right)$. *But* $\alpha < xy$

$$\Rightarrow f(y) - f(x) > 4^x(4^{y-x} - 1) - 4^{\frac{\alpha}{y}}(4^{y-x} - 1) =$$

$$\left.\begin{matrix} = (4^{y-x} - 1)\left(4^x - 4^{\frac{\alpha}{y}}\right) \\ y > x \text{ and } \frac{\alpha}{y} < \sqrt{\alpha} < x \end{matrix}\right\} \Rightarrow f(y) - f(x) > 0 \Leftrightarrow f(y) > f(x) \Leftrightarrow f \text{ is strictly increasing}$$

on $(\sqrt{\alpha}, +\infty)$. *Similar for* $(0, \sqrt{\alpha}) \Rightarrow f(x) = 4^x + 25^{\frac{1}{x}}$ *is strictly convexe (1)*

Let $g: \mathbb{R}_+^* \to (0, +\infty), g(x) = 4^{x+\frac{\alpha}{x}}$; $4^x \cdot 25^{\frac{1}{x}} = 4^x \cdot 4^{\frac{\alpha}{x}} = 4^{x+\frac{\alpha}{x}}, \alpha = \log_4 25$

$$g(x) = 4^{x+\frac{\alpha}{x}} \Rightarrow \frac{d}{dx}f(x) = 4^{x+\frac{\alpha}{x}} \ln 4 \left(1 - \frac{\alpha}{x^2}\right)$$

$$\frac{d}{dx^2}f(x) = \ln 4\left(1-\frac{\alpha}{x^2}\right)\frac{d}{dx}f(x) = 4^{x+\frac{\alpha}{x}}\ln 4\alpha\frac{2x}{x^4} =$$

$$\left.\begin{array}{l} = \underbrace{4^{x+\frac{\alpha}{x}}\ln 4}_{>0}\underbrace{\left(1-\frac{\alpha}{x^2}+\alpha\frac{2}{x^3}\right)} \\ 1-\frac{\alpha}{x^2}+\alpha\frac{2}{x^3} = \frac{x^3-\alpha x+2x}{x^3} > 0 \end{array}\right\} \Rightarrow g''(x) > 0 \Rightarrow g \text{ is strictly convexe (2)}$$

$E = f(x) + g(x)$, which is a sum of 2 strictly convexe functions $\Rightarrow E$ has maximum 2 solutions which are $x = 2$ and $x = \log_4 5$

12. Find:

$$\Omega = \int \left(x^{10}+\sqrt{1+x^{20}}\right)^{\frac{21}{10}} dx, x \in \mathbb{R}$$

Proposed by Nishant Kumar – Jamshedpur – India

Solution 1 by proposer

$$\int\left(x^{10}+\sqrt{1+x^{20}}\right)^{\frac{21}{10}} dx \text{ putting } x^{10}+\sqrt{1+x^{20}} = t$$

using the fact

$$\left[10x^9+\frac{10x^{19}}{\sqrt{1+x^{20}}}\right]dx = dt$$

$$\left(\sqrt{1+x^{20}}+x^{10}\right)\left(\sqrt{1+x^{20}}-x^{10}\right) = 1$$

$$\frac{10x^9\left(\sqrt{1+x^{20}}+x^{10}\right)}{\sqrt{1+x^{20}}}dx = dt,\ x^{10}+\sqrt{1+x^{20}} = t$$

$$\sqrt{1+x^{20}}-x^{10} = \frac{1}{t},\ \frac{10\left(t^2-1\right)^{\frac{9}{10}}\cdot t\cdot(2t)}{(2t)^{\frac{9}{10}}\cdot\left(t^2+1\right)}dx = dt$$

On adding we get $\sqrt{1+x^{20}} = \frac{t^2+1}{2t}$. On substractiong we get $x^{10} = \frac{t^2-1}{2t}$

$$dx = \frac{(2t)^{\frac{9}{10}}\cdot\left(t^2+1\right)}{20\cdot t^2\left(t^2-1\right)^{\frac{9}{10}}}dt,\ x^9 = \left\{\frac{t^2-1}{2t}\right\}^{\frac{9}{10}}$$

$$\int \frac{t^{\frac{21}{10}}\cdot(2t)^{\frac{9}{10}}(t^2+1)dt}{20t^2\cdot(t^2-1)^{\frac{9}{10}}}=\frac{1}{20}\int\frac{t^{\frac{1}{10}}\cdot(2t)^{\frac{9}{10}}(t^2+1)}{(t^2-1)^{\frac{9}{10}}}dt$$

$$\frac{1}{2^{\frac{1}{10}}\cdot 20}\int\frac{2t\cdot(t^2+1)}{(t^2-1)^{\frac{9}{10}}}dt=\frac{1}{2^{\frac{1}{10}}\cdot 20}\left[10(t^2-1)^{\frac{1}{10}}(t^2+1)-\int 2t\cdot(t^2-1)^{\frac{1}{10}}dt\right]$$

$$\frac{1}{2^{\frac{1}{10}}\cdot 20}\left[10(t^2-1)^{\frac{1}{10}}(t^2+1)-\frac{10}{11}(t^2-1)^{\frac{11}{10}}\right]$$

$$\frac{1}{2^{\frac{11}{10}}}\left[(t^2-1)^{\frac{1}{10}}(t^2+1)-\frac{(t^2-1)^{\frac{11}{10}}}{11}\right]+c$$

where $t=\left[x^{10}+\sqrt{1+x^{20}}\right]$

Solution 2 by Ravi Prakash - New Delhi – India

$$\Omega=\int\left(\sqrt{x^{20}+1}+x^{10}\right)^{\frac{21}{10}}$$

$$=\int\frac{\left[\left(\sqrt{x^{20}+1}+x^{10}\right)\left(\sqrt{x^{20}+1}-x^{10}\right)\right]^{\frac{21}{10}}}{\left[\sqrt{x^{20}+1}-x^{10}\right]^{\frac{21}{10}}}dx=$$

$$=\int\left[x^{10}\left(\sqrt{1+x^{-20}}-1\right)\right]^{\left(-\frac{21}{10}\right)}dx=\int x^{-21}\left[\sqrt{1+x^{-20}}-1\right]^{-\frac{21}{10}}dx$$

Put $\sqrt{1+x^{-20}}=t\Rightarrow x^{-20}=t^2-1,\ -20x^{-21}dx=2t\,dt$

$$\therefore\ \Omega=\int\left(-\frac{1}{20}\right)(t+1)^{-\frac{21}{20}}(2t)\,dt=(t+10)^{-\frac{1}{20}}(2t)-2\int(t+1)^{-\frac{1}{20}}dt$$

$$=\frac{2t}{(t+1)^{\frac{1}{20}}}-\frac{2\times 20}{19}(t+1)^{\frac{19}{20}}+c$$

$$=\frac{2\sqrt{1+x^{-20}}}{\left[1+\sqrt{x^{-20}+1}\right]^{\frac{1}{20}}}-\frac{40}{9}\left(1+\sqrt{x^{-20}+1}\right)^{\frac{19}{20}}+c$$

Solution 3 by Myagmarsuren Yadamsuren – Ulaanbataar – Mongolia

$$\int\left(x^{10}+\sqrt{1+x^{20}}\right)^{\frac{21}{10}}dx=\int\left[\frac{\left(\sqrt{1+x^{20}}+x^{10}\right)\cdot\left(\sqrt{1+x^{20}}-x^{10}\right)}{\sqrt{1+x^{20}}-x^{10}}\right]^{\frac{21}{10}}dx$$

$$=\int\frac{1}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{21}{10}}}dx=\int\frac{1}{x^{21}\cdot\left(\sqrt{1+\frac{1}{x^{20}}}-1\right)^{\frac{21}{10}}}dx=$$

$$=-\frac{1}{20}\int\frac{d\left(1+\frac{1}{x^{20}}\right)}{\left(\sqrt{1+\frac{1}{x^{20}}}-1\right)^{\frac{21}{10}}}=\left(1+\frac{1}{x^{20}}=y\right)=$$

$$=-\frac{1}{20}\cdot\int\frac{dy}{\left(\sqrt{y}-1\right)^{\frac{21}{10}}}=-\frac{1}{20}\left[-\frac{20}{11}\cdot\int\sqrt{y}\cdot d\left(\sqrt{y}-1\right)^{-\frac{11}{10}}\right]=$$

$$=\frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}}-\int\left(\sqrt{y}-1\right)^{-\frac{11}{10}}d\left(\sqrt{y}\right)\right)=$$

$$=\frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}}-\frac{1}{2}\cdot\int\frac{1}{\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{\frac{11}{10}}}dy\right)=$$

$$=\frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}}-\frac{1}{2}\cdot\left[-20\cdot\int d\left(\sqrt{y}-1\right)^{-\frac{1}{10}}\right]\right)=$$

$$=\frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}}+10\cdot\left(\sqrt{y}-1\right)^{-\frac{1}{10}}+C\right)=$$

$$=\frac{1}{11}\cdot\left[\frac{\sqrt{y}}{\left(\sqrt{y}-1\right)^{\frac{11}{10}}}+\frac{10}{\left(\sqrt{y}-1\right)^{\frac{1}{10}}}+C\right]=$$

$$=\frac{1}{11}\cdot\left[\frac{\sqrt{1+\frac{1}{x^{20}}}}{\left(\sqrt{1+\frac{1}{x^{20}}}-1\right)^{\frac{11}{10}}}+\frac{10}{\left(\sqrt{1+\frac{1}{x^{20}}}\right)^{\frac{1}{10}}}\right]+C=$$

$$=\frac{1}{11}\cdot\left[\frac{x\cdot\sqrt{1+x^{20}}}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{11}{10}}}+\frac{10x}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{1}{10}}}\right]+C=$$

$$=\frac{1}{11}\left[\frac{x\cdot\sqrt{1+x^{20}}}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{11}{10}}}+\frac{10x}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{1}{10}}}\right]+C$$

Solution 4 by Yen Tung Chung – Taichung – Taiwan

$$\int\left(x^{10}+\sqrt{1+x^{20}}\right)^{\frac{21}{10}}=\int\left(\frac{(1+x^{20})-x^{20}}{\sqrt{1+x^{20}}-x^{10}}\right)^{\frac{21}{10}}dx=$$

$$=\int\left(\sqrt{1+x^{20}}-x^{10}\right)^{-\frac{21}{10}}dx=\int\left(\sqrt{1+x^{-20}}-1\right)^{-\frac{21}{10}}\cdot x^{-21}dx$$

$$\left(u=\sqrt{1+x^{-20}}-1\Rightarrow x^{-20}=u^2+2u,-20x^{-21}dx=(2u+2)du\right)$$

$$=\int u^{-\frac{21}{20}}\cdot\left[-\frac{1}{10}(u+1)\right]du=-\frac{1}{10}\int\left(u^{-\frac{1}{20}}+u^{-\frac{21}{20}}\right)du=$$

$$=-\frac{1}{10}\left(\frac{20}{19}u^{\frac{19}{20}}-20u^{-\frac{1}{20}}\right)+C=$$

$$=-\frac{2}{19}\left(1-\sqrt{1+x^{-20}}\right)^{\frac{19}{20}}+2\left(1-\sqrt{2+x^{-20}}\right)^{-\frac{1}{20}}+C$$

13. Find:

$$\Omega = \int \left(x^{10} + \sqrt{1 + x^{20}}\right)^{\frac{21}{10}} dx, x \in \mathbb{R}$$

Proposed by Nishant Kumar – Jamshedpur – India

Solution 1 by proposer

$\int\left(x^{10} + \sqrt{1 + x^{20}}\right)^{\frac{21}{10}} dx$ ***putting*** $x^{10} + \sqrt{1 + x^{20}} = t$, ***using the fact***

$$\left[10x^9 + \frac{10x^{19}}{\sqrt{1+x^{20}}}\right] dx = dt; \left(\sqrt{1 + x^{20}} + x^{10}\right)\left(\sqrt{1 + x^{20}} - x^{10}\right) = 1$$

$$\frac{10x^9\left(\sqrt{1+x^{20}}+x^{10}\right)}{\sqrt{1+x^{20}}} dx = dt,\ x^{10} + \sqrt{1 + x^{20}} = t$$

$$\sqrt{1 + x^{20}} - x^{10} = \frac{1}{t}, \qquad \frac{10(t^2 - 1)^{\frac{9}{10}} \cdot t \cdot (2t)}{(2t)^{\frac{9}{10}} \cdot (t^2 + 1)} dx = dt$$

On adding we get $\sqrt{1 + x^{20}} = \frac{t^2+1}{2t}$***. On substractiong we get*** $x^{10} = \frac{t^2-1}{2t}$

$$dx = \frac{(2t)^{\frac{9}{10}} \cdot (t^2 + 1)}{20 \cdot t^2 (t^2 - 1)^{\frac{9}{10}}} dx;\ x^9 = \left\{\frac{t^2 - 1}{2t}\right\}^{\frac{9}{10}}$$

$$\int \frac{t^{\frac{21}{10}} \cdot (2t)^{\frac{9}{10}}(t^2 + 1)dt}{20t^2 \cdot (t^2 - 1)^{\frac{9}{10}}} = \frac{1}{20}\int \frac{t^{\frac{1}{10}} \cdot (2t)^{\frac{9}{10}}(t^2 + 1)}{(t^2 - 1)^{\frac{9}{10}}} dt$$

$$\frac{1}{2^{\frac{1}{10}} \cdot 20}\int \frac{2t \cdot (t^2 + 1)}{(t^2 - 1)^{\frac{9}{10}}} dt = \frac{1}{2^{\frac{1}{10}} \cdot 20}\left[10(t^2 - 1)^{\frac{1}{10}}(t^2 + 1) - \int 2t \cdot (t^2 - 1)^{\frac{1}{10}} dt\right]$$

$$\frac{1}{2^{\frac{1}{10}} \cdot 20}\left[10(t^2 - 1)^{\frac{1}{10}}(t^2 + 1) - \frac{10}{11}(t^2 - 1)^{\frac{11}{10}}\right]$$

$$\frac{1}{2^{\frac{11}{10}}}\left[(t^2 - 1)^{\frac{1}{10}}(t^2 + 1) - \frac{(t^2-1)^{\frac{11}{10}}}{11}\right] + c, \textbf{\textit{where }} t = \left[x^{10} + \sqrt{1 + x^{20}}\right]$$

Solution 2 by Ravi Prakash - New Delhi – India

$$\Omega=\int\left(\sqrt{x^{20}+1}+x^{10}\right)^{\frac{21}{10}}dx=$$

$$=\int\frac{\left[\left(\sqrt{x^{20}+1}+x^{10}\right)\left(\sqrt{x^{20}+1}-x^{10}\right)\right]^{\frac{21}{10}}}{\left[\sqrt{x^{20}+1}-x^{10}\right]^{\frac{21}{10}}}dx=$$

$$=\int\left[x^{10}\left(\sqrt{1+x^{-20}}-1\right)\right]^{\left(-\frac{21}{10}\right)}dx=\int x^{-21}\left[\sqrt{1+x^{-20}}-1\right]^{-\frac{21}{10}}dx$$

Put $\sqrt{1+x^{-20}}=t\Rightarrow x^{-20}=t^2-1;-20x^{-21}dx=2t\,dt$

$$\therefore\Omega=\int\left(-\frac{1}{20}\right)(t+1)^{-\frac{21}{20}}(2t)\,dt=$$

$$=(t+10)^{-\frac{1}{20}}(2t)-2\int(t+1)^{-\frac{1}{20}}dt=\frac{2t}{(t+1)^{\frac{1}{20}}}-\frac{2\times 20}{19}(t+1)^{\frac{19}{20}}+c=$$

$$=\frac{2\sqrt{1+x^{-20}}}{\left[1+\sqrt{x^{-20}+1}\right]^{\frac{1}{20}}}-\frac{40}{9}\left(1+\sqrt{x^{-20}+1}\right)^{\frac{19}{20}}+c$$

Solution 3 by Myagmarsuren Yadamsuren – Ulaanbataar – Mongolia

$$\int\left(x^{10}+\sqrt{1+x^{20}}\right)^{\frac{21}{10}}dx=\int\left[\frac{\left(\sqrt{1+x^{20}}+x^{10}\right)\cdot\left(\sqrt{1+x^{20}}-x^{10}\right)}{\sqrt{1+x^{20}}-x^{10}}\right]^{\frac{21}{10}}dx$$

$$=\int\frac{1}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{21}{10}}}dx=\int\frac{1}{x^{21}\cdot\left(\sqrt{1+\frac{1}{x^{20}}}-1\right)^{\frac{21}{10}}}dx=$$

$$=-\frac{1}{20}\int\frac{d\left(1+\frac{1}{x^{20}}\right)}{\left(\sqrt{1+\frac{1}{x^{20}}}-1\right)^{\frac{21}{10}}}=\left(1+\frac{1}{x^{20}}=y\right)=$$

$$= -\frac{1}{20}\cdot\int\frac{dy}{\left(\sqrt{y}-1\right)^{\frac{21}{10}}} = -\frac{1}{20}\left[-\frac{20}{11}\cdot\int\sqrt{y}\cdot d\left(\sqrt{y}-1\right)^{-\frac{11}{10}}\right] =$$

$$= \frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}} - \int\left(\sqrt{y}-1\right)^{-\frac{11}{10}}d\left(\sqrt{y}\right)\right) =$$

$$= \frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}} - \frac{1}{2}\cdot\int\frac{1}{\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{\frac{11}{10}}}dy\right) =$$

$$= \frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}} - \frac{1}{2}\cdot\left[-20\cdot\int d\left(\sqrt{y}-1\right)^{-\frac{1}{10}}\right]\right) =$$

$$= \frac{1}{11}\cdot\left(\sqrt{y}\cdot\left(\sqrt{y}-1\right)^{-\frac{11}{10}} + 10\cdot\left(\sqrt{y}-1\right)^{-\frac{1}{10}} + C\right) =$$

$$= \frac{1}{11}\cdot\left[\frac{\sqrt{y}}{\left(\sqrt{y}-1\right)^{\frac{11}{10}}} + \frac{10}{\left(\sqrt{y}-1\right)^{\frac{1}{10}}} + C\right] =$$

$$= \frac{1}{11}\cdot\left[\frac{\sqrt{1+\frac{1}{x^{20}}}}{\left(\sqrt{1+\frac{1}{x^{20}}}-1\right)^{\frac{11}{10}}} + \frac{10}{\left(\sqrt{1+\frac{1}{x^{20}}}\right)^{\frac{1}{10}}}\right] + C =$$

$$= \frac{1}{11}\cdot\left[\frac{x\cdot\sqrt{1+x^{20}}}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{11}{10}}} + \frac{10x}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{1}{10}}}\right] + C =$$

$$= \frac{1}{11}\left[\frac{x\cdot\sqrt{1+x^{20}}}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{11}{10}}} + \frac{10x}{\left(\sqrt{1+x^{20}}-x^{10}\right)^{\frac{1}{10}}}\right] + C$$

Solution 4 by Yen Tung Chung – Taichung – Taiwan

$$\int\left(x^{10}+\sqrt{1+x^{20}}\right)^{\frac{21}{10}}=\int\left(\frac{(1+x^{20})-x^{20}}{\sqrt{1+x^{20}}-x^{10}}\right)^{\frac{21}{10}}dx=$$

$$=\int\left(\sqrt{1+x^{20}}-x^{10}\right)^{-\frac{21}{10}}dx=\int\left(\sqrt{1+x^{-20}}-1\right)^{-\frac{21}{10}}\cdot x^{-21}dx$$

$$\left(u=\sqrt{1+x^{-20}}-1\Rightarrow x^{-20}=u^2+2u,-20x^{-21}dx=(2u+2)du\right)$$

$$=\int u^{-\frac{21}{20}}\cdot\left[-\frac{1}{10}(u+1)\right]du=-\frac{1}{10}\int\left(u^{-\frac{1}{20}}+u^{-\frac{21}{20}}\right)du=$$

$$=-\frac{1}{10}\left(\frac{20}{19}u^{\frac{19}{20}}-20u^{-\frac{1}{20}}\right)+C$$

$$=-\frac{2}{19}\left(1-\sqrt{1+x^{-20}}\right)^{\frac{19}{20}}+2\left(1-\sqrt{2+x^{-20}}\right)^{-\frac{1}{20}}+C$$

14. $\Omega=\lim_{n\to\infty}\sin\left(\frac{1}{n}\int_n^{n+1}\frac{dx}{\sqrt{(x-n)(n+1-x)}}\right)$

True or false:

$-1<\Omega<1$

Proposed by Daniel Sitaru – Romania

Solution by Togrul Ehmedov-Baku-Azerbaidjian

$$I_n=\int_n^{n+1}\frac{dx}{\sqrt{(x-n)(n+1-x)}}=\lim_{\substack{\varphi_1\to0^+\\\varphi_2\to0^+}}\int_{n+\varphi_1}^{n+1-\varphi_2}\frac{dx}{\sqrt{(x-n)(n+1-x)}}=$$

$$=\lim_{\substack{\varphi_1\to0^+\\\varphi_2\to0^+}}\int_{n+\varphi_1}^{n+1-\varphi_2}\left(\sqrt{\frac{n+1-x}{x-n}}+\sqrt{\frac{x-n}{n+1-x}}\right)dx$$

$$I=\int\left(\sqrt{\frac{n+1-x}{x-n}}+\sqrt{\frac{x-n}{n+1-x}}\right)dx$$

$$\frac{n+1-x}{x-n}=t^2\Rightarrow x=n+\frac{1}{1+t^2}\Rightarrow dx=-\frac{2t}{(1+t^2)^2}dt$$

$$I=-\int\left(t+\frac{1}{t}\right)\left(\frac{2t}{(1+t^2)^2}dt\right)=-2\arctan t=-2\arctan\sqrt{\frac{n+1-x}{x-n}}$$

$$I_n=\lim_{\substack{\varphi_1\to 0^+\\ \varphi_2\to 0^+}}\left(-2\arctan\sqrt{\frac{n+1-x}{x-n}}\right)\Bigg|_{n+\varphi_1}^{n+1-\varphi_2}=$$

$$-2\lim_{\substack{\varphi_1\to 0^+\\ \varphi_2\to 0^+}}\left(\arctan\sqrt{\frac{\varphi_2}{1-\varphi_2}}-\arctan\sqrt{\frac{1-\varphi_1}{\varphi_1}}\right)=-2\cdot\left(-\frac{\pi}{2}\right)=\pi$$

$$\Omega=\lim_{n\to\infty}\sin\left(\frac{1}{n}I_n\right)=\lim_{n\to\infty}\sin\left(\frac{\pi}{n}\right)=0$$

True or false

$$-1<\Omega<1$$

15. True or false: If $m\in\mathbb{N}^*$ then:

$$I_m=\int_0^1\frac{\sqrt{e^x}(x+3)}{(x+1)\sqrt{e^x}+m}dx<1$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$I_m=\int_0^1\frac{\sqrt{e^x}(x+3)}{(x+1)\sqrt{e^x}+m}dx$$

Put $t = (x+1)\sqrt{e^x} + m$, ***When*** $x = 0, t = m+1$

$$x = 1, t = m + 2\sqrt{e}$$

$\frac{dt}{dx} = (x+1)\frac{1}{2}\sqrt{e^x} + \sqrt{e^x} = \frac{1}{2}(x+3)\sqrt{e^x}$. ***Thus,***

$$I_m = 2\int_{m+1}^{m+2\sqrt{e}} \frac{dt}{t} = 2\ln\left(\frac{m+2\sqrt{e}}{m+1}\right);\ I_m < 1$$

$$\Leftrightarrow \ln\left(\frac{m+2\sqrt{e}}{m+1}\right) < \frac{1}{2} \Leftrightarrow m + 2\sqrt{e} < (m+1)\sqrt{e}$$

$$\Leftrightarrow \sqrt{e} < \left(\sqrt{e}-1\right)m \Leftrightarrow m < (m-1)\sqrt{e}$$

Not true for $m = 1, 2$. ***True, for*** $m \geq 3$

Solution 2 by Redwane El Mellass-Morocco

$$I_m = \int_0^1 \frac{(x+1)\sqrt{e^x} + m + 2\sqrt{e^x} - m}{(x+1)\sqrt{e^x} + m} dx = \int_0^1 1 - \frac{m - 2\sqrt{e^x}}{(x+1)\sqrt{e^x} + m} dx =$$

$$= 1 - \int_0^1 \frac{m - 2\sqrt{e^x}}{(x+1)\sqrt{e^x} + m} dx$$

$$\begin{cases} 2\sqrt{e^x} - m < 3 - m \\ (x+1)\sqrt{e^x} + m \geq 2 + m \end{cases} \Rightarrow \frac{2\sqrt{e^x} - m}{(x+1)\sqrt{e^x}} \begin{cases} < 0\ if\ m = 1,2 \\ > 0\ if\ m \geq 3 \end{cases} \Rightarrow$$

$\Rightarrow I_{m>2} < 1$ ***and*** $I_{1\leq m\leq 2} > 1$ ***(false)***

Solution 3 by Saptak Bhattacharya – Kolkata – India

By MVT, $I_m = \frac{e^{\frac{c}{2}}(c+3)}{(c+1)e^{\frac{c}{2}} + m}$ ***for some*** $c \in (0,1)$; $I_m < 1$ ***implies*** $e^{\frac{c}{2}} < \frac{m}{2}$

For $m = 1$ ***and*** 2. ***However, we have*** $e^{\frac{c}{2}} < \frac{1}{2}$ ***and*** $e^{\frac{c}{2}} < 1$

which is impossible if $c \in (0,1)$

Solution 4 by Soumava Chakraborty – Kolkata – India

Let $(x+1)\sqrt{e^x}+m=z(m\in N)$***. We have,*** $(x+3)\sqrt{e^x}dx=2dt$

$$\therefore I(m)=2\int\frac{dt}{t}\ \text{... from limits } (1+m) \text{ to } \left(2\sqrt{3}+m\right)$$

$$=2\ln|t|\ \text{... from limits } (1+m) \text{ to } \left(2\sqrt{e}+m\right)$$

$$=2\left[\ln\left(2\sqrt{e}+m\right)-\ln(1+m)\right]=2\ln\left[\frac{2\sqrt{e}+m}{1+m}\right]$$

$$\therefore I(m)<1\Leftrightarrow\ln\left[\frac{2\sqrt{e}+m}{1+m}\right]<\frac{1}{2}=\ln\sqrt{e}$$

$$\Leftrightarrow\frac{2\sqrt{e}+m}{1+m}<\sqrt{e}\Leftrightarrow 2\sqrt{e}+m<\sqrt{e}+m\sqrt{e}$$

$$\Leftrightarrow\sqrt{e}<m\left(\sqrt{e}-1\right)\Leftrightarrow\frac{\sqrt{e}}{\sqrt{e}-1}<m\Leftrightarrow 2\cdot 54\ \textbf{(approx.)}<m\Leftrightarrow m\geq 3$$

$$\therefore I(m)<1 \text{ is true if } m\geq 3 \text{ and } I(m)>1 \text{ for } m=1,2$$

Solution 5 by Yen Tung Chung – Vietnam

$$\because\frac{d}{dx}\left[2\ln\left((x+1)e^{\frac{x}{2}}+m\right)\right]=2\left(\frac{e^{\frac{x}{2}}+\frac{1}{2}(x+1)e^{\frac{x}{2}}}{(x+1)e^{\frac{x}{2}}+m}\right)=\frac{(x+3)\sqrt{e^x}}{(x+1)\sqrt{e^x}+m}$$

$$\therefore I_m=\int_0^1\frac{\sqrt{e^x}(x+3)}{(x+1)\sqrt{e^x}+m}dx=2\int_0^1\frac{\frac{1}{2}(x+3)e^{\frac{x}{2}}}{(x+1)e^{\frac{x}{2}}+m}dx=2\int_0^1\frac{d\left((x+1)e^{\frac{x}{2}}+m\right)}{(x+1)e^{\frac{x}{2}}+m}$$

$$=2\left(\ln\left|(x+1)e^{\frac{x}{2}}+m\right|\right)\Big|_0^1=2\left(\ln\left(2\sqrt{e}+m\right)-\ln(1+m)\right)=2\ln\left(\frac{2\sqrt{e}+m}{1+m}\right)$$

$$I_m<1\Rightarrow 2\ln\left(\frac{2\sqrt{e}+m}{1+m}\right)<1\Rightarrow\frac{2\sqrt{e}+m}{1+m}<\sqrt{e}\Rightarrow m>\frac{\sqrt{e}}{\sqrt{e}-1}\approx 2.54$$

16. If $\Omega = \int \frac{(x^2+1)^2}{\left(x^{\frac{113}{25}}+\frac{11}{3}x^{\frac{63}{25}}+11x^{\frac{13}{25}}\right)^5}dx = u\left(x^a+ex^b+fx^c\right)^k+Q$

Find: $\vartheta = -\frac{5}{k}(a+b+c)$

Proposed by Nishant Kumar-Jamshedpur-India

Solution 1 by proposer

$$\int \frac{(x^2+1)^2}{\left(x^{\frac{113}{25}}+\frac{11}{3}x^{\frac{63}{25}}+11\,x^{\frac{13}{25}}\right)^5}dx = \int \frac{x^4+2x^2+1}{\left(x^{\frac{113}{25}}+\frac{11}{3}x^{\frac{63}{25}}+11\,x^{\frac{13}{25}}\right)^5}dx$$

Multiplying $x^{\left(-\frac{3}{5}\right)}$ ***to numerator and denominator***

$$= \int \frac{x^{\frac{17}{5}}+2x^{\frac{7}{5}}+x^{-\frac{3}{5}}}{\left(x^{\frac{22}{5}}+\frac{11}{3}x^{\frac{12}{5}}+11\,x^{\frac{2}{5}}\right)^5}dx.\ \textit{Putting}\quad x^{\frac{22}{5}}+\frac{11}{3}x^{\frac{12}{5}}+11\,x^{\frac{2}{5}}=t$$

$$\left(\frac{22}{5}x^{\frac{17}{5}}+\frac{44}{5}x^{\frac{7}{5}}+\frac{22}{5}x^{-\frac{3}{5}}\right)dx = dt,\ \left(x^{\frac{17}{5}}+2x^{\frac{7}{5}}+x^{-\frac{3}{5}}\right)dx = \frac{5}{22}dt$$

$$\frac{5}{22}\int\frac{dt}{t^5} = -\frac{5}{88t^4} + \textit{constant}\ (Q)$$

$$-\frac{5}{88t^4}+Q = -\frac{5}{88}\left(x^{\frac{22}{5}}+\frac{11}{3}x^{\frac{12}{5}}+11\,x^{\frac{2}{5}}\right)^{-4}$$

Therefore $\Omega = -\frac{5}{88}\left(x^{\frac{22}{5}}+\frac{11}{3}x^{\frac{12}{5}}+11\,x^{\frac{2}{5}}\right)^{-4}+Q$

Comparing constants we get $a=\frac{22}{5}$, $b=\frac{12}{5}$, $c=\frac{2}{5}$, $k=-4$

$$\vartheta = -\frac{5}{k}(a+b+c) = \frac{5}{4}\left(\frac{22}{5}+\frac{12}{5}+\frac{2}{5}\right);\ \vartheta = 9.$$

Solution 2 by Yen Tung Chung-Taichung-Taiwan

$$\int \frac{(x^2+1)^2}{\left(x^{\frac{113}{25}}+\frac{11}{3}x^{\frac{63}{25}}+11\,x^{\frac{13}{25}}\right)^5}dx = \int \frac{x^4+2x^2+1}{x^{\frac{3}{5}}\left(x^{\frac{22}{5}}+\frac{11}{3}x^{\frac{12}{5}}+11x^{\frac{2}{5}}\right)^5}dx$$

$$= \int \frac{1}{\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^5} \cdot \left(x^{\frac{17}{5}} + 2x^{\frac{7}{5}} + x^{-\frac{3}{5}}\right)dx$$

$$\left(\textit{let} = x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}} \Rightarrow du = \frac{22}{5}\left(x^{\frac{17}{5}} + 2x^{\frac{7}{5}} + x^{-\frac{3}{5}}\right)dx\right)$$

$$= \int \frac{1}{u^5} \cdot \frac{5}{22}du = -\frac{5}{22} \cdot \frac{1}{4}u^{-4} + Q = -\frac{5}{88}\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^{-4} + Q$$

$$\therefore \vartheta = -\frac{5}{(-4)}\left(\frac{22}{5} + \frac{12}{5} + \frac{2}{5}\right) = 9$$

Solution 3 by Ravi Prakash - New Delhi – India

$$\textit{Write}\left(x^{\frac{113}{25}} + \frac{11}{3}x^{\frac{63}{25}} + 11x^{\frac{13}{25}}\right)^5 = \left(x^{\frac{2}{25}}\right)^5\left[x^{\frac{110}{25}} + \frac{11}{3}x^{\frac{60}{25}} + 11x^{\frac{10}{25}}\right]^5 =$$

$$= x^{\frac{3}{5}}\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^5. \textit{ Now,}$$

$$\Omega = \int \frac{x^{-\frac{3}{5}}(x^4+2x^2+1)}{\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^5}dx = \int \frac{x^{\frac{17}{5}} + 2x^{\frac{7}{5}} + x^{-\frac{3}{5}}}{\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^5}dx$$

$$\textit{Put } x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}} = t$$

$$dt = \left(\frac{22}{5}x^{\frac{17}{5}} + \frac{11}{3}x\frac{12}{5}x^{\frac{7}{5}} + \frac{22}{5}x^{-\frac{3}{5}}\right)dx = \frac{22}{5}\left(x^{\frac{17}{5}} + 2x^{\frac{7}{5}} + x^{-\frac{3}{5}}\right)dx$$

$$\therefore \Omega = \frac{5}{22}\int \frac{dt}{t^5} = \frac{5}{22}\int t^{-5}\,dt = -\frac{5}{88}t^{-4} + c =$$

$$= -\frac{5}{88}\left(x^{\frac{22}{5}} + \frac{11}{3}x^{\frac{12}{5}} + 11x^{\frac{2}{5}}\right)^{-4} + c \quad \therefore a = \frac{22}{5}, b = \frac{12}{5}, c = \frac{2}{5}, k = -4$$

$$\textit{Thus } \vartheta = -\frac{5}{k}(a+b+c) = \frac{-5}{-4}\left(\frac{22}{5} + \frac{12}{5} + \frac{2}{5}\right) = \frac{5}{4}\left(\frac{36}{5}\right) = 9$$

17. Find:

$$\Omega = \int_0^\infty \frac{x^2+2}{x^6+1}dx$$

Proposed by Togrul Ehmedov-Baku-Azerbaidian

Solution by Togrul Ehmedov-Baku-Azerbaidian

$$\Omega = \int_0^\infty \frac{x^2+2}{x^6+1}dx = \underbrace{\int_0^\infty \frac{x^2}{x^6+1}dx}_{I_1} + \underbrace{2\int_0^\infty \frac{1}{x^6+1}dx}_{I_2}$$

$$I_1 = \int_0^\infty \frac{x^2}{x^6+1}dx;\ x^6 = y \Rightarrow dx = \frac{1}{6}y^{-\frac{5}{6}}dy$$

$$I_1 = \int_0^\infty \frac{x^2}{x^6+1}dx = \frac{1}{6}\int_0^\infty \frac{y^{-\frac{1}{2}}}{y+1}dy$$

$$\frac{1}{y+1} = 1-t \Rightarrow dy = \frac{1}{(1-t)^2}dt$$

$$I_1 = \frac{1}{6}\int_0^\infty t^{-\frac{1}{2}}(1-t)^{-\frac{1}{2}}dt = \frac{1}{6}\frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}\right)} = \frac{1}{6}\Gamma^2\left(\frac{1}{2}\right) = \frac{\pi}{6}$$

$$I_2 = 2\int_0^\infty \frac{1}{x^6+1}dx = 2\left[\frac{\pi}{6\sin\left(\frac{\pi}{6}\right)}\right] = \frac{2\pi}{3}$$

$$\Omega = \int_0^\infty \frac{x^2+2}{x^6+1}dx = I_1 + I_2 = \frac{\pi}{6} + \frac{2\pi}{3} = \frac{5\pi}{6}$$

18. **Let $f:[a,b]\to(0,\infty)$ be a continuos function.**

If $m=\min f(x), M=\max f(x)$ then:

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{1}{f(x)}dx\right)\le\frac{\left((m+M)(b-a)\right)^2}{4mM}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Tran Van Hien-Vietnam

$f[a,b]\to(0,R)$ f ***continuous*** $\max f(x)=M;\min f(x)=m\forall x\in[a;b]$

Prove that

$$\int_a^b f(x)\,dx\int_a^b\frac{dx}{f(x)}\le\frac{(M+m)^2(b-a)^2}{4Mm}$$

$$f(x)=\sqrt{Mm}g(x)\Rightarrow m\le f(x)\le M\Rightarrow 1\le g(x)\le\sqrt{\frac{M}{m}}$$

$$\left(\int_a^b f(x)\,dx\right)\left(\int_a^b\frac{dx}{f(x)}\right)=\left(\int_a^b g(x)\sqrt{Mm}dx\right)\left(\int_a^b\frac{dx}{\sqrt{Mm}g(x)}\right)=$$

$$=\left(\int_a^b g(x)dx\right)\left(\int_a^b\frac{dx}{g(x)}\right)$$

We have $ab\le\frac{(a+b)^2}{4}$ ***and*** $\int_a^b f(x)\,dx=f(c)(b-a)$ ***With*** $c\in[a,b]$

$$\Rightarrow\left(\int_a^b g(x)\,dx\right)\left(\int_a^b\frac{dx}{g(x)}\right)\le\frac{\int_a^b\left(g(x)+\frac{1}{g(x)}\right)^2}{4}=\frac{(b-a)^2\left(\frac{1}{g^2(c)}+g^2(c)\right)}{4}$$

Put $h(t)=t^2+\frac{1}{t^2};\frac{t^2-1}{t^2}\Rightarrow h(t)\le h\left(\sqrt{\frac{M}{m}}\right)$

$$\Rightarrow g^2(x) + \frac{1}{g^2(x)} \le \frac{M}{m} + \frac{m}{M}$$

$$\Rightarrow \int_a^b \left(g(x) + \frac{1}{g(x)}\right)^2 dx = \int_a^b \left(\frac{1}{g^2(x)} + 2 + g^2(x)\right) dx \le$$

$$\le \frac{\left(\frac{m}{M} + 2\frac{M}{m}\right)(b-a)^2}{4} \le \frac{(b-a)^2(M+m)^2}{4Mm}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$m \le f(x) \le M, a \le x \le b \Rightarrow (m - f(x))(M - f(x)) \le 0$$

$$\Rightarrow mM - (m+M)f(x) + f(x)^2 \le 0 \Rightarrow \frac{mM}{f(x)} - (m+M) + f(x) \le 0$$

$$[\because f(x) > 0] \Rightarrow \frac{mM}{f(x)} + f(x) \le (m+M)$$

$$\Rightarrow mM \int_a^b \frac{dx}{f(x)} + \int_a^b f(x) dx \le (m+M)(b-a)$$

Let $J = mM \int_a^b \frac{dx}{f(x)}, H = \int_a^b f(x)\, dx$

$$J + H \le (m+M)(b-a) \Rightarrow J^2 + JH \le (m+M)(b-a)J$$

$$\Rightarrow JH \le \frac{1}{4}(m+M)^2(b-a)^2 - \left[\frac{1}{2}(m+M)(b-a) - J\right]^2$$

$$\Rightarrow JH \le \frac{1}{4}(m+M)^2(b-a)^2$$

$$\Rightarrow mM \int_a^b f(x)\, dx \int_a^b \frac{1}{f(x)} dx \le \frac{(m+M)^2}{4mM}(b-a)^2$$

Solution 3 by Soumitra Mandal-Chandar Nagore-India

$$m = \min f(x) \le f(x) \le \max f(x) = M$$

$$\Rightarrow (f(x)-m)(f(x)-M)\le 0 \Rightarrow f^2(x)+Mm\le (M+m)f(x)$$

$\Rightarrow f(x)+\frac{Mm}{f(x)}\le m+M$, *since,* $f(x)\neq 0$, f *is continuous hence* f *is* $R-$

Integrable

$$\Rightarrow \int_a^b f(x)\,dx + Mm\int_a^b \frac{dx}{f(x)} \le (M+m)\int_a^b dx = (M+m)(b-a)$$

$$\Rightarrow (b-a)(M+m) \overset{AM\ge GM}{\ge} 2\sqrt{Mm\left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{dx}{f(x)}\right)}$$

$$\therefore \left(\int_a^b f(x)\,dx\right)\left(\int_a^b \frac{dx}{f(x)}\right) \le \frac{\left((b-a)(M+m)\right)^2}{4Mm}$$

19. If $0<a<b$ then:

$$\frac{2}{\pi}\ln\left(\frac{b}{a}\right)b-a<\frac{\pi}{2}\int_a^b \frac{dx}{\arctan x}<\frac{\pi}{2}\ln\left(\frac{b}{a}\right)+b-a$$

Proposed by Daniel Sitaru – Romania

Solution by Francis Fregeau-Quebec-Canada:

Let $f(x)=\frac{\pi}{\frac{4}{\pi x}+2}=\frac{x\pi^2}{4+2\pi x}$ *and* $h(x)=\frac{\pi}{\frac{\pi}{x}+2}=\frac{\pi x}{2x+\pi}$

For $x>0$, *both* $f(x)$ *and* $h(x)$ *are strictly increasing and*

continuos on the interval $\left(0,\frac{\pi}{2}\right)$

Let $\phi(x)=f(x)-\arctan(x)$; $\psi(x)=h(x)-\arctan(x)$

$$f(0)-\arctan(0)\to 0;\ h(0)-\arctan(0)\to 0$$

$$f(\infty)-\arctan(\infty)\to 0;\ h(\infty)-\arctan(\infty)\to 0$$

$$\phi'(x)=\frac{\pi^2}{(\pi x+2)^2}-\frac{1}{(x^2+1)};\ \phi'(x)=0\Rightarrow x=\frac{x^2-4}{4\pi}=\delta$$

$$\phi(\delta)>0;\phi'(\delta-\epsilon)>0;\phi'(\delta+\epsilon)<0 \text{ for } \epsilon>0$$

Hence $\phi(x)$ can be considered as bounded monotone sequence on the intervals $(0,\delta]$ and (δ,∞), $0<\phi(x)<\delta$ for $x>0$

$\Rightarrow \arctan(x)<f(x)$ **for** $x>0$. ***Similarly:***

$$\psi'(x)=\frac{\pi^2}{(2x+\pi)^2}-\frac{1}{(x^2+1)};\ \psi'(x)=0\Rightarrow x=\frac{4\pi}{\pi^2-4}=\sigma$$

$$\psi(\sigma)<0;\ \psi'(\sigma-\epsilon)<0;\ \psi'(\sigma+\epsilon)>0$$

Hence $\psi(x)$ can be considered as bounded monotone sequence on the intervals $(0,\sigma]$ and (σ,∞)

$$-\sigma<\psi(x)<0 \text{ for } x>0\Rightarrow h(x)<\arctan(x) \text{ for } x>0$$

Let $x>0;\ h(x)<\arctan(x)<f(x)$

$$\Rightarrow \frac{\pi}{\frac{\pi}{x}+2}<\arctan(x)<\frac{\pi}{\frac{4}{\pi x}+2}\Rightarrow \frac{2}{\pi}\frac{1}{x}+1<\frac{\pi}{2}\frac{1}{\arctan(x)}<\frac{\pi}{2}\frac{1}{x}+1$$

Let $0<a<b$

$$\int_a^b\left(\frac{2}{\pi}\frac{1}{x}+1\right)dx<\frac{\pi}{2}\int_a^b\frac{1}{\arctan(x)}dx<\int_a^b\left(\frac{\pi}{2}\frac{1}{x}+1\right)dx$$

$$\Rightarrow \frac{2}{\pi}\ln\left(\frac{b}{a}\right)+b-a<\frac{\pi}{2}\int_a^b\frac{1}{\arctan(x)}dx<\frac{\pi}{2}\ln\left(\frac{b}{a}\right)+b-a$$

20. If $[a,b]\subset\left(0,\frac{\pi}{2}\right)$ then:

$$\int_a^b \sin x\,dx > \sqrt{b^2+1}-\sqrt{a^2+1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\cos a-\cos b > \sqrt{b^2+1}-\sqrt{a^2+1}$$

$$\Leftrightarrow \cos a+\sqrt{a^2+1} > \cos b+\sqrt{b^2+1}, \forall\, a,b\in\left(0,\frac{\pi}{2}\right) \quad \textbf{(1)}$$

$$\text{Let } f(x)=\cos x+\sqrt{x^2+1}, \forall x\in\left(0,\frac{\pi}{2}\right);\ f'(x)=-\sin x+\frac{x}{\sqrt{x^2+1}}$$

$$\text{Now, } \forall x\in\left(0,\frac{\pi}{2}\right), x<\tan x \Rightarrow x^2+1<1+\tan^2 x$$

$$\Rightarrow x^2+1<\sec^2 x \Rightarrow \frac{1}{x^2+1}>\cos^2 x \Rightarrow \frac{1}{x^2+1}>1-\sin^2 x$$

$$\Rightarrow \sin^2 x>1-\frac{1}{x^2+1} \Rightarrow \sin^2 x>\frac{x^2}{x^2+1} \Rightarrow \frac{x}{\sqrt{x^2+1}}-\sin x<0$$

$$\Rightarrow f'(x)<0 \Rightarrow f(x)=\cos x+\sqrt{x^2+1} \text{ is decreasing on } \left(0,\frac{\pi}{2}\right)$$

$$b>a; f(b)<f(a) \Rightarrow \cos a+\sqrt{a^2+1}>\cos b+\sqrt{b^2+1}$$

$\Rightarrow$ ***(1) is proved.***

Solution 2 by Kunihiko Chikaya-Tokyo-Japan

$$\int_a^b \sin x\,dx > \sqrt{b^2+1}-\sqrt{a^2+1};\ a,b\in\left(0,\frac{\pi}{2}\right);\ \tan x>x \quad \left(0<x<\frac{\pi}{2}\right)$$

$$\int_a^b \sin x\,dx > \int_a^b \frac{x}{\sqrt{x^2+1}}dx$$

$$= \int_a^b \frac{1}{\sqrt{x^2+1}}(x^2+1)'dx = \left[\sqrt{x^2+1}\right]_a^b = \sqrt{b^2+1} - \sqrt{a^2+1}$$

$$\Leftrightarrow \left(\frac{\cos x}{\sin x}\right)^2 < \frac{1}{x^2} \Leftrightarrow \frac{1}{\sin^2 x} < 1 + \frac{1}{x^2} \Leftrightarrow \sin x > \frac{x}{\sqrt{x^2+1}} \quad \left(0 < x < \frac{\pi}{2}\right)$$

21. If $f,g:[a,b]\to(0,\infty)$ integrable, such that $f(x)+g(x)\le 8$ then:

$$\int_a^b \frac{f(x)\sqrt{g(x)}+g(x)\sqrt{f(x)}}{f(x)-\sqrt{f(x)g(x)}+g(x)}dx \le 4(b-a)$$

Proposed by Daniel Sitaru – Romania

Solution by Anas Adlany - El Jadida – Morroco

We have for all $t,z>0$; $\frac{t\sqrt{z}+z\sqrt{t}}{t+z-\sqrt{zt}} = \sqrt{zt}\cdot\frac{\sqrt{z}+\sqrt{t}}{\underbrace{t+z-\sqrt{zt}}_{\ge\sqrt{zt}}} \le \sqrt{z}+\sqrt{t} \le \sqrt{2(z+t)}$

(because $x+y\le\sqrt{2}\sqrt{x^2+y^2}$*). So put* $z=f(x)$ *and* $t=g(x)$ *to find:*

$$\frac{f(x)\sqrt{g(x)}+g(x)\sqrt{f(x)}}{f(x)-\sqrt{f(x)g(x)}+g(x)} \le \sqrt{2}\sqrt{\left(f(x)+g(x)\right)} \le 4 \Rightarrow$$

$$\Rightarrow \int_a^b \frac{f(x)\sqrt{g(x)}+g(x)\sqrt{f(x)}}{f(x)-\sqrt{f(x)g(x)}+g(x)}dx \le 4(b-a)$$

22. If $f:[0,1]\to(0,\infty)$, f derivable, f' continuous,

$f'(x)=f'(1-x), \forall x\in[0,1]$ then:

$$\int_0^1 f(x)\,dx \ge \sqrt{f(0)\cdot f(1)}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Safal Das Biswas – Chinsurah – India

(i) $f:[0,1]\to(0,\infty)$; ***(ii)*** f ***is derivable; (iii)*** f' ***is continuous***

(iv) $f'(x)=f'(1-x)\forall x\in[0,1]$. ***Let,*** $\int_0^1 f(x)\,dx$,

then applying the property that $\int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx$

on I ***we have*** $I=\int_0^1 f(1-x)\,dx$. ***Thus we have***

$2I=\int_0^1 f(x)\,dx+\int_0^1 f(1-x)\,dx$. ***Which finally results***

$$I=\frac{1}{2}\int_0^1\{f(x)+f(1-x)\}\,dx.$$

Now let assume that $f(x)+f(1-x)=g(x)$ ***so we have that***

$I=\frac{1}{2}\int_0^1 g(x)\,dx$. ***Now,*** $g'(x)=f'(x)-f'(1-x)$.

Now, $f'(x)=f'(1-x)\ \forall x\in[0,1]$ ***is given hence***

$g'(x)=f'(x)-f'(1-x)=0\ x\in[0,1]$.

Hence $g(x)$ ***must be a constant function*** $\forall\ x\in[0,1]$. ***As***

$I=\frac{1}{2}\int_0^1 g(x)\,dx$, ***then***

$$I=\frac{g(x)}{2}\int_0^1 dx=\frac{g(x)}{2}=\frac{f(x)+f(1-x)}{2}\ \forall x\in[0,1]$$

Then $I=\frac{f(x)+f(1-x)}{2}=\frac{f(0)+f(1)}{2}$

Thus by applying $A.M.\ge G.M.$ ***we have,*** $\frac{f(0)+f(1)}{2}\ge\sqrt{f(0)\cdot f(1)}$.

Hence we have, $I\ge\sqrt{f(0)\cdot f(1)}$. ***So,***

$$\int_0^1 f(x)\,dx\ge\sqrt{f(0)\cdot f(1)}$$

Solution 2 by Soumitra Moukherjee - Chandar Nagore – India

Let

$$I=\int_0^1 f(x)\,dx=[xf(x)]_{x=0}^{x=1}-\int_0^1 xf'(x)dx=f(1)-\int_0^1 xf'(1-x)\,dx$$

$$=f(1)+\int_0^1 xf'(1-x)d(1-x)=f(1)+\int_0^1 f'(1-x)d(1-x)-\int_0^1(1-x)f'(1-x)d(1-x)$$

Let $z=1-x$ **then,** $dx=-dz$ **and when** $x=0$**, then** $z=1$: $x=1$, $z=0$

$$=f(1)+\int_1^0 f'(z)\,dz-\int_1^0 zf'(z)dz=f(1)+f(0)+\int_1^0\left[\frac{d}{dz}(z)\int f'(z)dz\right]dz$$

$$=f(1)+f(0)-\int_0^1 f(z)\,dz=f(1)+f(0)-I.$$

So, $I=\frac{f(1)+f(0)}{2}\ge\sqrt{f(0)\cdot f(1)}$**.**

Proved.

23. If $0\le x\le a$, $0\le y\le b$, $c>0$

$$\Omega_1=\int_0^c\left(\int_0^c\sqrt{x^2+y^2-2ax+a^2}\,dx\right)dy,$$

$$\Omega_2=\int_0^c\left(\int_0^c\sqrt{x^2+y^2-2by+b^2}dx\right)dy$$

then $\Omega_1+\Omega_2\le(a+b)c^2$

Proposed by Daniel Sitaru – Romania

Solution by Subhajit Chattopadhyay-Bolpur-India

$$\Omega_1+\Omega_2=\int_0^c\int_0^c\left(\sqrt{x^2+y^2-2ax+a^2}+\sqrt{x^2+y^2-2by+b^2}\right)dx\,dy$$

Now, $\sqrt{x^2+y^2-2ax+a^2}+\sqrt{x^2+y^2-2by+b^2}=$

$$=\sqrt{(a-x)^2+y^2}+\sqrt{(b-y)^2+x^2}$$

Again, $\sqrt{(a-x)^2+y^2}\le a-x+y;\ [0\le x\le a]$

and $\sqrt{(b-y)^2+x^2}\le b-y+x$

Hence, $\sqrt{(a-x)^2+y^2}+\sqrt{(b-y)^2+x^2}\le a-x+y+b-y+x=a+b$

$$\Omega_1+\Omega_2\le\int_0^c\int_0^c(a+b)\,dx\,dy=(a+b)c^2$$

24. If $0\le x\le 3, 0\le y\le 4, a>0$

$$\Omega_1=\int_0^a\left(\int_0^a\sqrt{x^2+y^2-6x+9}\,dx\right)dy$$

$$\Omega_2=\int_0^a\left(\int_0^a\sqrt{x^2+y^2-8y+16}\,dx\right)dy$$

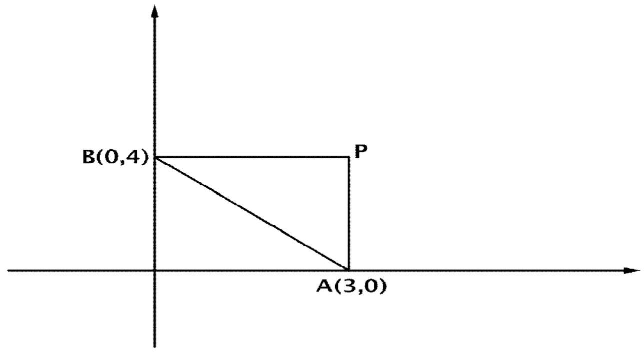
then: $\Omega_1+\Omega_2>5a^2$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

$$PA+PB=\sqrt{x^2+y^2-6x+9}+\sqrt{x^2+y^2-8y+16}$$

$$=\sqrt{(x-3)^2+y^2}+\sqrt{(x^2)+(y-4)^2}\ge\sqrt{3^2+y^2}=5$$



$$\Omega_1+\Omega_2=\int_0^a\left(\int_0^a\left(\sqrt{x^2+y^2-6x+9}+\sqrt{x^2+y^2-8y+16}\right)dx\right)dy\geq$$

$$\geq\int_0^a\left(\int_0^a 5dx\right)dy=5a^2$$

Solution 2 by SK Rejuan – West Bengal – India

$$0\leq x\leq 3, 0\leq y\leq 9, a>0$$

$$\Omega_1=\int_0^a\left(\int_0^a\sqrt{x^2+y^2-6x+9}\,dx\right)dy=\int_0^a\left(\int_0^a\sqrt{(3-x)^2+y^2}\,dx\right)dy\geq$$

$$\geq\int_0^a\left\{\int_0^a\frac{1}{\sqrt{2}}(3-x+y)dx\right\}dy=\frac{1}{\sqrt{2}}\int_0^a\left[\int_0^a\{(3+y)-x\}dx\right]dy=$$

$$=\frac{1}{\sqrt{2}}\int_0^a\left\{\left[(3+y)x-\frac{x^2}{2}\right]_0^a\right\}dy=\frac{1}{\sqrt{2}}\int_0^a\left\{(3+y)a-\frac{a^2}{2}\right\}dy=$$

$$=\frac{1}{\sqrt{2}}\left[3a+\frac{y^2}{2}a-\frac{a^2}{2}y\right]_0^a=\frac{1}{\sqrt{2}}\left(3a^2+\frac{a^3}{2}-\frac{a^2}{2}\right)=\frac{3}{\sqrt{2}}a^2$$

$$\Rightarrow\Omega_1\geq\frac{3}{\sqrt{2}}a^2\qquad (1)$$

$$\Omega_2 = \int_0^a \left(\int_0^a \sqrt{x^2 + y^2 - 8y + 16}\, dx \right) dy = \int_0^a \left(\int_0^a \sqrt{x^2 + (4-y)^2} dx \right) dy =$$

$$\geq \int_0^a \left\{ \int_0^a \frac{1}{\sqrt{2}} (x + 4 - y) dx \right\} dy$$

$$\Rightarrow \Omega_2 \geq \frac{1}{\sqrt{2}} \int_0^a \left[\int_0^a \{x + (4-y)\}\, dx \right] dy = \frac{1}{\sqrt{2}} \int_0^a \left\{ \left[\frac{x^2}{2} + (4-y)x \right]_0^a \right\} dy =$$

$$= \frac{1}{\sqrt{2}} \int_0^a \left\{ \frac{a^2}{2} + (4-y)a \right\} dy = \frac{1}{\sqrt{2}} \left[\frac{a^2}{2} y + (4ay) - \frac{y^2}{2} a \right]_0^a =$$

$$= \frac{1}{\sqrt{2}} \left(\frac{a^2}{2} + 4a^2 - \frac{a^3}{2} \right) = 2\sqrt{2}a^2$$

$$\Rightarrow \Omega_2 \geq 2\sqrt{2}a^2 \quad \textit{(2)}$$

$$\Omega_1 + \Omega_2 \geq \frac{3}{\sqrt{2}} a^2 + 2\sqrt{2}a^2 = \left(\frac{3+9}{\sqrt{2}} \right) a^2 \Rightarrow \Omega_1 + \Omega_2 \geq \frac{7}{\sqrt{2}} a^2 = \left(\frac{7}{2} \right) \sqrt{2} a^2$$

$$\Rightarrow \Omega_1 + \Omega_2 \geq 3 \cdot 5\sqrt{2}a^2 > 5a^2 \Rightarrow \Omega_1 + \Omega_2 > 5a^2$$

25. $\int_0^1 \int_0^1 (x^2 + 34y^2 - 10xy - 6y + 2)^2 dx\, dy \geq 1$

Proposed by Sameer Shihab-Riyadh-Saudi Arabia

Solution by Ravi Prakash-New Delhi-India

$$x^2 + 34y^2 - 10xy - 6y + 2 =$$

$$= (x^2 + 25y^2 - 10xy) + (9y^2 - 6y + 1) + 1 =$$

$$= (x - 5y)^2 + (3y - 1)^2 + 1 \geq 1$$

$$\int_0^1 \int_0^1 (x^2 + 34y^2 - 10xy - 6y + 2)^2 dx\, dy \geq 1$$

26. If $n \in \mathbb{N}, n \geq 2, n$ – fixed, $f: [1, n] \to (0, \infty), f$ – integrable, $i \in \overline{1, n-1}$

$$\Omega = \int_1^n f(x)\,dx,\ 0 \leq \alpha \leq \min\left(\int_i^{i+1} f(x)\,dx\right) \leq \max\left(\int_i^{i+1} f(x)\,dx\right) \leq \beta$$

then:

$$(n-1)\alpha\beta + \sum_{i=1}^{n-1}\left(\int_i^{i+1} f(x)\,dx\right)^2 \leq (\alpha+\beta)\Omega$$

Proposed by Daniel Sitaru - Romania

Solution by Soumitra Moukherjee-Chandar Nagore-India

We know,

$$\min\left(\int_i^{i+1} f(x)\,dx\right) \leq \int_i^{i+1} f(x)\,dx \leq \max\left(\int_i^{i+1} f(x)\,dx\right)$$

$$\alpha \leq \min\left(\int_i^{i+1} f(x)\,dx\right) \leq \int_i^{i+1} f(x)\,dx \leq \max\left(\int_i^{i+1} f(x)\,dx\right) \leq \beta$$

$$\left(\int_i^{i+1} f(x)\,dx - \alpha\right)\left(\int_i^{i+1} f(x)dx - \beta\right) \leq 0$$

$$\Rightarrow \alpha\beta + \left(\int_i^{i+1} f(x)\,dx\right)^2 \leq (\alpha+\beta)\int_i^{i+1} f(x)\,dx$$

$$\Rightarrow \sum_{i=1}^{n-1}\alpha\beta + \sum_{i=1}^{n-1}\left(\int_i^{i+1} f(x)\,dx\right)^2 \leq (\alpha+\beta)\sum_{i=1}^{n-1}\int_i^{i+1} f(x)\,dx$$

$$(n-1)\alpha\beta+\sum_{i=1}^{n-1}\left(\int_i^{i+1} f(x)\,dx\right)^2\le(\alpha+\beta)\Omega$$

27. If $f:[a,b]\to(0,\infty)$, $a<b$, continuous,

$m=\min f(x)$, $M=\max f(x)$, $n\in\mathbb{N}^*$ then:

$$\left(\frac{m}{M}\right)^{\frac{n(n+1)}{2}}(b-a)^{2n}\le\prod_{k=1}^{n}\left(\int_a^b f^k(x)\,dx\right)\left(\int_a^b\frac{1}{f^k(x)}dx\right)\le\left(\frac{M}{m}\right)^{\frac{n(n+1)}{2}}(b-a)^{2n}$$

Proposed by Daniel Sitaru – Romania

Solution by Anas Adlany-El Jadida-Morroco

We have for all $x\in[a,b];\ 0<m\le f(x)\le M\Rightarrow\begin{cases}m^k\le f^k(x)\le M^k\\ \frac{1}{M^k}\le\frac{1}{f^k(x)}\le\frac{1}{m^k}\end{cases}$

$$\Rightarrow\begin{cases}m^k(b-a)\le\int_a^b f^k(x)\le M^k(b-a)\\ \frac{1}{M^k}(b-a)\le\int_a^b\frac{1}{f^k(x)}\le\frac{1}{m^k}(b-a)\end{cases}\Rightarrow\left(\frac{m}{M}\right)^k(b-a)^2\le$$

$$\le\left(\int_a^b f^k(x)\right)\left(\int_a^b\frac{1}{f^k(x)}\right)\le\left(\frac{M}{m}\right)^k(b-a)^2$$

$$\Rightarrow\prod_{k=1}^{n}\left(\left(\frac{m}{M}\right)(b-a)^2\right)\le\prod_{k=1}^{n}\left(\int_a^b f^k(x)\right)\left(\int_a^b\frac{1}{f^k(x)}\right)\le\prod_{k=1}^{n}\left(\left(\frac{M}{m}\right)^k(b-a)^2\right)$$

$$\Rightarrow\left(\left(\frac{m}{M}\right)^{\frac{n(n+1)}{2}}(b-a)^{2n}\right)\le\prod_{k=1}^{n}\left(\int_a^b f^k(x)\right)\left(\int_a^b\frac{1}{f^k(x)}\right)\le\left(\left(\frac{M}{m}\right)^{\frac{n(n+1)}{2}}(b-a)^{2n}\right)$$

since $\sum_{k=1}^{n}k=\frac{n(n+1)}{2}$.

28. If $e\le a\le c\le b\le e^2$ then:

$$(b-a)\int_a^c \frac{x}{\log x}dx\le (c-a)\int_a^b \frac{x}{\log x}dx$$

Proposed by Daniel Sitaru – Romania

Solution Soumitra Moukherjee – Chandar Nagore – India

Let $f(t)=(t-a)\int_a^b \frac{x}{\log x}dx+(b-a)\int_t^a \frac{x}{\log x}$ *for all* $t\in[e,e^2]$

$$f'(t)=\int_a^b \frac{x}{\log x}dx-(b-a)\frac{t}{\log t}$$

$$\int_a^b \frac{x}{\log x}dx=\left[\frac{x^2}{\log x}\right]_a^b+\int_a^b \frac{x(1-\log x)}{(\log x)^2}dx$$

where: $\int_a^b \frac{x(1-\log x)}{(\log x)^x}dx\ge 0$. *Let* $\varphi(t)=\frac{t^2}{\log t}$ *for all* $t\in[e,e^2]$,

$\varphi'(t)=\frac{t(2\log t-1)}{(\log t)^2}\ge 0$ *for all* $t\in[e,e^2]$

so, $\varphi(t)$ *is continuous on* $[e,e^2]$, $\varphi'(t)\ge 0$ *for all* $t\in[e,e^2]$

so, for $a\le t\le b$, $\varphi(a)\le\varphi(b)$

$\frac{b^2}{\log b}-\frac{a^2}{\log a}\ge 0$ *where* $a,b\in[e,e^2]$, *so,* $\int_a^b \frac{x}{\log x}dx\ge(b-a)\frac{t}{\log t}$

where $t\in[a,b]$; $f'(t)\ge 0$ *for all* $t\in[a,b]\subset[e,e^2]$

so, $f(t)$ *is increasing and for* $c\in[a,b]\subset[e,e^2]$, $f(c)\ge 0$

$$(c-a)\int_a^b \frac{x}{\log x}dx\ge(b-a)\int_a^c \frac{x}{\log x}dx$$

(proved)

29. If $0 < a < b < 1$ then:

$$\frac{1}{b-a}\int_a^b\left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right)dx \ge \left(1+\frac{4}{\pi}\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

Let $0 < a, b < 1$. ***For*** $a \le x \le b$,

$$\frac{1}{\sin^{-1}x}+\frac{1}{\cos^{-1}x} \ge \frac{4}{\sin^{-1}x+\cos^{-1}x} = \frac{8}{\pi}$$

and $\sqrt{\sin^{-1}x\cos^{-1}x} \le \frac{1}{2}(\sin^{-1}x+\cos^{-1}x) = \frac{\pi}{4} \Rightarrow \frac{1}{\sin^{-1}x\cos^{-1}x} \ge \frac{16}{\pi^2}$

$$\left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right) \ge 1+\frac{8}{\pi}+\frac{16}{\pi^2}$$

$$\Rightarrow \frac{1}{b-a}\int_a^b\left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right)dx \ge \frac{b-a}{b-a}\left(1+\frac{8}{\pi}+\frac{16}{\pi^2}\right) = \left(1+\frac{4}{\pi}\right)^2$$

Solution 2 by Soumava Pal –Kolkata- India

$$\sin^{-1}x+\cos^{-1}x \ge 2\cdot\sqrt{\sin^{-1}x\cos^{-1}x}$$

(AM – GM)

$$\Rightarrow \frac{\pi}{2} \ge 2\sqrt{\sin^{-1}x\cos^{-1}x} \Rightarrow \frac{1}{\sqrt{\sin^{-1}x\cos^{-1}x}} \ge \frac{4}{\pi} \Rightarrow \frac{1}{\sin^{-1}x\cos^{-1}x} \ge \left(\frac{4}{\pi}\right)^2$$

$$\Rightarrow \frac{\frac{\pi}{2}+1}{\sin^{-1}x\cos^{-1}x} \ge \left(\frac{4}{\pi}\right)^2\left(\frac{\pi}{2}+1\right) = 2\cdot\frac{4}{\pi}\cdot 1+\left(\frac{4}{\pi}\right)^2$$

$$\Rightarrow 1+\frac{\sin^{-1}x+\cos^{-1}x+1}{\sin^{-1}x\cos^{-1}x} \ge 1+2\cdot\frac{4}{\pi}\cdot 1+\left(\frac{4}{\pi}\right)^2$$

$$\Rightarrow 1+\frac{1}{\sin^{-1}x}+\frac{1}{\cos^{-1}x}+\frac{1}{\sin^{-1}x\cos^{-1}x} \ge \left(1+\frac{4}{\pi}\right)^2$$

$$\Rightarrow \int_a^b \left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right)dx \geq \int_a^b \left(1+\frac{4}{\pi}\right)^2 dx = \left(1+\frac{4}{\pi}\right)^2 (b-a)$$

$$\Rightarrow \frac{1}{b-a}\int_a^b \left(1+\frac{1}{\sin^{-1}x}\right)\left(1+\frac{1}{\cos^{-1}x}\right) \geq \left(1+\frac{4}{\pi}\right)^2$$

30. If $a, b, c \in (2, \infty)$, $\Omega(a) = \int_0^1 \frac{1-x^2}{1+ax^2+x^4}\, dx$ then:

$$2bc\Omega(a) + 2c\Omega(b) + 2ab\Omega(c) < a^2 + b^2 + c^2$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

$$\textit{For } a \geq 2, \textit{let } \Omega(a) = \int_0^1 \frac{1-x^2}{1+ax^2+x^4}dx$$

$$\Omega'(a) = \int_0^1 \frac{(1-x^2)(-1)x^2}{(1+ax^2+x^4)^2}dx < 0 \Rightarrow \Omega(a) \textit{ is strictly decreasing on } [2,\infty).$$

$$\textit{Also, } \Omega(2) = \int_0^1 \frac{1-x^2}{1+2x^2+x^4}dx = \int_0^1 \frac{-1-x^2+2}{(1+x^2)^2}dx = \int_0^1 \left[\frac{2}{(1+x^2)^2} - \frac{1}{1+x^2}\right]dx$$

But,

$$\frac{\pi}{4} = \int_0^1 \frac{dx}{1+x^2} = \frac{x}{1+x^2}\Big|_0^1 + \int_0^1 \frac{x(2x)}{(1+x^2)^2}dx = \frac{1}{2} + \int_0^1 \frac{2(x^2+1)-2}{(1+x^2)^2}dx =$$

$$= \frac{1}{2} + \frac{\pi}{4} - \Omega(2) \Rightarrow \Omega(2) = \frac{1}{2}. \textit{ Thus, } 0 < \Omega(a) < \frac{1}{2} \ \forall a > 2. \textit{ Now,}$$

$$2bc\Omega(a) + 2ca\Omega(b) + 2ab\Omega(c) <$$

$$< bc + ca + ab \leq \frac{1}{2}(b^2+c^2) + \frac{1}{2}(c^2+a^2) + \frac{1}{2}(a^2+b^2) = a^2 + b^2 + c^2$$

31. $\int_0^1 \log^2(1+e^x)\,dx < \left(\int_0^1 \log(1+e^x)\,dx\right)^2 + \frac{1}{12}$

Proposed by Daniel Sitaru – Romania

Solution 1 by Daniel Sitaru-Romania

Let be $f:[0,1]\to\mathbb{R};\ f(x)=\log(1+e^x)$.

$$f'(x)=\frac{e^x}{1+e^x};\ f''(x)=\frac{(e^x)'(e^x+1)-e^x(e^x+1)'}{(e^x+1)^2}=\frac{e^x}{(e^x+1)^2}>0$$

$M=\max f'(x)=f'(1)=\frac{e}{(e+1)^2}$. **Let be** $g,h:[0,1]\to\mathbb{R}$;

$$g(x)=\log(1+e^x)-\frac{e}{(e+1)^2}\cdot x;\ h(x)=\log(1+e^x)+\frac{e}{(e+1)^2}\cdot x$$

$$g'(x)=f'(x)-\frac{e}{(e+1)^2}\le 0\Rightarrow g \text{ decreasing}$$

$$h'(x)=f'(x)+\frac{e}{(e+1)^2}\ge 0\Rightarrow h \text{ increasing}$$

By Cebyishev – integral form:

$$\int_0^1 (g(x)\cdot h(x))\,dx\le\left(\int_0^1 g(x)dx\right)\left(\int_0^1 h(x)dx\right)$$

$$\int_0^1\left(\log(1+e^x)-\frac{e}{(e+1)^2}\cdot x\right)\left(\log(1+e^x)+\frac{e}{(e+1)^2}\cdot x\right)dx\le$$

$$\le\int_0^1\left(\log(1+e^x)-\frac{e}{(e+1)^2}\cdot x\right)dx\cdot\int_0^1\left(\log(1+e^x)+\frac{e}{(e+1)^2}\cdot x\right)dx$$

$$\int_0^1\log^2(1+e^x)\,dx-\frac{e^2}{(e+1)^4}\cdot\frac{x^3}{3}\Big|_0^1\le\left(\int_0^1\log(1+e^x)\,dx\right)^2-\frac{1}{4}\cdot\frac{e^2}{(e+1)^4}$$

$$\int_0^1 \log^2(1+e^x)\,dx - \frac{e^2}{3(e+1)^4} \le \left(\int_0^1 \log(1+e^x)\,dx\right)^2 - \frac{e^2}{4(e+1)^2}$$

$$\int_0^1 \log^2(1+e^x)\,dx \le \left(\int_0^1 \log(1+e^x)\,dx\right)^2 + \frac{e^2}{3(e+1)^4} - \frac{e^2}{4(e+1)^4}$$

$$\int_0^1 \log^2(1+e^x)\,dx \le \left(\int_0^1 \log(1+e^x)\,dx\right)^2 + \frac{e^2}{12(e+1)^4} <$$

$$< \left(\int_0^1 \log(1+e^x)\,dx\right)^2 + \frac{1}{12}$$

Solution 2 by Soumitra Moukherjee - Chandar Nagore – India

Applying A.M $\ge$ G.M:

$$\frac{1}{12}\left(\int_0^1 \log(1+e^x)\,dx\right)^2 \ge \frac{1}{\sqrt{3}}\left(\int_0^1 \log(1+e^x)\,dx\right)$$

we need to show:

$$\frac{1}{\sqrt{3}}\int_0^1 \log(1+e^x)\,dx > \int_0^1 \log^2(1+e^x)\,dx$$

$$\Leftrightarrow \frac{1}{\sqrt{3}} > \log(1+e^x) \text{ for all } x \in [0,1]$$

$$0 \le x \le 1 \Leftrightarrow 2 \le \log(1+e^x) \le \log(1+e) \approx 0.5703 \text{ and } \frac{1}{\sqrt{3}} \approx 0.5773$$

$$\text{so, } \frac{1}{\sqrt{3}} > \log(1+e^x) \text{ for all } x \in [0,1], \text{ so,}$$

$$\frac{1}{12}+\left(\int_0^1 \log(1+e^x)\,dx\right)^2 > \int_0^1 \log^2(1+e^x)\,dx$$

(proved)

32. **Determine all functions f with the following property: They are defined for all real numbers except $\frac{1}{3}$ and $-\frac{1}{3}$, and for each of those real numbers the equality $f\left(\frac{x+1}{1-3x}\right)+f(x)=x$ holds.**

Germany NMO

Solution 1 by Marian Ursarescu-Romania

$$f\left(\frac{x+1}{1-3x}\right)+f(x)=x_1\ \ \forall x\neq\frac{1}{3}, x\neq-\frac{1}{3}$$

$$x\to\frac{x+1}{1-3x}\Rightarrow f\left(\frac{\frac{x+1}{1-3x}+1}{1-3\frac{(x+1)}{1-3x}}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x}\Rightarrow$$

$$f\left(\frac{x-1}{3x+1}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x}\quad \textbf{(1)}$$

$$x\to\frac{x-1}{3x+1}\Rightarrow f\left(\frac{\frac{x-1}{3x+1}+1}{1-3\left(\frac{x+1}{3x+1}\right)}\right)+f\left(\frac{x-1}{3x+1}\right)=\frac{x-1}{3x+1}\Rightarrow$$

$$\Rightarrow f(x)+f\left(\frac{x-1}{3x+1}\right)=\frac{x-1}{3x+1}\quad \textbf{(2)}$$

From hypothesis and (1) and (2) $\Rightarrow$

$$\begin{cases} f\left(\frac{x+1}{1-3x}\right)+f(x)=x \\ f\left(\frac{x-1}{3x+1}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x} \\ f(x)+f\left(\frac{x-1}{3x+1}\right)=\frac{x-1}{3x+1} \end{cases}$$

_ _ _ _ _ _ _ _ _ _ _ _ _ _ _ _

$$\oplus\, 2\left(f(x)+f\left(\frac{x+1}{1-3x}\right)+f\left(\frac{x-1}{3x+1}\right)\right)=x+\frac{x+1}{1-3x}+\frac{x+1}{3x+1}$$

$$\Rightarrow f(x)+f\left(\frac{x+1}{1-3x}\right)+f\left(\frac{x-1}{3x+1}\right)=\frac{1}{2}\left(x+\frac{x+1}{1-3x}+\frac{x+1}{3x+1}\right)$$

$$f\left(\frac{x-1}{3x+1}\right)+f\left(\frac{x+1}{1-3x}\right)=\frac{x+1}{1-3x}$$

— — — — — — — — — — — — — — — — — — —

$$\ominus\, f(t)=\frac{1}{2}\left(x+\frac{x+1}{1-3x}+\frac{x+1}{3x+1}\right)-\frac{x+1}{1-3x}\dots$$

Solution 2 by Abdallah El Farissi-Bechar-Algerie

$x\neq\frac{1}{3},-\frac{1}{3}$ ***and*** $f\left(\frac{1+x}{1-3x}\right)+f(x)=x$***. We have*** $\begin{cases} f\left(\frac{x+1}{1-3x}\right)+f(x)=x \\ f\left(\frac{x+1}{1-3x}\right)+f\left(\frac{x-1}{1+3x}\right)=\frac{x+1}{1-3x} \\ f(x)+f\left(\frac{x-1}{1+3x}\right)=\frac{x-1}{1+3x}\end{cases}$

then $f(x)=\frac{1}{2}\left(x-\frac{1+x}{1-3x}+\frac{x-1}{1+3x}\right)$

33. Find all continuous functions $f:\mathbb{R}\to\mathbb{R}$ such that:

$$f(x)+f(3x)+f(9x)=91x^2+26x+3$$

Proposed by Marian Ursarescu-Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

$$f(x)+f(3x)+f(9x)=91x^2+26x+3 \quad \textit{(3)}$$

Put $g(x)=f(x)-x^2-2x-1$***. We have***

(1) $\Rightarrow g(x)+x^2+2x+1+g(3x)+9x^2+6x+1+g(9x)+81x^2+18x+1=$

$=91x^2+26x+3\Rightarrow g(x)+g(3x)+g(9x)=0$ ***(2)***

Put $x\to 3x$***, we have (2)***$\Rightarrow g(3x)+g(9x)+g(27x)=0$ ***(3)***

(2) and (3) $\Rightarrow g(x)=g(27x)$ ***(4)***

Put $x\to\frac{x}{27}$***, we have (4)*** $\Rightarrow g(x)=g\left(\frac{x}{27}\right)$ ***(5)***

Put $x\to\frac{x}{27}$***, we have (5)*** $\Rightarrow g\left(\frac{x}{27}\right)=g\left(\frac{x}{27^2}\right)$

Similarly, we have $g(x) = g\left(\frac{x}{27}\right) = g\left(\frac{x}{27^2}\right) = \cdots = g\left(\frac{x}{27^n}\right)\ \forall n \in \mathbb{N}$.

The sequence (u_n) such that $u_0 = x, u_{n+1} = \frac{x}{27^n}$. We have $\lim_{n\to+\infty} u_n = 0$

We have $g(u_0) = g(u_1) = \cdots = g(u_n) = g(u_n + 1) = \cdots = g(\lim_{n\to\infty} u_n) = g(0)$

Put $x \to 0$, we have (2) $\Rightarrow 3g(0) = 0 \Rightarrow g(0) = 0 \Rightarrow g(x) = 0\ \forall x \in \mathbb{R}$

So, $f(x) = x^2 + 2x + 1\quad \forall x \in \mathbb{R}$

We have (1) $\Rightarrow x^2 + 2x + 1 + 9x^2 + 6x + 1 + 81x^2 + 18x + 1 = 91x^2 + 26x + 3$

(True). Therefore $f(x) = x^2 + 2x + 1\ \forall x \in \mathbb{R}$

34. If $a, b, c \in (0, \infty)$ then:

$$\left(1+\frac{b}{a}\right)^c\left(\int_0^1 e^{-x^2}\,dx\right)^{c+1} + \left(1+\frac{a}{b}\right)\left(\int_0^1 e^{x^2}\,dx\right)^{c+1} \geq \left(\int_0^1 (e^{-x^2}+e^{x^2})dx\right)^{c+1}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

Let $A = \int_0^1 e^{-x^2}\,dx, B = \int_0^1 e^{x^2}\,dx$. Consider

$$f(t) = \left(1+\frac{1}{t}\right)^c A^{c+1} + (1+t)^c B^{c+1}, t > 0$$

$$f'(t) = c\left(1+\frac{1}{t}\right)^{c-1}\left(-\frac{1}{t^2}\right)A^{c+1} + c(1+t)^{c-1}B^{c+1} =$$

$$= c\left(1+\frac{1}{t}\right)^{c-1}\frac{B^{c+1}}{t^2}\left[t^{c+1} - \left(\frac{A}{B}\right)^{c+1}\right]$$

Note that $f'(t) < 0$ for $0 < t < \frac{A}{B}$, $f'(0) > 0$ for $t > \frac{A}{B}$

$f(t)$ is least when $t = \frac{A}{B}$. Thus, $f(t) \geq f\left(\frac{A}{B}\right)\ \forall\ t > 0 \Rightarrow f\left(\frac{a}{b}\right) \geq f\left(\frac{A}{B}\right)$

$$\Rightarrow \left(1+\frac{b}{a}\right)^c\left(\int_0^1 e^{-x^2}\,dx\right)^{c+1} + \left(1+\frac{a}{b}\right)^c\left(\int_0^1 e^{x^2}\,dx\right)^{c+1} \geq$$

$$\ge\left(1+\frac{A}{B}\right)^c B^{c+1}+\left(1+\frac{B}{A}\right)A^{c+1}=(A+B)^cB+(A+B)^cA=$$

$$=(A+B)^{c+1}=\left(\int_0^1\left(e^{-x^2}+e^{x^2}\right)dx\right)^{c+1}$$

Solution 2 by Soumitra Moukherjee - Chandar Nagore – India

Applying Holder's Inequality

$$\left\{\left(1+\frac{b}{a}\right)^c\left(\int_0^1 e^{-x^2}\,dx\right)^{c+1}+\left(1+\frac{a}{b}\right)^c\left(\int_0^1 e^{x^2}\,dx\right)\right\}\left\{\left(\frac{a}{a+b}+\frac{b}{a+b}\right)\left(\frac{a}{a+b}+\frac{b}{a+b}\right)\dots(c\ times)\right\}$$

$$\ge\left\{\sqrt[c+1]{\left(1+\frac{b}{a}\right)^c\left(\frac{a}{a+b}\right)^c\left(\int_0^1 e^{-x^2}\right)^{c+1}}+\sqrt[c+1]{\left(1+\frac{a}{b}\right)^c\left(\frac{b}{a+b}\right)^c\left(\int_0^1 e^{x^2}\right)^{c+1}}\right\}^{c+1}$$

$$\left(1+\frac{b}{a}\right)^2\left(\int_0^1 e^{-x^2}\right)^{c+1}+\left(1+\frac{a}{b}\right)^c\left(\int_0^1 e^{x^2}\,dx\right)^{c+1}\ge\left(\int_0^1\left(e^{-x^2}+e^{x^2}\right)dx\right)^{c+1}$$

35. Let $f:[0,1]\to(-1,1)$ be a continuous function so that

$$\int_0^1 f(x)\,dx\notin\{-1,1\}$$

Prove that:

$$\frac{e^{\int_0^1 f(x)dx}}{1+e^{\int_0^1 f(x)dx}}\le\int_0^1\frac{e^{f(x)}}{1+e^{f(x)}}dx$$

Proposed by Soumitra Mandal - Kolkata – India

Solution by Daniel Sitaru – Romania

$$g(x)=\frac{e^{f(x)}}{1+e^{f(x)}}$$

$$e^{\frac{1}{1-0}\int_0^1 \log(g(x))dx} \le \frac{1}{1-0}\int_0^1 g(x)\,dx$$ *– (AM-GM – integral form)*

$$e^{\int_0^1 (f(x)-\log(1+e^{f(x)}))dx} \le \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}}dx$$

$$\frac{e^{\int_0^1 f(x)dx}}{e^{\int_0^1 (\log(1+e^{f(x)}))dx}} \le \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}}dx \quad (1)$$

$$\varphi(x) = \log(1+e^x), \varphi'(x) = \frac{e^x}{1+e^x}, \varphi'(x) = \frac{e^x}{(1+e^x)^2} > 0$$

$$\log\left(1+e^{\int_0^1 f(x)dx}\right) \overset{Jensen}{\le} \int_0^1 (\log(1+e^{f(x)}))\,dx$$

$$\frac{1}{1+e^{\int_0^1 f(x)dx}} \le \frac{e^{\int_0^1 f(x)dx}}{e^{\int_0^1 (\log(1+e^{f(x)}))dx}} \quad (2)$$

By (1), (2): $\frac{1}{1+e^{\int_0^1 f(x)dx}} \le \int_0^1 \frac{e^{f(x)}}{1+e^{f(x)}}dx$

36. If $a, b \in \mathbb{R}$ then:

$$2(b-a)^2\sqrt{e^{a+b}} \le 8\left(\sqrt{e^a}-\sqrt{e^b}\right)^2 \le (e^a+e^b)(b-a)^2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal -Kolkata – India

Let $f(x) = \sqrt{e^x}$ ***for all*** $x \in [a,b]$ ***where*** $a < b$

[WLOG let us assume $a < b$***]***

$f''(x) = \frac{\sqrt{e^x}}{4} > 0$, ***hence*** $f(x)$ ***is strictly convex***

Applying HERMITE – HADAMARD Inequality

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}$$

$$\Rightarrow e^{\frac{a+b}{4}}\le\frac{1}{b-a}\int_a^b\sqrt{e^x}\,dx\le\frac{\sqrt{e^a}+\sqrt{e^b}}{2}$$

$$\Rightarrow e^{\frac{a+b}{4}}\le\frac{2}{b-a}\left(\sqrt{e^b}-\sqrt{e^a}\right)\le\frac{\sqrt{e^a}+\sqrt{e^b}}{2}$$

$$\Rightarrow e^{\frac{a+b}{2}}\le\frac{4}{(b-a)^2}\left(\sqrt{e^b}-\sqrt{e^a}\right)^2\le\frac{\left(\sqrt{e^a}+\sqrt{e^b}\right)^2}{4}\le\frac{e^b+e^a}{2}$$

$$\Rightarrow 2(b-a)^2\sqrt{e^{a+b}}\le 8\left(\sqrt{e^b}-\sqrt{e^a}\right)^2\le\left(e^b+e^a\right)(b-a)^2$$

Solution 2 by Sameer Shihab-Riyadh-Saudi Arabia

Let $f(x)=\sqrt{e^x}$. since $f(x)$ is concave up

$$f\left(\frac{a+b}{2}\right)\le\frac{1}{b-a}\int_a^b f(x)\,dx\le\frac{f(a)+f(b)}{2}$$

$$\sqrt{e^{\frac{a+b}{2}}}\le\frac{1}{b-a}\int_a^b\sqrt{e^x}\,dx\le\frac{\sqrt{e^a}+\sqrt{e^b}}{2}$$

$$\sqrt{e^{\frac{a+b}{2}}}\le\left[\frac{1}{b-a}2\sqrt{e^x}\right]_a^b\le\frac{\sqrt{e^a}+\sqrt{e^b}}{2}$$

$$\sqrt{e^{\frac{a+b}{2}}}\le\frac{2}{b-a}\left(\sqrt{e^b}-\sqrt{e^a}\right)\le\frac{\sqrt{e^a}+\sqrt{e^b}}{2}$$

$$e^{\frac{a+b}{2}}\le\frac{4}{(b-a)^2}\left(\sqrt{e^b}-\sqrt{e^a}\right)^2\le\left(\frac{\sqrt{e^a}+\sqrt{e^b}}{2}\right)^2$$

but $\frac{\left(\sqrt{e^a}+\sqrt{e^b}\right)^2}{4}\le\frac{e^a}{2}+\frac{e^b}{2}$ ***Bergstrom's inequality***

$$e^{\frac{a+b}{2}} \leq \frac{4}{(b-a)^2}\left(\sqrt{e^b}-\sqrt{e^a}\right)^2 \leq \frac{e^a}{2}+\frac{e^b}{2}$$

$$2(b-a)^2 \cdot e^{\frac{b+a}{2}} \leq 8\left(\sqrt{e^b}-\sqrt{e^a}\right) \leq (b-a)^2\left(e^a+e^b\right)$$

Solution 3 by Vinhhop Tran-Quang Tri-Vietnam

For the LHS inequality: $2(b-a)^2\sqrt{e^{a+b}} \leq 8\left(\sqrt{e^a}-\sqrt{e^b}\right)^2$

$$\Leftrightarrow \left(\frac{b-a}{2}\right)^2 \leq \frac{e^a-2e^{\frac{a+b}{2}}+e^b}{e^{\frac{a+b}{2}}} \Leftrightarrow \left(\frac{b-a}{2}\right)^2 \leq e^{\frac{a-b}{2}}+e^{-\frac{a-b}{2}}-2$$

$$\Leftrightarrow e^t+e^{-t}-t^2-2 \geq 0, t=\frac{a-b}{2}.$$

WLOG, assume that $t \geq 0$***. Let consider function*** $f(t)=e^t+e^{-t}-t^2-2$, $t \in [0;\ +\infty)$

We need to prove $f(t) \geq 0\ \forall t \geq 0$

$f'(t)=e^t-e^{-t}-2t, f''(t)=e^t+e^{-t}-2>0\ \forall\in(0;\infty)$***. So the function*** $f'(t)$ ***increases on the interval*** $[0;+\infty)$***, implies*** $f'(t)>f'(0)=0\ \forall t\in(0;+\infty)$***. This shows that*** $f(t)$ ***is an increasing function on the interval*** $[0;\ +\infty)$***. Hence,*** $f(t)\geq f(0)=0\ \forall\ t\geq 0.$

2. For the RHS ***inequality*** $8\left(\sqrt{e^a}-\sqrt{e^b}\right)^2 \leq \left(e^a+e^b\right)(b-a)^2$

$$\Leftrightarrow 8\frac{\left(\sqrt{e^a}-\sqrt{e^b}\right)^2}{e^a+e^b} \leq (b-a)^2 \Leftrightarrow 8\frac{e^a-2e^{\frac{a+b}{2}}+e^b}{e^a+e^b} \leq (b-a)^2$$

$$\Leftrightarrow 8-16\frac{e^{\frac{a+b}{2}}}{e^a+e^b} \leq (b-a)^2 \Leftrightarrow 8-16\frac{1}{e^{\frac{a-b}{2}}+e^{-\frac{a-b}{2}}} \leq (b-a)^2$$

$$\Leftrightarrow \frac{4}{e^t+e^{-t}}+t^2-2 \geq 0, t=\frac{a-b}{2}.$$

WLOG, assume that $t \geq 0$***. Let consider function***

$g(t) = \frac{4}{e^t+e^{-t}} + t^2 - 2, t \in [0;\infty)$.*We need to prove* $g(t) \geq 0\ \forall t \geq 0$.

$$g'(t) = \frac{-4(e^t - e^{-t})}{(e^t + e^{-t})^2} + 2t, g''(t) = 2 - \frac{4}{e^t + e^{-t}} + 8\frac{(e^t - e^{-t})^2}{(e^t + e^{-t})^3} > 0$$

$\forall t \in (0; +\infty)$. ***So the function*** $g'(t)$ ***increases on the interval*** $[0; +\infty)$, ***implies*** $g'(t) > g'(0)\ \forall t \in (0; +\infty)$. ***This shows that*** $g(t)$ ***is an increasing function on the interval*** $[0; +\infty)$. ***Hence,*** $g(t) \geq g(0) = 0\ \forall\ t \geq 0$.

37. Find $m, n \in \mathbb{N}^*$ such that $x^2 - x + 3$ divide $(x + 2)^m - (x^2 + 2)^n, x \in \mathbb{R}$.

Proposed by Marian Ursarescu-Romania

Solution by Khanh Hung Vu-Ho Chi Minh-Vietnam

We have $(x + 2)^m - (x^2 + 2)^n = (x^2 - x + 3).Q(x)$ ***(1)***

Put $x = \frac{1+i\sqrt{11}}{2}$, ***we have (1)*** $\Rightarrow \left(\frac{5+i\sqrt{11}}{2}\right)^m = \left(\frac{-1+i\sqrt{11}}{2}\right)^n$ ***(2)***

Put $x = \frac{1-i\sqrt{11}}{2}$, ***we have (1)*** $\Rightarrow \left(\frac{5-i\sqrt{11}}{2}\right)^m = \left(\frac{-1-i\sqrt{11}}{2}\right)^n$ ***(3)***

Put $\frac{(2)}{(3)}$, ***we have*** $\left(\frac{7+5i\sqrt{11}}{18}\right)^m = \left(\frac{-5-i\sqrt{11}}{6}\right)^n \Rightarrow \left(\frac{-5-i\sqrt{11}}{6}\right)^{2m} = \left(\frac{-5-i\sqrt{11}}{6}\right)^n$ ***(4)***

Put α ***is the angle satisfy*** $\cos\alpha = \frac{-5}{6}$ ***and*** $\sin\alpha = \frac{-\sqrt{11}}{6}$

We have (4) $\Rightarrow \cos(2m\alpha) + i\cdot\sin(2m\alpha) = \cos(n\alpha) + i\cdot\sin(n\alpha)$

$$\Rightarrow \begin{cases}\cos(2m\alpha) = \cos(n\alpha) & (5)\\ \sin(2m\alpha) = \sin(n\alpha) & (6)\end{cases}$$

We have (5) $\Rightarrow \left[\begin{array}{ll}2m\alpha = n\alpha + k2\pi & (7)\\ 2m\alpha = -n\alpha + k2\pi & (8)\end{array}\right.$

Lemma: If $\frac{\pi}{\beta}$ ***is a rational number, we have*** $\cos\beta \in \left\{\pm 1; \pm\frac{1}{2}\right\}$. ***Prove***

We have $\frac{\pi}{\beta}$ ***is a rational number*** $\Rightarrow \beta = r\pi\ (r \in Q)$

With De Moivre's Formula we deduce that $\cos r\pi + i\cdot\sin r\pi$ ***and*** $\cos r\pi - i\cdot\sin r\pi$ ***are algebraic integers*** $\Rightarrow 2\cos r\pi$ ***is an algebraic integer.***

But $2\cos r\pi \in Q \Rightarrow 2\cos r\pi \in Z$

Now from $-2 \leq 2\cos r\pi \leq 2$ *so we have* $2\cos r\pi \in \{-2;\ -1; 0; 1; 2\}$

or $\cos\beta \in \left\{\pm 1;\ \pm\frac{1}{2}\right\}$

We have (7) $\Rightarrow (2m-n)\alpha = k2\pi \Rightarrow 2m-n = 0$ *(since* $\frac{\pi}{\alpha} \in I$*)* $\Rightarrow 2m = n$ *(9)*

We have (8) $\Rightarrow (2m+n)\alpha = k2\pi \Rightarrow 2m+n = 0$ *(since* $\frac{\pi}{\alpha} \in I$*)* $\Rightarrow -2m = n$ *(10)*

We have (10) $\Rightarrow -\sin(n\alpha) = \sin(n\alpha) \Rightarrow \sin(n\alpha) = 0 \Rightarrow n\alpha = q2\pi\ (q \in Z) \Rightarrow n = 0$

(since $\frac{\pi}{\alpha} \in I$*) (Absurd). We have (9)* $\Rightarrow \sin(n\alpha) = \sin(n\alpha)$ *(True)*

Therefore with $2m = n$, $x^2 - x + 3$ *divide* $(x+2)^m - (x^2+2)^n, x \in R$

38. If $0 < a < b < \frac{\pi}{2}$ then:

$$7\int_a^b x^3 \tan x\left(\sin\frac{x}{2} + \tan\frac{x}{2}\right)\sqrt{\sin x \tan x}\,dx > b^7 - a^7$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty – Kolkata – India

We first prove that $\forall x \in \left(0, \frac{\pi}{2}\right), \sin x > x\sqrt{\cos x}$

$$\Leftrightarrow \sin^2 x > x^2\cos x \Leftrightarrow 1 - \cos^2 x > x^2 \cos x$$

$$\Leftrightarrow 1 > \cos x\,(x^2 + \cos x) \Leftrightarrow \sec x > x^2 + \cos x \quad (1)$$

Let $f(x) = \sec x - x^2 - \cos x \quad \forall\, x \in \left[0, \frac{\pi}{2}\right)$

$$f'(x) = \sec x \tan x - 2x + \sin x = g(x) \quad \textit{(say)}$$

$$g'(x) = \sec^3 x + \tan x \sec x \tan x - 2 + \cos x$$

$$= \sec x\,(1 + \tan^2 x) + \sec x \tan^2 x - 2 + \cos x$$

$$= (\sec x + \cos x - 2) + 2\sec x \tan^2 x$$

$$\overset{A-G}{\geq} 2\sqrt{\sec x \cos x} - 2 + 2\sec x \tan^2 x$$

$$= 2 - 2 + 2\sec x\tan^2 x = 2\sec x\tan^2 x \ge 0$$

$$\therefore g(x) = f'(x) \ge g(0) = 0 \Rightarrow f(x) \ge f(0) = 0$$

$$\therefore \sec x - x^2 - \cos x \ge 0, \textit{ equality at } x = 0$$

$$\Rightarrow \forall x \in \left(0, \frac{\pi}{2}\right), \sec x - x^2 - \cos x > 0$$

$$\Rightarrow \sin x > x\sqrt{\cos x} \quad \textit{(from (1))}$$

$$\Rightarrow \sin^2 x > x^2 \cos x \Rightarrow \frac{\sin^2 x}{\cos x} > x^2 \Rightarrow \sin x \tan x > x^2 \Rightarrow \sqrt{\sin x \tan x} > x$$

$$\textit{Also, } \sin\frac{x}{2} + \tan\frac{x}{2} \overset{A-G}{\ge} 2\sqrt{\sin\frac{x}{2}\tan\frac{x}{2}} > 2\left(\frac{x}{2}\right) = x$$

$$\therefore x^3 \tan x \left(\sin\frac{x}{2} + \tan\frac{x}{2}\right)\sqrt{\sin x \tan x}$$

$$> x^3 \cdot x \cdot x \cdot x = x^6 \left(\because \forall x \in \left(0, \frac{\pi}{2}\right), \tan x > x\right)$$

$$\therefore LHS > 7\int_a^b x^6\, dx = 7\left(\frac{1}{7}\right)[x^7]_a^b = b^7 - a^7$$

Solution 2 by Daniel Sitaru – Romania

$$\sin x \tan x - \left(2\tan\frac{x}{2}\right)^2 = \frac{\sin^2 x}{\cos x} - \frac{4\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}} = \frac{4\sin^2\frac{x}{2}}{\cos x\cos^2\frac{x}{2}}\left(\cos^2\frac{x}{2} - 1\right)^2 > 0$$

$$\sqrt{\sin x \tan x} > 2\tan\frac{x}{2} > 2 \cdot \frac{x}{2} = x, \quad \textit{(1)}$$

$$f: \left(0, \frac{\pi}{2}\right) \to \mathbb{R}, f(x) = \sin x + \tan x - 2x, f'(x) = \cos x + \tan^2 x - 1$$

$$f''(x) = \frac{\sin x\,(2 - \cos^3 x)}{\cos^3 x} > 0, \inf f'(x) = \inf f(x) = 0 \to f(x) > 0$$

$$\sin x + \tan x > 2x \to \sin\frac{x}{2} + \tan\frac{x}{2} > x, \quad \textit{(2)}$$

$$\tan x > x, \quad \textit{(3)}$$

By multiplying (1), (2), (3) →

$$x^3\tan x\left(\sin\frac{x}{2}+\tan\frac{x}{2}\right)\sqrt{\sin x\tan x}>x^6$$

$$\int_a^b x^3\tan x\left(\sin\frac{x}{2}+\tan\frac{x}{2}\right)\sqrt{\sin x\tan x}\,dx>\frac{1}{7}(b^7-a^7)$$

39. Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be integrable and satisfying

$f(xt+(1-t)y)\le tf(x)+(1-t)f(y)$ where $x,y\in\mathbb{R}^+$ and $t\in(0,1)$

Show that:

$$\frac{1}{\ln a}\int_1^a f(x^4)\,dx+\frac{1}{\ln b}\int_1^b f(x^4)\,dx\ge 2\int_0^1 (ab)^{\frac{x}{2}}f(a^{4x}-a^{3x}b^x+a^{2x}b^{2x}-a^xb^{3x}+b^{4x})dx$$

where $a,b>0$.

Proposed by Soumitra Mandal-Chandar Nagore-India

Solution by Daniel Sitaru – Romania

$$x=a^y\to\frac{1}{\log a}\int_1^a f(x^4)\,dx=\int_0^1 a^yf(a^{4y})\,dy$$

$$\frac{1}{\log a}\int_1^a f(x^4)\,dx+\frac{1}{\log b}\int_1^b f(x^4)\,dx=$$

$$=\int_0^1\left(a^xf(a^{4x})+b^xf(b^{4x})\right)dx\overset{JENSEN}{\ge}$$

$$\ge\int_0^1(a^x+b^x)f\left(\frac{a^{5x}+b^{5x}}{a^x+b^x}\right)dx\overset{AM-GM}{\ge}$$

$$\geq 2\int_0^1 \sqrt{a^x b^x}\, f(a^{4x} - a^{3x}b^x + a^{2x}b^{2x} - a^x b^{3x} + b^{4x})dx$$

40. If $0 < a < b < \frac{\pi}{2}$ then:

$$\int_a^b \frac{\cos x}{\sin x + 4x\cos x}dx < \frac{1}{5}\left(\log\frac{b}{a}\right)^{\frac{4}{5}} \cdot \sqrt[5]{\log\left(\frac{\sin b}{\sin a}\right)}$$

Proposed by Daniel Sitaru – Romania

Solution by Abdallah El Farissi – Bechar – Algerie

$$\int_a^b \frac{\cos x}{\sin x + 4x\cos x}dx \overset{AM-GM}{\leq} \frac{1}{5}\int_a^b \frac{1}{x^{\frac{4}{5}}}\left(\frac{\cos x}{\sin x}\right)^{\frac{1}{5}}dx$$

and by Holder inequality

$$\int_a^b \frac{\cos x}{\sin x + 4x\cos x}dx \leq \frac{1}{5}\left(\int_a^b \frac{dx}{x}\right)^{\frac{4}{5}}\left(\int_a^b \frac{\cos x}{\sin x}dx\right)^{\frac{1}{5}} = \frac{1}{5}\left(\ln\frac{b}{a}\right)^{\frac{4}{5}}\left(\ln\frac{\sin b}{\sin a}\right)^{\frac{1}{5}}$$

$$< \left(\ln\frac{b}{a}\right)^{\frac{4}{5}}\left(\ln\frac{\sin b}{\sin a}\right)^{\frac{1}{5}}$$

41. For $a, b, c \in (0, \infty)$; $a < b < c$; $f: [0, a] \to [0, b]$; $g: [0, b] \to [0, c]$ continuous, bijectifs and strictly increasing functions prove that:

$$\frac{1}{c}\int_0^a (g \circ f)^2(x)dx + \frac{1}{a}\int_0^c (f^{-1} \circ g^{-1})^2(x)dx \leq ac$$

Proposed by Daniel Sitaru – Romania

Solution by Daniel Sitaru – Romania

$$(g\circ f)(x)\in[0,c]\Rightarrow(g\circ f)(x)\le c;(\forall)x\in[0,a]$$

$$\frac{1}{c}\int_0^a(g\circ f)^2(x)dx\le\frac{1}{c}\int_0^a c\cdot(g\circ f)(x)\,dx=\int_0^a(g\circ f)(x)\,dx$$

$$(f^{-1}\circ g^{-1})(x)\in[0,a]\Rightarrow(f^{-1}\circ g^{-1})(x)\le a(\forall)x\in[0,c]$$

$$\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})^2(x)\,dx\le\frac{1}{a}\int_0^c a(f^{-1}\circ g^{-1})\,(x)dx=\int_0^c(f^{-1}\circ g^{-1})\,(x)dx$$

$$\frac{1}{c}\int_0^a(g\circ f)^2\,(x)dx\le\int_0^a(g\circ f)\,(x)dx$$

$$\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})^2\,(x)dx\le\int_0^c(f^{-1}\circ g^{-1})(x)\,dx$$

$$\frac{1}{c}\int_0^a(g\circ f)^2\,(x)dx+\frac{1}{a}\int_0^c(f^{-1}\circ g^{-1})^2\,(x)dx\le$$

$$\le\int_0^a(g\circ f)(x)\,dx+\int_0^c(f^{-1}\circ g^{-1})\,(x)dx=ac$$

42. If $0<a<1, f:\left[a,\frac{1}{a}\right]\to\mathbb{R}, f$ convexe and increasing function then:

$$\frac{1-a^2}{a}f\left(\frac{1+a^2}{2a}\right)\le\int_a^{\frac{1}{a}}\left(\frac{1+x^2}{2x^2}\right)f(x)dx\le\frac{1-a^2}{2a}\left(f(a)+f\left(\frac{1}{a}\right)\right)$$

Proposed by Abdallah El Farissi-Bechar-Algerie

Solution by proposer

f is increasing convex function on $\left[a,\frac{1}{a}\right]$ then $f\left(\frac{1}{x}\right)$ is convex too, then by H-H inequality we have

$$2f\left(\frac{1+a^2}{2a}\right)\le\frac{a}{1-a^2}\left(\int_a^{\frac{1}{a}}f(x)\,dx+\int_a^{\frac{1}{a}}f\left(\frac{1}{x}\right)dx\right)\le\left(f(a)+f\left(\frac{1}{a}\right)\right)$$

in the second integral we use the changment $y=\frac{1}{x}$ we get the inequality

43. If $a,b,c\in\left(0,\frac{\pi}{4}\right)$ then:

$$0\le\int_0^a\left(\int_0^b\left(\int_0^c\left(\sum(\tan x+2\tan y\tan z)+4\prod\tan x\right)dx\right)dy\right)dz\le abc$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal – Kolkata – India

Let $f:[0,c]\to\mathbb{R}^+$ defined by

$$f(x)=\tan x\,(4\tan y\tan z-2\tan y-2\tan z+1)+\tan y+\tan z-2\tan y\tan z$$

for all $x\in[0,c]$. Now,

$$f'(x)=\sec^2x\,(4\tan y\tan z-2\tan y-2\tan z+1)\ge 0 \text{ since}$$

$$x\in(0,c)\subseteq\left(0,\frac{\pi}{4}\right)$$

and $y,z\in\left(0,\frac{\pi}{4}\right)$. So, f is continuous on $[0,c]$ and $f'(x)\ge 0$ hence

f is increasing on $[0,c]$. So, $f\left(\frac{\pi}{4}\right)\ge f(c)\ge f(x)\ge f(0)$

$$\Rightarrow 4\tan y\tan z - 2\tan y - 2\tan z + 1 \ge f(x)$$
$$\ge \tan y + \tan z - 2\tan y\tan z$$
$$\Rightarrow (2\tan y - 1)(2\tan z - 1) \ge f(x) \ge \frac{1}{2} - \frac{1}{2}(2\tan y - 1)(2\tan z - 1)$$
$$\Rightarrow 1 \ge f(x) \ge 0 \textit{ for all } y, z \in \left(0, \frac{\pi}{4}\right)$$
$$\therefore 0 \le \int_0^a \left(\int_0^b \left(\int_0^c \left(\sum(\tan x + 2\tan y\tan z) + 4\prod \tan x\right) dx\right) dy\right) dz \le abc$$

44. If $a, b, c > 0$ then:

$$e^{\sum a\int_b^c \frac{x^4+1}{x^6+1}dx} \le \left(\frac{c}{b}\right)^a \cdot \left(\frac{a}{c}\right)^b \cdot \left(\frac{b}{a}\right)^c$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nishant Kumar-Jamsedhpur-India

$$a\int_b^c \frac{x^4+1}{x^6+1}dx = a\left[\tan^{-1} c + \frac{1}{3}\tan^{-1} c^8 - \tan^{-1} b - \frac{1}{3}\tan^{-1} b^8\right]$$

$$\textit{let } f(x) = \tan^{-1} x + \frac{1}{3}\tan^{-1} x^8 - \log x;\ x > 0;$$

$$f'(x) = \frac{1}{1+x^2} + \frac{x^8}{1+x^6} - \frac{1}{x} < 0;\ f(x) \textit{ is decreasing } \forall\ x > 0.$$

$$\textit{If } a \le b;\ f(a) \ge f(b)$$

$$\tan^{-1} a + \frac{1}{8}\tan^{-1} a^8 - \log a \ge \tan^{-1} b + \frac{1}{3}\tan^{-1} b^8 - \log b$$

$$\tan^{-1} b + \frac{1}{3}\tan^{-1} b^8 - \tan^{-1} a - \frac{1}{3}\tan^{-1} a^8 \le \log b - \log a$$

$$\int_a^b \frac{x^4+1}{x^6+1}dx \le \log\frac{b}{a};\ c\int_a^b \frac{x^4+1}{x^6+1}dx \le c\log\frac{b}{a} = \log\left(\frac{b}{a}\right)^c$$

$$e^{c\int_a^b \frac{x^4+1}{x^6+1}dx} \leq \left(\frac{b}{a}\right)^c \quad (1)$$

similarly for $b \leq c = e^{a\int_a^b \frac{x^4+1}{x^6+1}dx} \leq \left(\frac{c}{b}\right)^a$ ***(2)***

and for $c \leq a = e^{b\int_a^b \frac{x^4+1}{x^6+1}dx} \leq \left(\frac{a}{c}\right)^b$ ***(3)***

multiplying (1), (2) and (3) we get the result

$$e^{\sum a\int_b^c \frac{x^4+1}{x^6+1}dx} \leq \left(\frac{c}{b}\right)^a \cdot \left(\frac{a}{c}\right)^b \cdot \left(\frac{b}{a}\right)^c$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$x^6 + 1 - x - x^5 \text{ where } x \in (0, \infty)$$

$$= x^5(x-1) - (x-1) = (x^5 - 1)(x-1)$$

$$= (x-1)^2(x^4 + x^3 + x^2 + x + 1) \geq 0$$

$$\therefore \frac{x^4+1}{x^6+1} \leq \frac{1}{x}. \text{ Now, } e^{\sum a\int_b^c \frac{x^4+1}{x^6+1}dx} = \prod\left(e^{\int_b^c \frac{x^4+1}{x^6+1}dx}\right)^a \leq$$

$$\leq \prod_{cyc}\left(e^{\int_b^c \frac{dx}{x}}\right)^a = \prod_{cyc}\left(e^{\ln c - \ln b}\right)^a = \prod_{cyc}\left(\frac{c}{b}\right)^a$$

Solution 3 by Redwane El Mellass-Casablanca-Morocco

$$\textbf{Let } (x > 0) = \frac{x^4+1}{x^6+1}.$$

$$\therefore f(x) - \frac{1}{x} = \frac{x^5 + x - x^6 - 1}{x(x^6+1)} = \frac{-(x-1)^2(x^4 + x^3 + x^2 + x + 1)}{x(x^6+1)} \leq 0.$$

$$\Rightarrow f(x) \leq \frac{1}{x} \Rightarrow e^{\sum a\int_b^c f(x)dx} \leq e^{\sum a\int_b^c \frac{1}{x}dx} = e^{\sum a\ln\left(\frac{c}{b}\right)} = \prod\left(\frac{c}{b}\right)^a.$$

Solution 4 by Abdallah El Farissi-Bechar-Algerie

For all $x > 0$ ***we have*** $(1-x)^2(1 + x + x^2 + x^3 + x^4) \geq 0$ ***then***

$$(1-x)(1-x^5)\geq 0 \textit{ it follow that } \frac{1+x^4}{1+x^6}\leq\frac{1}{x}$$

$$\exp\left\{\sum a\int_b^c \frac{1+x^4}{1+x^6}dx\right\}\leq \exp\left\{\sum a\int_b^c \frac{1}{x}dx\right\}$$

$$=\exp\left\{a\int_b^c \frac{1}{x}dx\right\}\exp\left\{b\int_c^a \frac{1}{x}dx\right\}\exp\left\{c\int_a^b \frac{1}{x}dx\right\}$$

$$=\exp\left\{\ln\left(\frac{c}{b}\right)^a\right\}\exp\left\{\ln\left(\frac{a}{c}\right)^b\right\}\exp\left\{\ln\left(\frac{b}{a}\right)^c\right\}=\left(\frac{c}{b}\right)^a\left(\frac{a}{c}\right)^b\left(\frac{b}{a}\right)^c$$

Solution 5 by Ravi Prakash-New Delhi-India

$$\textit{For } x>0;\ \frac{x^4+1}{x^6+1}\leq\frac{1}{x}\Leftrightarrow x^5+x\leq x^6+1$$

$$\Leftrightarrow x^6-x^5+1-x\geq 0\Leftrightarrow x^5(x-1)-(x-1)\geq 0$$

$$\Leftrightarrow (x-1)(x^5-1)\geq 0\Leftrightarrow (x-1)^2(x^4+x^3+x^2+x+1)\geq 0$$

$$\therefore \int_b^c \frac{x^4+1}{x^6+1}dx\leq\int_b^c\frac{1}{x}dx=\ln\left(\frac{c}{b}\right)\Rightarrow e^{a\int_b^c\frac{x^4+1}{x^6+1}dx}\leq e^{a\ln\left(\frac{c}{b}\right)}=\left(\frac{c}{b}\right)^a$$

$$\textit{Similarly for other expressions. Thus } e^{\sum a\int_b^c\frac{x^4+1}{x^6+1}dx}\leq\left(\frac{c}{b}\right)^a\left(\frac{a}{c}\right)^b\left(\frac{b}{a}\right)^c$$

Solution 6 by Saptak Bhattacharya-Kolkata-India

$$\textit{By power mean: } \sqrt[6]{\frac{x^6+1}{2}}\geq\sqrt[4]{\frac{x^4+1}{2}}\Rightarrow\frac{(x^6+1)^2}{4}\geq\frac{(x^4+1)^3}{8}$$

$$\Rightarrow\left(\frac{x^4+1}{x^6+1}\right)^2\leq\frac{2}{(x^4+1)}\Rightarrow\frac{x^4+1}{x^6+1}\leq\frac{\sqrt{2}}{\sqrt{x^4+1}}$$

$$\textit{By AM}\geq\textit{GM: } x^4+1\geq 2x^2\Rightarrow\frac{1}{\sqrt{x^4+1}}\leq\sqrt{2}x.\ \textit{Thus,}\frac{x^4+1}{x^6+1}\leq\frac{1}{x}.\ \textit{So,}$$

$$e^{\sum a\int_b^c\frac{x^4+1}{x^6+1}dx}\leq\prod\left(e^{\int_b^c\frac{dx}{x}}\right)^6=\prod\left(\frac{c}{b}\right)^a$$

45. Prove that:

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\arctan x \arctan\frac{1}{x}\right)dx \le \frac{\pi^3}{96}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Abdallah Ell Farissi – Bechar – Algerie

Let $f(x) = \arctan(x)\arctan\left(\frac{1}{x}\right)$ ***for all*** $x \in (0,\infty)$ ***we have***

$$f'(x) = \frac{1}{x^2+1}\left(\arctan\left(\frac{1}{x}\right) - \arctan(x)\right)$$

then f ***increasing on*** $(0,1)$ ***and decreasing on*** $(0,+\infty)$ ***it follow that for all*** $x \in (0,+\infty), f(x) \le f(1) = \frac{\pi^2}{16}.$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\arctan(x)\arctan\left(\frac{1}{x}\right)dx \le \frac{\pi}{6}f(1) = \frac{\pi^3}{96}$$

Solution 2 by Anouy Chakraborty – India

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^{-1}x\tan^{-1}\frac{1}{x} \le \frac{\pi^3}{96};\ x \in \left[\frac{\pi}{6},\frac{\pi}{3}\right], \tan^{-1}x > 0$$

using AM ≤ AM, $\left(\tan^{-1}x\tan^{-1}\frac{1}{x}\right) \le \frac{\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)^2}{4}$ ***(*)***

$$\tan^{-1}\frac{1}{x} = \cos^{-1}x \ \times \tan^{-1}x + \cos^{-1}x = \frac{\pi}{2}$$

$$(*) \le \frac{\pi^2}{16}$$

It suffices to show that $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\frac{\pi^2}{16}dx \le \frac{\pi^3}{96} \Rightarrow \frac{\pi^2}{16} \times \frac{\pi}{6} \le \frac{\pi^3}{96}$*, which is true*

$$= \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \tan^{-1}x\tan^{-1}\frac{1}{x}dx \le \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{(\tan^{-1}x + \cos^{-1}x)^2}{4} = \frac{\pi^3}{96}$$

Solution 3 by Soumava Chakraborty – Kolkata – India

$$\textit{Let } \tan^{-1}x = \theta \therefore \tan^{-1}\frac{\pi}{6} \le \theta \le \tan^{-1}\frac{\pi}{3}$$

$$\therefore \frac{\pi}{2} - \tan^{-1}\left(\frac{\pi}{3}\right) \le \frac{\pi}{2} - \theta \le \frac{\pi}{2} - \tan^{-1}\left(\frac{\pi}{6}\right) \Rightarrow 0 < \frac{\pi}{2} - \theta < \frac{\pi}{2}$$

$$\textit{Now, } x = \tan\theta \Rightarrow \frac{1}{x} = \tan\left(\frac{\pi}{2} - \theta\right) \Rightarrow \tan^{-1}\left(\frac{1}{x}\right) = \frac{\pi}{2} - \theta \quad \left(\because 0 < \frac{\pi}{2} - \theta < \frac{\pi}{2}\right)$$

$$\therefore \tan^{-1}(x)\cdot\tan^{-1}\left(\frac{1}{x}\right) = \left(\sqrt{\theta\left(\frac{\pi}{2} - \theta\right)}\right)^2 \overset{AM-GM}{\le} \left(\frac{\theta + \frac{\pi}{2} - \theta}{2}\right)^2 = \frac{\pi^2}{16}$$

$$\therefore \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} (\tan^{-1}x)\left(\tan^{-1}\frac{1}{x}\right)dx \le \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\pi^2}{16}dx = \frac{\pi^2}{16}\left(\frac{\pi}{3} - \frac{\pi}{6}\right) = \frac{\pi^3}{96}$$

Solution 4 by Rovshan Pirguliyev – Sumgait – Azerbaidjian

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \arctan x \cdot \arctan\frac{1}{x}dx \le \frac{\pi}{96} \ (*)$$

$$x \in \left[\frac{\pi}{6}, \frac{\pi}{3}\right], \arctan x > 0, \arctan\frac{1}{x} > 0$$

$$\arctan x \cdot \arctan\frac{1}{x} \le \frac{(\arctan x + \arctan x)^2}{4}, \qquad \left(ab \le \frac{(a+b)^2}{4}\right)$$

$$f(x)=\arctan x+\arctan x;\ f'(x)=\frac{1}{1+x^2}-\frac{1}{1+x^2}=0\Rightarrow f(x)=\frac{\pi}{2}$$

$$\arctan x\cdot\arctan\frac{1}{x}\le\frac{\left(\frac{\pi}{2}\right)^2}{4}=\frac{\pi^2}{16}.$$ ***Hence (*) proved.***

Solution 5 by Soumitra Mandal - Chandar Nagore – India

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^{-1}\tan^{-1}\left(\frac{1}{x}\right)dx\le\frac{1}{4}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\left(\tan^{-1}x+\tan^{-1}\frac{1}{x}\right)^2dx$$

$$=\frac{\pi^2}{16}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}dx\left[\tan^{-1}x+\tan^{-1}\frac{1}{x}=\frac{\pi}{2}\right]=\frac{\pi^3}{96}$$

Solution 6 by Saptak Bhattacharya – Kolkata – India

$$\tan^{-1}x\cot^{-1}x\le\left(\frac{\tan^{-1}x+\cot^{-1}x}{2}\right)^2\ (AM\ge GM)=\frac{\pi^2}{16}.\ \text{So,}$$

$$\int_{\frac{\pi}{6}}^{\frac{\pi}{3}}\tan^{-1}x\cot^{-1}x\,dx\le\frac{\pi^2}{16}\cdot\left(\frac{\pi}{3}-\frac{\pi}{6}\right)=\frac{\pi^2}{96}$$

(Proved)

46. Let $f:[a,b]\to\mathbb{R}$ be continuously differentiable function and

$$\int_a^b f(x)\,dx=0.$$

Prove that

$$\left|\int_a^b xf(x)dx\right|\le\frac{(b-a)^3}{12}\max\{f'(x):x\in[a,b]\}$$

Proposed by Duong Viet Thong

Solution by Abdallah El Farissi – Bechar – Algerie

Let $M = \max\{f'(x), x \in [a,b]\}$ and $g(x) = f(x) - Mx$, clearly g is decreasing function then by Chebyshev inequality we have

$$\int_a^b xf(x)\,dx \le \frac{1}{b-a}\left(\int_a^b g(x)\,dx\right)\left(\int_a^b x\,dx\right)$$

then

$$\int_a^b xf(x)\,dx \le \frac{1}{b-a}\left(\int_a^b f(x) - Mxdx\right)\left(\int_a^b x\,dx\right) + M\int_a^b x^2\,dx$$

$$= M\left(\int_a^b x^2\,dx - \frac{1}{b-a}\left(\int_a^b x\,dx\right)^2\right) = M\frac{(b-a)^3}{12}.$$

47. Let $a, b, c > 0$. Prove that

$$a^2\int_0^b \frac{\arctan x}{x}dx + b^2\int_0^c \frac{\arctan x}{x}dx + c^2\int_0^a \frac{\arctan x}{x}dx < a^3 + b^3 + c^3$$

Proposed by Daniel Sitaru – Romania

Solution by Togrul Ehmedov – Baku – Azerbaidian

$$\frac{\tan^{-1}x - \tan^{-1}0}{x-0} = \frac{1}{1+\varepsilon^2} \le 1;\ \frac{\tan^{-1}x}{x} \le 1$$

$$\int_0^b \frac{\arctan x}{x}dx < \int_0^b dx = b$$

$$a^2\int_0^b \frac{\arctan x}{x}dx + b^2\int_0^c \frac{\arctan x}{x}dx + c^2\int_0^a \frac{\arctan x}{x}dx < a^2b + b^2c + c^2a$$

$$\begin{cases} a^3 + a^3 + b^3 \ge 3a^2b \\ b^3 + b^3 + c^3 \ge 3b^2c \\ c^3 + c^3 + a^3 \ge 3c^2a \end{cases} \Rightarrow a^3 + b^3 + c^3 \ge a^2b + b^3c + c^2a$$

$$a^2\int_0^b \frac{\arctan x}{x}dx + b^2\int_0^c \frac{\arctan x}{x}dx + c^2\int_0^a \frac{\arctan x}{x}dx < a^3 + b^3 + c^3$$

48. If $0 < a < b < \frac{\pi}{2}$ then:

$$\frac{1}{2}\int_a^b (\sin(\cos x) + \tan(\cos x))\,dx > \sin b - \sin a$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

$$\textit{Let } f(x) = \sin x + \tan x - 2x, 0 \le x \le 1$$

$$f'(x) = \cos x + \sec^2 x - 2 \ge 2\sqrt{\cos x \sec^2 x} - 2 = 2\left(\sqrt{\sec x} - 1\right) > 0$$

$$\textit{for } 0 < x < 1. \textit{ Thus, } f(x) > f(0) = 0, \textit{ for } 0 < x < 1$$

$$\Rightarrow \sin(\cos x) + \tan(\cos x) > 2\cos x, 0 < x < \frac{\pi}{2}$$

$$\Rightarrow \frac{1}{2}\int_a^b [\sin(\cos x) + \tan(\cos x)]\,dx > \int_a^b \cos x\,dx = \sin b - \sin a$$

$$\left[0 < a < b < \frac{\pi}{2}\right]$$

Solution 2 by Nishant Kumar – Jamshedpur – India

$$\textit{We know } \sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\sin x \approx x \text{ *for smaller value of* } x;\ \tan x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \cdots$$

$$\tan x > x \text{ *for smaller value of* } x:\ \sin x + \tan x > 2x$$

$$\sin(\cos x) + \tan(\cos x) > 2\cos x$$

$$\int_a^b \sin(\cos x) + \tan(\cos x)\,dx > 2\int_a^b \cos x\,dx$$

$$\textit{As } 0 < a < b < \frac{\pi}{2}$$

$$\int_a^b \sin(\cos x) + \tan(\cos x)\,dx > 2(\sin b - \sin b)\begin{Bmatrix} 0 < \cos a < 1 \\ 0 < \cos b < 1 \end{Bmatrix}$$

Solution 3 by Richdad Phuc-Hanoi-Vietnam

By Katsuura inequality: $\sin x + \tan x > 2x,\ x \in \left(0, \frac{\pi}{2}\right)$

$$\sin(\cos x) + \tan(\cos x) > 2\cos x$$

$$\int_a^b \sin(\cos x) + \tan(\cos x)\,dx > 2\int_a^b \cos x\,dx = 2(\sin b - \sin b)$$

$$\frac{1}{2}\int_a^b (\sin(\cos x) + \tan(\cos x))\,dx > \sin b - \sin a$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

We know $\sin x \ge x - \frac{x^3}{6}$ ***for all*** $x \ge 0$ ***and*** $\tan x \ge x + \frac{x^3}{3}$ ***for all*** $x \ge 0$

$$\therefore \frac{1}{2}\int_a^b (\sin(\cos x) + \tan(\cos x))\,dx \ge \frac{1}{2}\int_a^b \left(\cos x - \frac{\cos^3 x}{6} + \cos x + \frac{\cos^3 x}{3}\right)dx$$

$$= \frac{1}{2}\int_a^b \left(2\cos x + \frac{\cos^3 x}{6}\right)dx > \int_a^b \cos x\,dx = \sin b - \sin a$$

49. $a.\ 2\left(\int_3^4 e^{-x^2}dx\right)^2 + 2\left(\int_2^3 e^{-x^2}dx\right)^2 \geq \left(\int_2^4 e^{-x^2}dx\right)^2$

$$b.\left(\int_2^4 e^{-x^2}dx\right)^2 \geq \left(\int_3^4 e^{-x^2}\right)^2 + \left(\int_2^3 e^{-x^2}dx\right)^2$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

Let $a = \int_3^4 e^{-x^2}dx, b = \int_2^3 e^{-x^2}dx,\ a + b = \int_2^4 e^{-x^2}dx$

(a) $2a^2 + 2b^2 = (a+b)^2 + (a-b)^2 \geq (a+b)^2$

(b) $(a+b)^2 = a^2 + b^2 + 2ab \geq a^2 + b^2\ \ [\because ab > 0]$

50. If $a, b, c \geq 0$

$$\Omega(a) = \int_0^a \sqrt{\frac{x(x^2+x+1)}{(x+1)(x^4+x^2+1)}}dx$$

then:

$$\left(\Omega(a) + \Omega(b) + \Omega(c)\right)^3 \geq \prod \log\left(\sqrt{a^3+1} + \sqrt{a^3}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\Omega(a) = \int_0^a \frac{\sqrt{x(x^2+x+1)}}{\sqrt{x+1}\sqrt{x^4+x^2+1}}$$

$$x^4 + x^2 + 1 = (x^2+1)^2 - x^2 = (x^2+x+1)(x^2-x+1)$$

$$\therefore \Omega(a)=\int_0^a\frac{\sqrt{x}}{\sqrt{(x+1)(x^2-x+1)}}dx=\int_0^a\frac{\sqrt{x}}{\sqrt{\left(x^{\frac{3}{2}}\right)^2+1}}dx$$

$$\textit{Put } x^{\frac{3}{2}}=t, \frac{3}{2}\sqrt{x}dx=dx$$

$$\therefore \Omega(a)=\frac{2}{3}\int_0^{a^{\frac{3}{2}}}\frac{dt}{\sqrt{t^2+1}}=\frac{2}{3}\ln\left(t+\sqrt{t^2+1}\right)\Big|_0^{a^{\frac{3}{2}}}=\frac{2}{3}\ln\left(\sqrt{a^3+1}+\sqrt{a^3}\right)$$

$$\textit{Now, } \Omega(a)+\Omega(b)+\Omega(c)\ge 3\left[\left(\Omega(a)\right)\left(\Omega(b)\right)\left(\Omega(c)\right)\right]^{\frac{1}{3}}$$

$$\Rightarrow [\Omega(a)+\Omega(b)+\Omega(c)]^3\ge 27\Omega(a)\Omega(b)\Omega(c)$$

$$=8\prod\left[\log\sqrt{a^3+1}+\sqrt{a^3}\right]\ge\prod\left[\log\sqrt{(a^3+1)}+\sqrt{a^3}\right]$$

Solution 2 by Saptak Bhattacharya-Kolkata-India

$$4>1\Rightarrow 4(x+1)(x^4+x^2+1)>(x+1)(x^4+x^2+1)$$

$$\Rightarrow 4(x+1)(x^2+x+1)(x^2-x+1)>(x+1)(x^4+x^2+1)$$

$$\Rightarrow 4(x^3+1)(x^2+x+1)>(x+1)(x^4+x^2+1)$$

$$\Rightarrow \frac{x(x^2+x+1)}{(x+1)(x^4+x^2+1)}>\frac{x}{4(x^3+1)}$$

$$\Rightarrow 3\sqrt{\frac{x(x^2+x+1)}{(x+1)(x^4+x^2+1)}}>\frac{3x^2}{2\sqrt{x^3(x^3+1)}}$$

$$3\sqrt{\frac{x(x^2+x+1)}{(x+1)(x^4+x^2+1)}}>\frac{1}{\sqrt{x^3+1}+\sqrt{x^3}}\left(\frac{3x^2}{2\sqrt{x^3+1}}+\frac{3x^2}{2\sqrt{x^3}}\right)$$

$$\Rightarrow \textit{Integrating from } 0 \textit{ to } a,$$

$$3\Omega(a) > \int_0^a \frac{1}{\sqrt{x^3+1}+\sqrt{x^2}}\left(\frac{3x^2}{2\sqrt{x^2+1}}+\frac{3x^2}{2\sqrt{x^3}}\right)dx$$

$$\Rightarrow 3\Omega(a) > \int_0^{\ln\left(\sqrt{a^3+1}+\sqrt{a^3}\right)} d\left\{\ln\left(\sqrt{x^3+1}+\sqrt{x^3}\right)\right\}$$

$$\Rightarrow 3\Omega(a) > \ln\left(\sqrt{a^3+1}+\sqrt{a^3}\right).$$ ***Thus,***

$$27\prod\Omega(a) > \prod\left(\ln\sqrt{a^3+1}+\sqrt{a^3}\right).$$ ***But by AM ≥ GM;***

$$\left(\sum\Omega(a)\right)^3 \ge 27\prod\Omega(a) > \prod\ln\left(\sqrt{a^3+1}+\sqrt{a^3}\right)$$

51. $\int_0^1 \sqrt{x}\sin x\,dx > \frac{49}{135}$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

Let $f(x) = \sin x - \left(x-\frac{x^3}{6}\right), 0 \le x \le 1;\ f'(x) = \cos x - \left(1-\frac{1}{2}x^2\right)$

$f''(x) = -\sin x + x > 0$ ***for*** $0 < x < 1 \Rightarrow f'(x) \uparrow$ ***on*** $[0,1]$

$\Rightarrow f'(x) > f'(0) = 0$ ***for*** $0 < x \le 1 \therefore f(x) \uparrow$ ***on*** $[0,1]$

$\Rightarrow f(x) > 0$ ***for*** $0 < x \le 1 \therefore \sin x > x - \frac{1}{6}x^3, 0 < x \le 1$

$$\Rightarrow \int_0^1 \sqrt{x}\sin x\,dx > \int_0^1 \left(x^{\frac{3}{2}} - \frac{1}{6}x^{\frac{7}{2}}\right)dx = \frac{2}{5} - \frac{2}{6\times 9} = \frac{49}{135}$$

Solution 2 by Saptak Bhattacharya-Kolkata-India

$\int_0^1 \sqrt{x}\sin x\,dx = 2\int_0^1 \frac{x\sin x}{2\sqrt{x}}dx$;***Put*** $\sqrt{x} = t$;

$$= 2\int_0^1 t^2 \sin t^2\, dt$$

Now, $\sin k > k - \frac{k^3}{3!}$*. So,* $\sin t^2 > t^2 - \frac{t^6}{6} \Rightarrow 2t^2 \sin t^2 > 2t^4 - \frac{2t^8}{6}$

$$\Rightarrow 2\int_0^1 t^2 \sin t^2\, dt > 2\int_0^1 \left(t^4 - \frac{t^8}{6}\right) dt = \frac{2 \times 49}{54 \times 5} = \frac{49}{135}$$

52. $a, b \in \left(0, \frac{\pi}{2}\right), a < b$

Prove that exists $\alpha, \beta \in (a, b)$ such that:

$$\left(\int_a^b \tan x\, dx\right)\left(\int_a^b \cot x\, dx\right) \le \frac{(b-a)^2}{\sin\beta \cos\alpha}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = \frac{1}{\cos a}$ *for all* $x \in \left(0, \frac{\pi}{2}\right)$ *and* $g(x) = 1$ *for all* $x \in \left(0, \frac{\pi}{2}\right)$

1. $f(x)$ *and* $g(x)$ *are integrable on* $\left(0, \frac{\pi}{2}\right)$

2. $g(x)$ *keeps the same sign on* $\left(0, \frac{\pi}{2}\right)$

$$\therefore \int_a^b \tan x\, dx \le \int_a^b \frac{dx}{\cos x} = \frac{1}{\cos x}(b-a)$$

where $\alpha \in (a, b)$

Similarly, $\int_a^b \cot x\, dx \le \int_a^b \frac{dx}{\sin x} = \frac{b-a}{\sin\beta}$*. where* $\beta \in (a, b)$

$$\therefore \left(\int_a^b \tan x\,dx\right)\left(\int_a^b \cot x\,dx\right) \le \frac{(b-a)^2}{\sin\beta\cos\alpha}$$

53. $0 < p < q < \frac{\pi}{2}; f:\mathbb{R}\to\mathbb{R}$ **a continuous function,**

$$\int_{p\sin t}^{q\cos t} f(x)\,dx \le \int_{\frac{p}{\sqrt{2}}}^{\frac{q}{\sqrt{2}}} f(x)\,dx, \forall t \in \left(0,\frac{\pi}{2}\right)$$

Prove that: $qf\left(\frac{q}{\sqrt{2}}\right) + pf\left(\frac{p}{\sqrt{2}}\right) = 0$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Let $F(x) = \int_0^x f(t)\,dt$

As f **is continuous on** $\mathbb{R}$, F **is differentiable on** $\mathbb{R}$, **and** $F'(x) = f(x)$

Let $G(t) = \int_{p\sin t}^{q\cos t} f(x)\,dx, t \in \left(0,\frac{\pi}{2}\right) = F(q\cos t) - F(p\sin t)$

we have G **is differentiable on** $\left(0,\frac{\pi}{2}\right)$ **and** $G(t) \le G\left(\frac{\pi}{4}\right)\ \forall t \in \left(0,\frac{\pi}{2}\right)$

$\Rightarrow G$ **attains maximum value at** $t = \frac{\pi}{4}$

$$\therefore G'\left(\frac{\pi}{4}\right) = 0 \Rightarrow -q\sin\left(\frac{\pi}{4}\right)F'\left(\frac{q}{\sqrt{2}}\right) - b\cos\frac{\pi}{4}F'\left(\frac{p}{\sqrt{2}}\right) = 0$$

$$\Rightarrow qf\left(\frac{q}{\sqrt{2}} + bf\left(\frac{p}{\sqrt{2}}\right)\right) = 0.$$

54. **From the book “Math Phenomenon”**

$$\int_{-1}^{1}\sqrt{1-x^2}\cos^{-1}x\,dx > \frac{e^2}{4}$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\textit{Let } \cos^{-1}x = \theta, \theta \in [0,\pi]$$

$$\int_{-1}^{1}\sqrt{1-x^2}\cos^{-1}x\,dx \Rightarrow x = \cos\theta \quad (\because \textit{in } [0,\pi], \sin\theta \geq 0)$$

$$\Rightarrow \sqrt{1-x^2} = \sqrt{\sin^2\theta} = \sin\theta\,;\ dx = -\sin\theta d\theta = \int_{\pi}^{0}\theta\sin\theta(-\sin\theta)\,d\theta$$

$$= -\frac{1}{2}\int_{0}^{\pi}\theta(-2\sin\theta)d\theta = -\frac{1}{2}\int_{0}^{\pi}\theta(\cos 2\theta - 1)d\theta$$

$$= -\frac{1}{2}\left(\int_0^\pi \theta\cos\theta\,d\theta - \int_0^\pi \theta d\theta\right) \rightarrow \textit{(1)}$$

$$\int \theta\cos 2\theta\,d\theta = \theta\int \cos 2\theta\,d\theta - \frac{1}{2}\int \sin 2\theta\,d\theta$$

$$= \frac{\theta}{2}(\sin 2\theta) + \frac{1}{4}\cos 2\theta + c \therefore \int_0^\pi \theta\cos 2\theta\,d\theta = \left[\frac{\theta\sin 2\theta}{2} + \frac{\cos 2\theta}{4}\right]_0^\pi$$

$$= \left(\frac{\pi}{2}\sin 2\pi + \frac{\cos 2\pi}{4}\right) - \left(\frac{0\cdot\sin 0}{2} + \frac{\cos 0}{4}\right) = 1 - 1 = 0$$

$$\therefore \textit{(1)} \Rightarrow \int_{-1}^{1}\sqrt{1-x^2}\cos^{-1}x\,dx = -\frac{1}{2}\left(0 - \frac{1}{2}[\theta^2]_0^\pi\right)$$

$$= -\frac{1}{2}\left(-\frac{\pi^2}{2}\right) = \frac{\pi^2}{4} \because \pi > e \therefore \frac{\pi^2}{4} > \frac{e^2}{4}$$

55. **Solve for real numbers:**

$$\begin{cases} x^2+\sqrt{y^2+12}=\sqrt{y^2+60} \\ y^2+\sqrt{z^2+12}=\sqrt{z^2+60} \\ z^2+\sqrt{x^2+12}=\sqrt{x^2+60} \end{cases}$$

Proposed at Spanish-TST

Solution 1 by Abdallah El Farissi-Bechar-Algerie

Let $f(x)=\sqrt{x+60}-\sqrt{x+12}, x\in[0,+\infty[$***, we note that*** $x=4$ ***is an unique fixed point of*** f ***(***f ***is contractant function on*** $[0,+\infty[$,

$$|f(x)-f(y)|\leq\left(\frac{1}{4\sqrt{15}}+\frac{1}{4\sqrt{3}}\right)|x-y|\leq\frac{1}{2}|x-y|)$$

Let $g(x)=f\circ f\circ f(x)$ ***we have*** $x=4$ ***is an unique fixed point of*** g ***(***f ***is contractant function then*** g ***too)***

$$\begin{cases} x^2=\sqrt{y^2+60}-\sqrt{y^2+12} \\ y^2=\sqrt{z^2+60}-\sqrt{z^2+12} \\ z^2=\sqrt{x^2+60}-\sqrt{x^2+12} \end{cases}\Rightarrow\begin{cases} x^2=f(y^2) \\ y^2=f(z^2) \\ z^2=f(x^2) \end{cases}\Rightarrow\begin{cases} x^2=g(x^2) \\ y^2=g(y^2) \\ z^2=g(z^2) \end{cases}$$

then $x^2=y^2=z^2=4$***. The set of solutions of the system is***

$$A=\{(x,y,z),x,y,z\in\{-2,2\}\}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

Find all $x,y,z\in\mathbb{R}$ ***satisfying:***

$$x^2+\sqrt{y^2+12}=\sqrt{y^2+60}\rightarrow \textbf{(a)}$$

$$y^2+\sqrt{z^2+12}=\sqrt{z^2+60}\rightarrow \textbf{(b)}$$

$$z^2+\sqrt{x^2+12}=\sqrt{x^2+60}\rightarrow \textbf{(c)}$$

$$\sqrt{a^2+60}-\sqrt{a^2+12} > \textbf{or} < 4\Leftrightarrow\sqrt{a^2+60} > \textbf{or} < 4+\sqrt{a^2+12}$$

$$\Leftrightarrow a^2+60 > \textbf{or} < 16+a^2+12+8\sqrt{a^2+12}\Leftrightarrow 4 > \textbf{or} < \sqrt{a^2+12}$$

$$\Leftrightarrow 16 > \textbf{or} < a^2+12\Leftrightarrow a^2 < \textbf{or} > 4$$

$\therefore \sqrt{a^2+60}-\sqrt{a^2+12}>4 \Leftrightarrow a^2<4 \rightarrow$ ***(1)***

and $\sqrt{a^2+60}-\sqrt{a^2+12}<4 \Leftrightarrow a^2>4 \rightarrow$ ***(2)***

Let us assume $x^2>4 \therefore$ ***(a)*** $\Rightarrow \sqrt{y^2+60}-\sqrt{y^2+12}>4 \Rightarrow y^2<4$ ***(by (1))***

$\therefore$ ***(b)*** $\Rightarrow \sqrt{z^2+60}-\sqrt{z^2+12}<4 \Rightarrow z^2>4$ ***(by (2))***

$\therefore$ ***(c)*** $\Rightarrow \sqrt{x^2+60}-\sqrt{x^2+12}>4 \Rightarrow x^2<4$ ***(by (1)), thus leading to a condition. Hence,*** $x^2 \ngtr 4 \rightarrow$ ***(i)***

Similarly, if we assume $x^2<4$***, we shall obtain*** $x^2>4$***, this again leading to a contradiction. Hence*** $x^2 \nless 4 \rightarrow$ ***(ii)***

(i), (ii) $\Rightarrow x^2=4 \therefore$ ***(c)*** $\Rightarrow z^2=4 \therefore$ ***(b)*** $\Rightarrow y^2=4$

$$\therefore \begin{pmatrix} x=2\\ y=2\\ z=2\end{pmatrix}, \begin{pmatrix} x=2\\ y=2\\ z=-2\end{pmatrix}, \begin{pmatrix} x=2\\ y=-2\\ z=2\end{pmatrix}, \begin{pmatrix} x=2\\ y=-2\\ z=-2\end{pmatrix}, \begin{pmatrix} x=-2\\ y=2\\ z=2\end{pmatrix}, \begin{pmatrix} x=-2\\ y=2\\ z=-2\end{pmatrix}, \begin{pmatrix} x=-2\\ y=-2\\ z=2\end{pmatrix}, \begin{pmatrix} x=-2\\ y=-2\\ z=-2\end{pmatrix}$$

are all possible solutions.

56. If $a,b,c>0$ then:

$$\int_{-a}^{a}\frac{e^{x^2}+e^{-x^2}}{2^x+1}dx+\int_{-b}^{b}\frac{e^{x^2}+e^{-x^2}}{3^x+1}+\int_{-c}^{c}\frac{e^{x^2}+e^{-x^2}}{5^x+1}dx>6\sqrt[3]{abc}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Chris Kyriazis-Greece

Let's say that $I_1=\int_{-a}^{a}\frac{e^{x^2}+e^{-x^2}}{2^x+1}dx$ ***(1)***

We transform as: $u=-x$***. Then easy*** $I_1=\int_{-a}^{a}\frac{2^x\left(e^x+e^{-x^2}\right)}{2^x+1}dx$ ***(2)***

So by (1) + (2)

$$2I_1 = \int_{-a}^{a}\left(e^x + e^{-x^2}\right)dx \geq \int_{-a}^{a} 2dx = 4a \Rightarrow I_1 \geq 2a$$

(since $e^x + e^{-x^2} \geq 2, \forall x \in \mathbb{R}$). Working on the same idea, we have

$$I_2 = \int_{-b}^{b}\frac{x^{x^2} + e^{-x^2}}{3^x + 1}dx \Rightarrow \cdots 2I_2 \geq 4b = 0$$

$I_2 \geq 2b$ ***and finally if*** $I_3 = \int_{-c}^{c}\frac{e^{x^2}+e^{-x^2}}{5^x+1} \Rightarrow I_3 \geq 2c$

So: $I_1 + I_2 + I_3 = 2(a+b+c) \overset{AM-GM}{\geq} 6\sqrt[3]{abc}$

Solution 2 by Ravi Prakash-New Delhi-India

For $a > 0, k > 0$***, let*** $I = \int_{-k}^{k}\frac{e^{x^2}+e^{-x^2}}{a^x+1}$ ***(1)***

Put $x = -t$***, so that*** $I = \int_{k}^{-k}\frac{e^{t^2}+e^{-t^2}}{a^{-t}+1}(-1)dt = \int_{-k}^{k}\frac{a^x\left(e^{x^2}+e^{-x^2}\right)}{a^x+1}dx$ ***(2)***

Adding (1) and (2), we get

$$2I = \int_{-k}^{k}\left(e^{x^2} + e^{-x^2}\right)dx \Rightarrow I = \int_{0}^{k}\left(e^{x^2} + e^{-x^2}\right)dx \geq 2\int_{0}^{k} = 2k$$

$$\therefore \int_{-a}^{a}\frac{e^{x^2} + e^{-x^2}}{1 + 2^x}dx + \int_{-b}^{b}\frac{e^{x^2} + e^{-x^2}}{1 + 3^x}dx + \int_{-c}^{c}\frac{e^{x^2} + e^{-x^2}}{1 + 5^x}dx$$

$$\geq 2(a+b+c) \geq 6(abc)^{\frac{1}{3}}$$

57. **If $a > 0, f:[0,a] \to \mathbb{R}, f'(a) = f(a) = 0,$**

$$f \in C^2([0,a]), f'f'' \in C^1([0,a]) \text{ then:}$$

$$60\left(\int_0^a f(x)\,dx\right)^4 \le a^8\left(\int_0^a (f'(x))^2 dx\right)\left(\int_0^a (f''(x))^2 dx\right)$$

Proposed by Daniel Sitaru – Romania

Solution by Chris Kyriazis-Greece and independently by Soumitra Mandal-Chandar Nagore-India

$$f(a) = f'(a) = 0$$

$$\left(\int_0^a f(x)dx\right)^2 = \left([xf(x)]_{x=0}^{x=a} - \int_0^a xf'(x)dx\right)^2 \le \left(\int_0^a x^2dx\right)\left(\int_0^a (f'(x))^2dx\right)$$

$$= \frac{a^3}{3}\left(\int_0^a \left(f'(x)\right)^2 dx\right). \textit{ again,}$$

$$\left(\int_0^a f(x)dx\right)^2 = \left(\int_0^a xf'(x)dx\right)^2 = \left(\left[\frac{x^2}{2}f'(x)\right]_{x=0}^{x=a} - \frac{1}{2}\int_0^a x^2f''(x)dx\right)^2$$

$$= \left(\frac{1}{2}\int_0^a x^2f''(x)dx\right)^2 \le \frac{1}{4}\left(\int_0^a x^4dx\right)\left(\int_0^a (f''(x))^2dx\right) = \frac{a^5}{20}\left(\int_0^a (f''(x))^2dx\right)$$

$$\therefore \left(\int_0^a f(x)dx\right)^4 \le \frac{a^8}{60}\left(\int_0^a (f'(x))^2dx\right)\left(\int_0^a (f''(x))^2dx\right)$$

58. If $a, b, c, d > 0, a + b + c + d = \pi$

$$\Omega(a) = \int_a^{2a} \frac{\arctan(x+1)}{x} dx$$

then: $\Omega(a) + \Omega(b) + \Omega(c) + \Omega(d) < \pi(1 + \log 2)$

Proposed by Daniel Sitaru – Romania

Solution 1 by Anas Adlany-Zemamra-Morroco

We have known that for any $x \geq 0, \tan(x) \leq x$ ***that is:***

$$\sum \Omega(a) \leq \sum \int_a^{2a} \frac{\arctan(x+1)}{x} dx \leq \sum \int_a^{2a} \frac{x+1}{x} dx =$$

$$= \sum a + \ln(2) < \pi(1 + \log(2)).$$ ***Hence proved.***

Solution 2 by Michel Rebeiz-Hamra-Lebanon

$$f(x) = \tan^{-1}(x+1);\ f'(x) > 0$$

x	$-\infty$	∞
$f'(x)$	+	
$f(x)$	$-\frac{\pi}{2}$ ↗	$\frac{\pi}{2}$

$$f(x) < \frac{\pi}{2} \text{ and } a < x < 2a \to x > 0;\ \frac{f(x)}{x} = \frac{\frac{\pi}{2}}{x}$$

$$\int_a^{2a} \frac{f(x)}{x} dx < \frac{\pi}{2} \int_a^{2a} \frac{1}{x} dx;\ \Omega(a) < \frac{\pi}{2} \ln 2;\ \sum \Omega(a) < 4 \times \frac{\pi}{2} \ln 2$$

$$\sum \Omega(a) < 2\pi \ln 2 < \pi + \pi \ln 2$$

59. Find:

$$L=\lim_{n\to\infty}\left(\int_0^{\left(\frac{\pi}{2}\right)^n}\sin\left(\sqrt[n]{x}\right)dx\right)$$

Proposed by Regragui El Khammal-Morocco

Solution by Daniel Sitaru-Romania

$$x-\frac{x^3}{6}<\sin x\,,\ \sqrt[n]{x}-\frac{\sqrt[n]{x^3}}{6}<\sin\left(\sqrt[n]{x}\right)$$

$$\frac{x^{\frac{1}{n}+1}}{\frac{1}{n}+1}\Bigg|_0^{\left(\frac{\pi}{2}\right)^n}-\frac{1}{6}\cdot\frac{x^{\frac{3}{n}+1}}{\frac{3}{n}+1}\Bigg|_0^{\left(\frac{\pi}{2}\right)^n}<\int_0^{\left(\frac{\pi}{2}\right)^n}\sin\left(\sqrt[n]{x}\right)dx$$

$$\frac{\left(\frac{\pi}{2}\right)^{n+1}}{\frac{n+1}{n}}-\frac{1}{6}\cdot\frac{\left(\frac{\pi}{2}\right)^{n+3}}{\frac{n+3}{n}}<\int_0^{\left(\frac{\pi}{2}\right)^n}\sin\left(\sqrt[n]{x}\right)dx$$

$$\infty=\lim_{n\to\infty}\left(\frac{\pi}{2}\right)^{n+1}\cdot\left(\frac{1}{\frac{n+1}{n}}-\frac{\left(\frac{\pi}{2}\right)^2}{\frac{n+3}{n}}\right)\le L\Rightarrow L=\infty$$

60. $\Omega_n=\lim_{x\to 0}\left(\frac{1}{(2^x-1)^n}-\frac{1}{\left(\frac{x\log 2}{1!}+\frac{(x\log 2)^2}{2!}+\cdots+\frac{(x\log 2)^n}{n!}\right)^n}\right);n\in\mathbb{N}^*$

Find:

$$\Omega=\lim_{n\to\infty}\sum_{k=1}^{n}\Omega_k$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Let: $a = \frac{x\log 2}{1!} + \frac{x^2(\log 2)^2}{2!} + \cdots + \frac{x^n(\log 2)^n}{n!}$; $b = \frac{x^{n+1}(\log 2)^{n+1}}{(n+1)!} + \cdots$ *.Now,*

$$a + b = \sum_{k=1}^{\infty}\frac{(x\log 2)^k}{k!} = e^{x\log 2} - 1 = 2^x - 1 \Rightarrow (a+b)^n = (2^x - 1)^n$$

$$\Omega_n = \lim_{x\to 0}\left[\frac{1}{(a+b)^n} - \frac{1}{a^n}\right] = \lim_{x\to 0}\left[\frac{-\binom{n}{1}a^{n-1}b - \binom{n}{2}a^{n-2}b^2 - \cdots - b^n}{(a+n)^n a^n}\right]$$

Coefficient of x^{2n} *in* $a^{n-1}b$ *in the numerator:* $-\frac{(\log 2)^{n-1}(\log 2)^{n+1}}{(n+1)!}$

Coefficient of x^{2n} in $(a+b)^n a^n$ *is:* $\frac{(\log 2)^n(\log 2)^n}{(1!)^n(1!)^n}$

Also, on the terms except first in the numerator involve x^{3n} *and higher powers of* x. $\Omega_n = -\binom{n}{1}\frac{1}{(n+1)!}\ \forall\, n \ge 1$; $\Omega_n = -\frac{(n+1-1)}{(n+1)!}$;

$$\Omega_n = -\left(\frac{1}{n!} - \frac{1}{(n+1)!}\right) \Rightarrow \sum_{n=1}^{\infty}\Omega_n = -\left(\frac{1}{1!} - \frac{1}{2!}\right) - \left(\frac{1}{2!} - \frac{1}{3!}\right)\ldots \Rightarrow \Omega = -1$$

61. $A = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}, A^n = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}, n \in \mathbb{N}^*$

$$\Omega = \lim_{n\to\infty}\left(\frac{a_n + b_n}{c_n + d_n}\right)$$

Prove that: $\Omega < 1$

Proposed by Daniel Sitaru – Romania

Solution 1 by Francis Fregeau – Quebec – Canada

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \qquad \lambda_1 = \frac{1+\sqrt{5}}{2}; \lambda_2 = \frac{1-\sqrt{5}}{2}$$

$$\Rightarrow E_1 = \binom{-1+\sqrt{5}}{2}; E_2 = \binom{-1-\sqrt{5}}{2}$$

$$\therefore M = E_1 \cup E_2 \Rightarrow A = MDM^{-1};\ D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow A^n = MD^nM^{-1}$$

$$= \frac{1}{10}\begin{bmatrix} 5\lambda_1^n - \sqrt{5}\lambda_1^n + 5\lambda_2^n + \sqrt{5}\lambda_2^n & 5\lambda_1^n - 3\sqrt{5}\lambda_1^n + 5\lambda_2^n + 3\sqrt{5}\lambda_2^n \\ 2\sqrt{5}\lambda_1^n - 2\sqrt{5}\lambda_2^n & -5\lambda_1^n + \sqrt{5}\lambda_1^n - 5\lambda_2^n - \sqrt{5}\lambda_2^n \end{bmatrix}$$

$$\lim_{n\to\infty} \lambda_2^n = 0 \therefore L = \frac{1}{10}\lim_{n\to\infty} \frac{10\lambda_1^n - 4\sqrt{5}\lambda_1^n}{-5\lambda_1^n + 3\sqrt{5}\lambda_1^n} = \frac{10 - 4\sqrt{5}}{10(3\sqrt{5}-5)} < 1$$

Solution 2 by Manish Moryani – Satna – India

Let be the sequence $f_0 = 0; f_1 = 1; f_{n+2} = f_n + f_{n+1}$ ***with the characteristic equation*** $\lambda^2 - \lambda - 1 = 0$

$$\lambda_{1,2} = \frac{1 \pm \sqrt{5}}{2}; f_n = \alpha\left(\frac{1+\sqrt{5}}{2}\right)^n + \beta\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$$f_0 = \alpha + \beta = 0; f_1 = \alpha \cdot \frac{1+\sqrt{5}}{2} + \beta \cdot \frac{1-\sqrt{5}}{2} = 1$$

$$\alpha = \frac{2}{\sqrt{5}}; \beta = -\frac{2}{\sqrt{5}};\ f_n = \frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^n - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$A = \begin{pmatrix} f_0 & f_1 \\ f_1 & f_2 \end{pmatrix}$ ***and we prove by induction that:***

$$A^n = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}; n \geq 1;\ A^{n+1} = A^n \cdot A = \begin{pmatrix} f_{n-1} & f_n \\ f_n & f_{n+1} \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix} =$$

$$= \begin{pmatrix} f_n & f_{n-1} + f_n \\ f_{n-1} + f_n & f_n + f_{n+1} \end{pmatrix} = \begin{pmatrix} f_n & f_{n+1} \\ f_{n+1} & f_{n+2} \end{pmatrix} = \begin{pmatrix} a_n & b_n \\ c_n & d_n \end{pmatrix}$$

$$L = \lim_{n\to\infty}\left(\frac{a_n + b_n}{c_n + d_n}\right) = \lim_{n\to\infty} \frac{f_n + f_{n+1}}{f_{n+1} + f_{n+2}} = \lim_{n\to\infty} \frac{f_{n+2}}{f_{n+3}} =$$

$$= \lim_{n\to\infty} \frac{\frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+2} - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+2}}{\frac{2}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+3} - \frac{2}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+3}} = \frac{2}{\sqrt{5}+1} = \frac{\sqrt{5}-1}{2} < 1$$

62. Find:

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} e^{-\left(\frac{k}{n}\right)^2}}$$

Proposed by Daniel Sitaru – Romania

Solution by Togrul Ehmedov – Baku – Azerbaidian

$$\Omega = \lim_{n\to\infty} \sqrt[n]{\prod_{k=1}^{n} e^{-\left(\frac{k}{n}\right)^2}} = \lim_{n\to\infty} \sqrt[n]{e^{-\left[\frac{1^2+2^2+\cdots+n^2}{n^2}\right]}} =$$

$$= \lim_{n\to\infty} e^{-\left[\frac{1^2+2^2+\cdots+n^2}{n^3}\right]} = e^{-\lim\limits_{n\to\infty}\left(\left[\frac{n(n+1)(2n+1)}{6n^3}\right]\right)} = e^{-\frac{1}{3}}$$

63. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{1+n^{2016}} + \frac{1}{2+n^{2016}} + \cdots + \frac{1}{n^{4032}}\right)\cdot \ln n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumitra Mandal – Kolkata – India

$$\Omega = \lim_{n\to\infty}\left(\frac{1}{1+n^{2016}} + \frac{1}{2+n^{2016}} + \cdots + \frac{1}{n^{4032}}\right)\ln n$$

We have, $\frac{1}{1+n^{2016}} < 1, \frac{1}{2+n^{2016}} < \frac{1}{2}, \dots, \frac{1}{n^{4032}} < \frac{1}{n}$

Similarly, $-1<\frac{1}{1+n^{2016}},-\frac{1}{2}<\frac{1}{2+n^{2016}},\dots,-\frac{1}{n}<\frac{1}{n^{4032}}$

$$\therefore -\frac{1}{n}-\dots-\frac{1}{2}-1<\frac{1}{1+n^{2016}}+\frac{1}{2+n^{2016}}+\dots+\frac{1}{n^{4032}}<$$

$$<1+\frac{1}{2}+\dots+\frac{1}{n}$$

$$\therefore -\ln n\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)<\ln\left(\frac{1}{1+n^{2016}}+\frac{1}{2+n^{2016}}+\dots+\frac{1}{n^{4032}}\right)$$

$$<\ln n\left(1+\frac{1}{2}+\dots+\frac{1}{n}\right)$$

$$-\ln n\,(\gamma_n+\ln n)<\Omega<\ln n\,(\gamma_n+\ln n)$$

where $\gamma_n=1+\frac{1}{2}+\frac{1}{3}+\dots+\frac{1}{n}-\ln n$ ***Euler's Constant***

$|\Omega-\ln n\,(\ln n+\gamma_n)|<\varepsilon$ ***will hold. only if there exists a*** k ***such that***

$k=[\ln^2 n]+[\ln n\cdot\gamma_n]$ ***where*** $[x]$ =***gif function***

$$\therefore \lim_{n\to\infty}\Omega=\lim_{n\to\infty}\ln n\,(\ln n+\gamma_n)=\lim_{n\to\infty}(\ln n)^2+\gamma\lim_{n\to\infty}\ln n$$

where γ = ***Euler's Constant***

$$\therefore \lim_{n\to\infty}\left(\frac{1}{1+n^{2016}}+\frac{1}{2+n^{2016}}+\dots+\frac{1}{n^{4082}}\right)\ln n=$$

$$=\lim_{n\to\infty}(\ln n)^2+\gamma\lim_{n\to\infty}\ln n=+\infty$$

Solution 2 by Redwane El Mellass – Casablanca – Morocco

$\therefore$ ***for*** $n\geq 2$,

$$\sum_{k=1}^{n^{4036}-n^{2016}}\frac{1}{k+n^{2016}}\ln n>\frac{(n^{4032}-n^{2016})}{n^{4032}}\ln n=\left(1-\frac{1}{n^{2016}}\right)\ln n>\frac{\ln n}{2}$$

$$\Rightarrow \lim_{n\to+\infty}\sum_{k=1}^{n^{4036}-n^{2016}}\frac{1}{k+n^{2016}}\ln n=+\infty$$

64. Find:

$$\Omega = \lim_{n\to\infty} n\tan\left(\frac{\pi e n!}{5}\right)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Rovsen Pirguliyev – Sumgait – Azerbaidian

$$\Omega = \lim_{n\to\infty} ntan\left(e\cdot\frac{\pi}{5}\cdot n!\right) = \lim_{n\to\infty} ntan\left(\frac{\pi}{5}\cdot n!\cdot\sum_{k=0}^{n+1}\frac{1}{k!}+\frac{\pi}{5}\cdot n!\cdot\sum_{k=n+2}^{\infty}\frac{1}{k!}\right) =$$

$$= \lim_{n\to\infty} ntan\left(m\pi+\frac{\pi}{5(n+1)}\right) = \lim_{n\to\infty} ntan\left(\frac{\pi}{5(n+1)}\right) = \frac{\pi}{5}$$

Observation:

$$0 < n!\sum_{k=n+2}^{\infty}\frac{1}{k!} = n!\left(\frac{1}{(n+2)!}+\frac{1}{(n+3)!}+\cdots\right) =$$

$$= \frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots < \frac{1}{(n+1)(n+2)}+\frac{1}{(n+2)(n+3)}+\cdots$$

$$< \left(\frac{1}{n+1}-\frac{1}{n+2}\right)+\left(\frac{1}{n+2}-\frac{1}{n+3}\right)+\cdots < \frac{2}{n+1}$$

Solution 2 by Anas Adlany-El Jadida-Morocco

Using the fact that $e = 1+\frac{1}{2}+\cdots+\frac{1}{n!}+\cdots$***. We obtain***

$$\tan\left(\frac{\pi}{5}n!\,e\right) = \tan\left(\frac{\pi}{5}\underbrace{n!\left(1+\frac{1}{2}+\cdots+\frac{1}{n!}+\cdots\right)}_{\epsilon_n}\right) = \tan\left(\pi\cdot\frac{1}{5}\epsilon_n\right)$$

Where $\begin{cases}\epsilon_n = \frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\cdots\\ \epsilon_n\to 0\end{cases}$***. Since*** $\frac{1}{n+1}<\epsilon_n<\frac{1}{n}$***. Hence***

$$\lim_{\infty} n\tan\left(\frac{\pi}{5}n!\,e\right) = \lim_{\infty} n\epsilon_n\cdot\frac{\tan\left(\frac{\pi}{5}\epsilon_n\right)}{\epsilon_n} = \frac{\pi}{5}$$

Solution 3 by Shivam Sharma-New Delhi-India

$\Rightarrow \left(\frac{\pi}{5}\right)\lim_{n\to\infty}(nen!)\left(\frac{\tan\left(\frac{\pi en!}{5}\right)}{\frac{\pi en!}{5}}\right)$. Then, $L = \left(\frac{\pi}{5}\right)\lim_{n\to\infty}(nen!)$. As we know,

$e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$. Put $x = 1$, we get, $e = \sum_{k=0}^{\infty}\left(\frac{1}{k!}\right)$. Using this, we get,

$\Rightarrow \left(\frac{\pi}{5}\right)\lim_{n\to\infty}(n)$. Now, applying the ratio test, we get, $\frac{n+1}{n} = 1 + \frac{1}{n}$

As $n \to \infty$, so, $\lim_{n\to\infty}\left(1+\frac{1}{n}\right) \to 1$. So, we get, $L = \frac{\pi}{5}$

65. Let $a_n, b_n \in (0,\infty)$ and $n \geq 1$

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a \text{ and } b_n = \lim_{n\to\infty}\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\ldots\sqrt[n]{n!}}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1} - \frac{a_n\cdot b_n}{n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal – Kolkata – India

$$\lim_{n\to\infty}(a_{n+1}-a_n) = a \text{ and } b_n = \lim_{n\to\infty}\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}$$

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1} - \frac{a_n\cdot b_n}{n}\right) = \lim_{n\to\infty}\frac{c_{n+1}-c_n}{n+1-n}$$

where $c_n = \frac{a_n\cdot b_n}{n}$ for all $n \geq 1$

$$= \lim_{n\to\infty}\frac{c_n}{n}\ [applying\ Reverse\ Cesaro - Stolz] = \lim_{n\to\infty}\frac{a_n\cdot b_n}{n^2}$$

$$= \left(\lim_{n\to\infty}\frac{a_n}{n}\right)\left(\lim_{n\to\infty}\frac{b_n}{n}\right) = \left(\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n}\right)\left(\lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}}\right)$$

$$= \lim_{n\to\infty}(a_{n+1}-a_n)\cdot\lim_{n\to\infty}\left(\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n+1]{(n+1)!}}{(n+1)^{n+1}}\cdot\frac{n^n}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}\right)$$

Applying Cauchy D – Alembert's Theorem

$$= a\cdot\lim_{n\to\infty}\left(\frac{n^n}{(n+1)^{n+1}}\cdot\sqrt[n+1]{(n+1)!}\right)$$

$$= a\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}}{\left(1+\frac{1}{n}\right)^n} = \frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{(n+2)!}{(n+2)^{n+2}}\cdot\frac{(n+1)^{n+1}}{(n+1)!}\right)$$

(Cauchy D-Alembert's Theorem)

$$= \frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n+2}\cdot\frac{(n+1)^{n+1}}{(n+2)^{n+1}}\right) = \frac{a}{e}\cdot\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}} = \frac{a}{e^2}$$

66. Find:

$$\Omega = \lim_{n\to\infty}\sqrt[n^2]{2!!\sqrt{3!!}\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}}$$

Proposed by D.M. Bătinețu – Giurgiu, N. Stanciu – Romania

Solution 1 by Rozeta Atanasova – Skopje

$$\Omega = \lim_{n\to\infty}\sqrt[n^2]{2!!\prod_{k=2}^{n}\sqrt[k]{(2k-1)!!}} \Rightarrow \ln\Omega = \lim_{n\to\infty}\frac{\ln 2 + \sum_{k=2}^{n}\ln\sqrt[k]{(2k-1)!!}}{n^2}$$

Applying the First Cauchy's theorem on limits ⇒

$$\ln\Omega = \lim_{n\to\infty}\frac{\ln\sqrt[n]{(2n-1)!!}}{n} = \lim_{n\to\infty}\frac{\sum_{k=1}^{n}\ln(2k-1)}{n^2}$$

Applying the First Cauchy's theorem on limits $\Rightarrow$

$$\ln\Omega = \lim_{n\to\infty}\frac{\ln(2n-1)}{n} = 0 \Rightarrow \Omega = e^0 = 1$$

Solution 2 by Soumitra Mandal-Chandar Nagore – India

We know $(2n+1)!! = \frac{(2n+1)!}{2^n\cdot n!}$

$$\Omega = \lim_{n\to\infty}\sqrt[n^2]{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n]{(2n-1)!!}} = \lim_{n\to\infty}\sqrt[n]{c_n}$$

Where $c_n = \sqrt[n]{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n]{(2n-1)!!}}$

$$\therefore \Omega = \lim_{n\to\infty}\sqrt[n]{c_n} = \lim_{n\to\infty}\frac{c_{n+1}}{c_n}\ [\text{Cauchy D}-\text{Alembert}]$$

$$= \lim_{n\to\infty}\frac{\sqrt[n+1]{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n]{(2n+1)!!}}}{\sqrt[n]{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n]{(2n-1)!!}}}$$

$$= \lim_{n\to\infty}\left(\frac{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n+2]{(2n+3)!!}}{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n+1]{(2n+1)!!}}\cdot\frac{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n]{(2n-1)!!}}{2!!\sqrt{3!!}\sqrt[3]{5!!}\dots\sqrt[n+1]{(2n+1)!!}}\right)$$

[Cauchy D-Alembert]

$$= \lim_{n\to\infty}\frac{\sqrt[n+2]{(2n+3)!!}}{\sqrt[n+1]{(2n+1)!!}} = \lim_{n\to\infty}\left(\frac{(2n+5)!!}{(2n+3)!!}\cdot\frac{(2n+1)!!}{(2n+3)!!}\right)$$

[Cauchy D-Alembert]

$$= \lim_{n\to\infty}\left(\frac{\frac{(2n+5)!}{2^{n+2}(n+2)!}}{\frac{(2n+3)!}{2^{n+1}(n+1)!}}\cdot\frac{\frac{(2n+1)!}{2^n n!}}{\frac{(2n+3)!}{2^{n+1}(n+1)!}}\right) = \lim_{n\to\infty}\left(\frac{n}{n+1}\right)\cdot\frac{(n+2)(2n+3)}{(n+1)(2n+1)} = 1$$

Solution 3 by Dana Heuberger-Romania

We have: $l = e^{\lim_{n\to\infty}\frac{\ln x_n}{n^2}}$, *where* $x_n = \ln\left(2!!\cdot\sqrt{3!!}\cdot\sqrt[3]{5!!}\cdot\ldots\cdot\sqrt[n]{(2n-1)!!}\right)$.

Using Cesaro-Stolz theorem, we deduce:

$$\lim_{n\to\infty}\frac{\ln x_n}{n^2} = \lim_{n\to\infty}\frac{\ln x_{n+1}-\ln x_n}{n^2} = \lim_{n\to\infty}\frac{\ln\sqrt[n+1]{(2n+1)!!}}{2n+1} =$$

$$= \lim_{n\to\infty}\frac{\ln(2n+1)!!}{(n+1)(2n+1)} = \lim_{n\to\infty}\frac{y_n}{z_n},$$

where $y_n = \ln(2n+1)!!$ *and* $z_n = 2n^2+3n+1$.

Using again Cesaro-Stolz theorem, we obtain:

$$\lim_{n\to\infty}\frac{\ln x_n}{n^2} = \lim_{n\to\infty}\frac{y_n}{z_n} = \lim_{n\to\infty}\frac{y_{n+1}-y_n}{z_{n+1}-z_n} = \lim_{n\to\infty}\frac{\ln(2n+3)}{4n+5} = 0,$$

therefore $l = e^{\lim_{n\to\infty}\frac{\ln x_n}{n^2}} = e^0 = 1$.

67. Let $a_n, b_n \in (0,\infty)$ and $n \geq 1$, $\lim_{n\to\infty}(a_{n+1}-a_n) = a$

and

$$b_n = \lim_{n\to\infty}\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}$$

find:

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1}-\frac{a_n\cdot b_n}{n}\right)$$

Proposed by D.M. Bătinețu – Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$\lim_{n\to\infty}(a_{n+1}-a_n) = a$ *and* $b_n = \lim_{n\to\infty}\sqrt[n]{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}$

$$\Omega = \lim_{n\to\infty}\left(\frac{a_{n+1}\cdot b_{n+1}}{n+1}-\frac{a_n\cdot b_n}{n}\right) = \lim_{n\to\infty}\frac{c_{n+1}-c_n}{n+1-n}$$

where $c_n = \frac{a_n b_n}{n}$ *for all* $n \geq 1$

$$=\lim_{n\to\infty}\frac{c_n}{n}\overset{Reverse\ Caesaro-Stolz}{=}\lim_{n\to\infty}\frac{a_nb_n}{n^2}$$

$$=\left(\lim_{n\to\infty}\frac{a_n}{n}\right)\left(\lim_{n\to\infty}\frac{b_n}{n}\right)=\left(\lim_{n\to\infty}\frac{a_{n+1}-a_n}{n+1-n}\right)\left(\lim_{n\to\infty}\sqrt[n]{\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}{n^n}}\right)$$

$$=\lim_{n\to\infty}(a_{n+1}-a_n)\cdot\lim_{n\to\infty}\left(\frac{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n+1]{(n+1)!}}{(n+1)^{n+1}}\cdot\frac{n^2}{\sqrt{2!}\cdot\sqrt[3]{3!}\cdot\ldots\cdot\sqrt[n]{n!}}\right)$$

$$\overset{Applying\ Cauchy\ D-Alemberts\ Theorem}{=}a\cdot\lim_{n\to\infty}\left(\frac{n^n}{(n+1)^{n+1}}\cdot\sqrt[n+1]{(n+1)!}\right)$$

$$=a\cdot\lim_{n\to\infty}\frac{\sqrt[n+1]{\frac{(n+1)!}{(n+1)^{n+1}}}}{\left(1+\frac{1}{n}\right)^n}=\frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{(n+2)!}{(n+2)^{n+2}}\cdot\frac{(n+1)^{n+1}}{(n+1)!}\right)$$

[Cauchy D-Alemberts Theorem]

$$=\frac{a}{e}\cdot\lim_{n\to\infty}\left(\frac{n+1}{n+2}\cdot\frac{(n+1)^{n+1}}{(n+2)^{2n+1}}\right)=\frac{a}{e}\cdot\lim_{n\to\infty}\frac{1}{\left(1+\frac{1}{n+1}\right)^{n+1}}=\frac{a}{e^2}$$

68. Solve for real numbers:

$$\begin{cases}a,b,c>0\\ abc=1\\ a^4b+b^4c+c^4a=ab+bc+ca\end{cases}$$

Proposed by Carmen Chirfot-Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$a^4b+b^4c+c^4a=\frac{a^4}{ac}+\frac{b^4}{ab}+\frac{c^4}{bc}\overset{Berström}{\geq}\frac{(a^2+b^2+c^2)}{ac+ab+bc}\geq$$

$$\geq\frac{(ac+bc+ab)^2}{ac+bc+ab}\geq ac+bc+ab\quad \textbf{(1)}$$

$$ab+bc+ca=a^4b+b^4c+c^4a\geq\frac{(a^2+b^2+c^2)}{ac+ab+bc}\Rightarrow$$

$$\Rightarrow ac+ab+bc\geq a^2+b^2+c^2,\textbf{ but (1)}$$

$$\Rightarrow a^2+b^2+c^2\geq ac+ab+bc\Rightarrow a^2+b^2+c^2=ab+ac+bc$$

$$\text{“=”}\ a=b=c\Rightarrow abc=1\Rightarrow a^3=1, a=1, b=1, c=1$$

$$(a,b,c)=(1,1,1)$$

Solution 2 by Nguyen Thanh Nho-Vietnam

$$a,b,c>0; abc=1$$

$$\left(\frac{a^3}{c}+ca\right)+\left(\frac{b^3}{a}+ab\right)+\left(\frac{c^3}{b}+bc\right)\geq 2a^2+2b^2+2c^2\geq 2(ab+bc+ca)$$

$$\Rightarrow\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\geq ab+bc+ca$$

$$\Rightarrow abc\left(\frac{a^3}{c}+\frac{b^3}{a}+\frac{c^3}{b}\right)\geq 1(ab+bc+ca)$$

$$\Rightarrow a^4b+b^4c+c^4a\geq ab+bc+ca\ \textbf{Done!}$$

$$\text{“=”}\Leftrightarrow\begin{cases}a=b=c\\ abc=1\end{cases}\Leftrightarrow a=b=c=1$$

69. If $a,b\in\mathbb{N}, a<b$

$$\Omega(a,b)=\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{(a+i)(b+i)}$$

then:

$$\Omega(a,b)\geq\left(\frac{a!}{b!}\right)^{\frac{1}{b-a}}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash - New Delhi – India

$$\sum_{i=1}^{n}\frac{1}{(a+i)(b+i)}=\frac{1}{b-a}\sum_{i=1}^{n}\left(\frac{1}{a+i}-\frac{1}{b+i}\right)$$

$$=\frac{1}{b-a}\left[\frac{1}{a+1}+\frac{1}{a+2}+\cdots+\frac{1}{a+(b-a)}-\frac{1}{n+a+1}-\frac{1}{n+a+2}+\cdots+\frac{1}{b+n}\right]$$

$$=\frac{1}{b-a}\left[\frac{n}{(a+1)(n+a+1)}+\frac{n}{(a+2)(n+a+2)}+\cdots+\frac{n}{b(b+n)}\right]$$

$$=\frac{1}{b-a}\sum_{k=1}^{b-a}\left[\frac{1}{(a+k)\left(1+\frac{a+k}{n}\right)}\right]$$

$$\therefore \lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{(a+i)(b+i)}=\lim_{n\to\infty}\frac{1}{b-a}\sum_{k=1}^{b-a}\frac{1}{(a+k)\left(1+\frac{a+k}{n}\right)}$$

$$=\frac{1}{b-a}\sum_{k=1}^{b-a}\frac{1}{(a+k)}>\left(\prod_{k=1}^{b-a}\frac{1}{a+k}\right)^{\frac{1}{b-a}}=\left(\frac{a!}{b!}\right)^{\frac{1}{b-a}}$$

70. If $a,b,c>0, a+b+c=1$,

$$\Omega(a,b)=\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{\sqrt[n]{b}+\sqrt[n]{a+1}}\right)^{n}$$

then:

$$\sum\sqrt{b(a+1)}\cdot\Omega(a,b)\le 2$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Redwane El Mellas-Casablanca-Morocco

$$\therefore \Omega(a,b)=\lim_{t\to 0^+}\left(\frac{a^t+(b+1)^t}{b^t+(a+1)^t}\right)^{\frac{1}{t}}=\lim_{t\to 0^+} e^{\frac{\ln\left(a^t+(b+1)^t\right)-\ln\left(b^t+(a+1)^t\right)}{t}}$$

$$=e^{\left(\ln\left(a^t+(b+1)^t\right)-\ln\left(b^t+(a+1)^t\right)\right)'(0)}=e^{\frac{\ln a+\ln(b+1)-\ln b-\ln(a+1)}{2}=\frac{\ln\frac{a(b+1)}{b(a+1)}}{2}}=$$

$$=\sqrt{\frac{a(b+1)}{b(a+1)}}$$

$$(*)\sum\sqrt{b(a+1)}\,\Omega\,(a,b)=\sum\sqrt{a(b+1)}\overset{AM-GM}{\leq}\sqrt{3}\sqrt{\sum a(b+1)}=$$

$$=\sqrt{3}\sqrt{\sum ab+1}$$

$$\left(\sum a\right)^2=1=\sum a^2+2\sum ab\overset{AM-GM}{\geq}3\sum ab\Rightarrow\sum ab\leq\frac{1}{3}$$

Finally $\sum\sqrt{b(a+1)}\,\Omega(a,b)\leq\sqrt{3}\sqrt{\frac{1}{3}+1}=2$ ***with equality if***

$$a=b=c=\frac{1}{3}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

LEMMA: let $a_i,i=1,2,\dots,n$ ***be positive real numbers, then***

$$\lim_{x\to\infty}\left(\frac{\sqrt[x]{a_1}+\sqrt[x]{a_2}+\cdots+\sqrt[x]{a_n}}{n}\right)^{nx}=a_1\cdot a_2\cdot\ldots\cdot a_n$$

$$\therefore \Omega(a,b)=\lim_{n\to\infty}\left(\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{\sqrt[n]{b}+\sqrt[n]{a+1}}\right)^n=\lim_{n\to\infty}\left(\frac{\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{2}}{\frac{\sqrt[n]{a}+\sqrt[n]{b+1}}{2}}\right)^n=$$

$$=\frac{\sqrt{a(b+1)}}{\sqrt{b(a+1)}}$$

$$\therefore \sum_{cyc}\sqrt{b(a+1)}\cdot\Omega(a,b)=\sum_{cyc}\sqrt{a(b+1)}\overset{Cauchy-Schwarz}{\overbrace{\leq}}$$

$$\leq\sqrt{(a+b+c)(a+b+c+3)}=2\ \textbf{(proved)}$$

71. Find:

$$\Omega=\lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}\right)$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

Let $E_n=\sum_{k=0}^{n}\frac{1}{k!}$. ***Then*** $\lim_{n\to\infty}E_n=e$

$$\therefore \lim_{n\to\infty}\frac{\sqrt[n]{n!}}{n}=\lim_{n\to\infty}\sqrt[n]{\frac{n!}{n^n}}\overset{Cauchy-D'Alembert}{\cong}\lim_{n\to\infty}\left(\frac{(n+1)!}{(n+1)^{n+1}}\cdot\frac{n^n}{n!}\right)=\frac{1}{e}$$

$$\therefore \lim_{n\to\infty}\sqrt[n]{E_n}\overset{Cauchy-D'Alembert}{\cong}\lim_{n\to\infty}\frac{E_{n+1}}{E_n}=\frac{e}{e}=1$$

$$\therefore \left(\sqrt[n+1]{(n+1)!}-\sqrt[n]{n!\,E_n}\right)=\lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\cdot\sqrt[n]{E}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right)$$

where $u_n=\frac{\sqrt[n+1]{(n+1)!}}{\sqrt[n]{n!E_n}}$ ***for all*** $n\in\mathbb{N}$

$$\therefore \lim_{n\to\infty}u_n=\lim_{n\to\infty}\left(\frac{\sqrt[n+1]{(n+1)!}}{n+1}\cdot\frac{n+1}{n}\cdot\frac{1}{\frac{\sqrt[n]{n!}}{n}}\cdot\frac{1}{\sqrt[n]{E_n}}\right)$$

$$=\left(\lim_{n\to\infty}\frac{\sqrt[n+1]{(n+1)!}}{n+1}\right)\left(\lim_{n\to\infty}\frac{n+1}{n}\right)\left(\lim_{n\to\infty}\frac{1}{\frac{\sqrt[n]{n!}}{n}}\right)\left(\lim_{n\to\infty}\frac{1}{\sqrt[n]{E_n}}\right)=1$$

$\therefore$ ***as*** $u_n\to 1$ ***then*** $\frac{u_n-1}{\ln u_n}\to 1$ ***as*** $n\to\infty$

Now, $\lim_{n\to\infty} u_n^n = \lim_{n\to\infty}\left(\frac{1}{\frac{\sqrt[n+1]{(n+1)!}}{n+1}}\cdot\frac{1}{n+1}\cdot\frac{(n+1)!}{n!}\cdot\frac{1}{E_n}\right) = 1.$ *So,*

$$\lim_{n\to\infty}\ln u_n^n = \ln\left(\lim_{n\to\infty} u_n^n\right) = 0$$

$$\therefore \lim_{n\to\infty}\left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)}\right) =$$

$$= \lim_{n\to\infty}\left(\frac{\sqrt[n]{n!}}{n}\sqrt[n]{E_n}\cdot\frac{u_n-1}{\ln u_n}\cdot\ln u_n^n\right) = 0$$

(Ans:)

Solution 2 by Shivam Sharma-New Delhi-India

$$L = \lim_{n\to\infty}\left(\left((n+1)!\right)^{\frac{1}{n+1}} - \left(n!\left(1+\frac{1}{1!}+\frac{1}{2!}+\cdots+\frac{1}{n!}\right)\right)^{\frac{1}{n}}\right)$$

Applying Stirling's formula in the first limit, we get,

$$(n+1)! = \left(\frac{n+1}{e}\right)^{n+1}\sqrt{2\pi(n+1)}$$

$$\Rightarrow \lim_{n\to\infty}\left(\left(\left(\frac{n+1}{e}\right)^{n+1}\sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}} - \left(n!\left(\sum_{m=1}^{n}\frac{1}{m!}\right)\right)^{\frac{1}{n}}\right)$$

$$\Rightarrow \lim_{n\to\infty}\left(\frac{n+1}{e}\left(\sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}}\right) - e^0$$

Applying ratio test, we get, $\frac{n+1}{n}\to 1$, ***as*** $n\to\infty$, ***So*** $\frac{n+1}{e}\left(\sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}}\to 1$ ***as*** $n\to\infty$

$\Rightarrow 1-e^0 \Rightarrow 1-1$***(OR),*** $L=0$ ***(Q.E.D.)***

72. Find:

$$\Omega = \lim_{n\to\infty}\frac{1}{n!}\int_0^1\left((1-x)^n+\cos nx\right)e^x dx$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nirapada Pal-Jhargram-India

For all $x \in [0,1]$,

$$|[(1-x)^n + \cos nx]e^x| = |(1-x)^n + \cos nx||e^x| \text{ since } |AB| = |A||B|$$

$$\leq |(1-x)^n|e^x + |\cos nx|e^x \text{ since } e^x \geq 0 \text{ for all } x$$

$$\leq 2e^x \text{ since } (1-x)^n \leq 1 \text{ for all } x \in [0,1] \text{ and } |\cos nx| \leq 1 \text{ for all } n, \text{ for}$$

all $x \leq 2e$***. So the integrand is bounded. Hence***

$$\lim_{n\to\infty}\frac{1}{n!}\int_0^1 [(1-x)^n + \cos nx]e^x\,dx \leq \lim_{n\to\infty}\frac{1}{n!}\int_0^1 2e\,dx = \lim_{n\to\infty}\frac{2e}{n!} = 0$$

Solution 2 by Ravi Prakash-New Delhi-India

$$\text{For } 0 \leq x \leq 1, 0 \leq 1-x \leq 1 \Rightarrow 0 \leq (1-x)^n \leq 1, |\cos(nx)| \leq 1$$

$$\therefore |(1-x)^n + \cos(nx)| \leq |(1-x)^n| + |\cos(nx)| \leq 2$$

$$\Rightarrow \left|\int_0^1 \{(1-x)^n + \cos(nx)\}e^x\right| \leq \int_0^1 2e^x dx = 2(e-1)$$

$$\Rightarrow \left|\frac{1}{n!}\int_0^1 \{(1-x)^n + \cos(nx)\}e^x dx\right| \leq \frac{2}{n!}(e-1)$$

As $\frac{2}{n!}(e-1) \to 0$ ***as*** $n \to \infty$,

$$\lim_{n\to\infty}\frac{1}{n!}\int_0^1 [(1-x)^n + \cos(nx)]\,e^x dx = 0$$

Solution 3 by Shivam Sharma-New Delhi-India

$$\to \lim_{n\to\infty}\frac{1}{n!}\int_0^1 (1-x)^n e^x dx + \lim_{n\to\infty}\frac{1}{n!}\int_0^1 e^x\cos(nx)\,dx$$

As we can see the first integral is written in the form of "Incomplete gamma function", so we get,

$$\int_0^1 (1-x)^n e^x dx = e[\Gamma(n+1)\ \ \Gamma(n+1,1)]$$

And, As we know,

$$\int e^{ax}\cos(bx)\,dx\,\frac{e^{ax}}{a^2|b^2}\lfloor a\cos(bx)\mid b\sin(bx)\rfloor|C,$$

which can be proved by applying I.B.P. putting $a\mid,b\ n$

On putting the limits we get,

$$\int_0^1 e^x\cos(nx)\,dx = \frac{e(n\sin(n))|\,e(\cos(n))-1}{1+n^2}$$

Putting all the values, we get,

$$L = \lim_{n\to\infty}\frac{1}{n!}\left[e\big(\Gamma(n|1)-\Gamma(n|1,1)\big)\right]|\lim_{n\to\infty}\frac{1}{n!}\left[\frac{e(n\sin(n))+e(\cos(n))}{1+n^2}\right]$$

As we know, $\Gamma(n|1)\quad n!$***. Using this, we get,***

$$L = \lim_{n\to\infty}\frac{1}{n!}\left[e\left(n!\quad\frac{1}{n|1}\right)\right]+\lim_{n\to\infty}\frac{1}{n!}\left[\frac{e(n\sin(n))+e(\cos(n))}{1+n^2}\right]$$

$\to e(0)+0\to 0$ ***(OR)*** $L = 0$. ***(Q.E.D)***

73. If $a,b>0,|x|<1,|y|<1$

$$\Omega(x) = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{(-x)^i}{i+1}$$

then:

$$(a+b)\Omega\left(\frac{ax+by}{a+b}\right)\le a\Omega(x)+b\Omega(y)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\Omega(x)=\lim_{n\to\infty}\sum_{i=0}^{n}\frac{(-x)^i}{i+1}=\lim_{n\to\infty}\left(1-\frac{x}{2}+\frac{x^2}{3}+\cdots+\frac{(-1)^n x^n}{n+1}\right)$$

$$\therefore x\Omega(x)=\lim_{n\to\infty}\left(x-\frac{x^2}{2}+\frac{x^3}{3}+\cdots+\frac{(-1)^n x^{n+1}}{n+1}\right)=\log(1+x)$$

$$\therefore \Omega(x)=\frac{\log(1+x)}{x}\Rightarrow \Omega'(x)=\frac{1}{x^2+x}-\frac{\log(1+x)}{x^2}$$

$$\Omega''(x)=-\frac{2x+1}{(x^2+x)^2}-\frac{1}{x^2(1+x)}+\frac{2\log(1+x)}{x^3}=$$

$$=\frac{2(1+x)^2\log(1+x)-(3x^2+2x)}{x^3(1+x)^2}$$

Let $f(x)=2(1+x^2)\log(1+x)-(3x^2+2x)$ ***for all*** $x\in[0,1]$

$$f'(x)=2(1+x)+4(1+x)\log(1+x)-6x-2=$$

$$=(1+x)\left(\log(1+x)-\frac{x}{1+x}\right)\ge 0$$

$\left[\because \log(1+x)\ge\frac{x}{1+x}\ for\ all\ x\ge 0\right]$***. So,*** f ***is increasing*** $f(x)\ge f(0)=0$

$\therefore \Omega''(x)\ge 0$ ***for all*** $|x|<1$***. Hence,*** Ω ***is convex. So,***

$$\Omega\left(\frac{ax+by}{a+b}\right)\le\frac{a\Omega(x)}{a+b}+\frac{b\Omega(y)}{a+b}\therefore a\Omega(x)+b\Omega(y)\ge\Omega\left(\frac{ax+by}{a+b}\right)$$

74. Find:

$$L=\lim_{n\to\infty}\left(\sum_{k=1+100n}^{200n}\frac{1}{k}+\sum_{k=1+200n}^{400n}\frac{1}{k}+\sum_{k=1+300n}^{400n}\frac{1}{k}\right)$$

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Proposed by Daniel Sitaru – Romania

Solution 1 by Nirapada Pal-Jhargram-India

$$\lim_{n\to\infty}\left(\sum_{k=1+100n}^{200n}\frac{1}{k}+\sum_{k=1+200n}^{400n}\frac{1}{k}+\sum_{k=1+300n}^{400n}\frac{1}{k}\right)$$

$$=\lim_{n\to\infty}\left(\sum_{k=1+100n}^{400n}\frac{1}{k}+\sum_{k=1+300n}^{400n}\frac{1}{k}\right)=\lim_{n\to\infty}\left(\sum_{k=1}^{300n}\frac{1}{100n+k}+\sum_{k=1}^{100n}\frac{1}{300n+k}\right)$$

$$=300\lim_{p\to\infty}\frac{1}{p}\sum_{k=1}^{p}\frac{1}{100+300\frac{k}{p}}+100\lim_{q\to\infty}\frac{1}{q}\sum_{k=1}^{q}\frac{1}{100+300\frac{k}{q}}$$

$$=300\int_0^1\frac{dx}{100+300x}+100\int_0^1\frac{dx}{300+100x}$$

$$=\int_{x=0}^{x=1}\frac{d(100+300x)}{100+300x}+\int_{x=0}^{x=1}\frac{d(300+100x)}{300+100x}$$

$$=\ln(100+300x)\Big|_{x=0}^{x=1}\ln(300+100x)\Big|_{x=0}^{x=1}$$

$$=\ln 400-\ln 100+\ln 400-\ln 300=\ln\frac{16}{3}$$

Solution 2 by Daniel Sitaru – Romania

$$\textit{Let be } a_n=1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}$$

$$L=\lim_{n\to\infty}(a_{200n}-a_{100n}+a_{400n}-a_{200n}+a_{400n}-a_{300n})=$$

$$=\lim_{n\to\infty}(2a_{400n}-a_{100n}-a_{300n})=$$

$$=\lim_{n\to\infty}\left(2a_{400n}-2\log 400n-a_{100n}+\log 100n-a_{300n}+\log 300n+\log\frac{400n\cdot 400n}{100n\cdot 300n}\right)$$

$$=2\gamma-\gamma-\gamma+\log\frac{16}{3}=\log\frac{16}{3}$$

75. Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{\pi^2}{6}-\left(\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\right)\right)^n$$

Proposed by D.M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution by Shivam Sharma-New Delhi-India

$$L = \lim_{n\to\infty}\left[\frac{\pi^2}{6}-\left(\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}\dots+\frac{1}{n^2}\right)\right]^n \Rightarrow \lim_{n\to\infty}\left[\frac{\pi^2}{6}-\sum_{k=2}^{n}\left(\frac{1}{k^2}\right)\right]^n$$

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\pi^2}{6}-\left(H_n^{(2)}-1\right)\right]^n \Rightarrow \lim_{n\to\infty}\left[\frac{\pi^2}{6}-H_n^{(2)}+1\right]^n$$

As we know, $\psi^1(n+1)+H_n^{(2)}=\frac{\pi^2}{6}$ *(OR)* $H_2^{(2)}=\frac{\pi^2}{6}-\psi^1(n+1)$.

Using this, we get,

$$\Rightarrow \lim_{n\to\infty}\left[\frac{\pi^2}{6}+\psi^1(n+1)-\frac{\pi^2}{6}+1\right]^n \Rightarrow \lim_{n\to\infty}[\psi^1(n+1)+1]^n$$

As we know, $\psi^1(n+1)=\psi^1(n)-\frac{1}{n^2}$. *Using this, we get,*

$$\Rightarrow \lim_{n\to\infty}\left[\psi^1(n)-\frac{1}{n^2}+1\right]^n$$

Now, applying Root test, we get, $\left[\left(\psi^1(n)-\frac{1}{n^2}+1\right)^n\right]^{\frac{1}{n}} \to e$, *as* $n\to\infty$

Hence, $L=e$. *(Q.E.D)*

76. In $\Delta\, ABC$:

$$\alpha=\frac{\sin 2A}{\tan B+\tan C},\ \beta=\frac{\sin 2B}{\tan A+\tan C},\ \gamma=\frac{\sin 2C}{\tan A+\tan B}$$

Solve in real numbers:

$$\alpha x^2 - 2\beta x + \gamma = 0$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Peru

En un triángulo ABC.

$$\alpha = \frac{\operatorname{sen} 2A}{\tan C + \tan B}, \beta = \frac{\operatorname{sen} 2B}{\tan A + \tan C}, \gamma = \frac{\operatorname{sen} 2C}{\tan A + \tan B}$$

Resolver en $\mathbb{R}$: $\alpha x^2 - 2\beta x + \gamma = 0$. Recordar lo siguiente:

$$\tan x + \tan y = \frac{\operatorname{sen}(x+y)}{\cos x \cos y} \wedge \operatorname{sen} 2x = 2 \operatorname{sen} x \cos x$$

$$\Rightarrow \alpha = \frac{\operatorname{sen} 2A}{\frac{\operatorname{sen}(C+B)}{\cos C \cos B}} = \frac{\operatorname{sen} 2A \cos C \cos B}{\operatorname{sen} A} = 2 \cos A \cos B \cos C$$

De forma análoga se llega que:

$$\beta = 2 \cos A \cos B \cos C \bigwedge \gamma = 2 \cos A \cos B \cos C$$

$\Rightarrow$ Se puede concluir que: $\alpha = \beta = \gamma$

$$x = \frac{2\beta \pm \sqrt{4B^2 - 4\alpha\gamma}}{2\alpha} \rightarrow x = 1 \text{ (Raíz doble)} \Leftrightarrow \Delta = 0$$

77. Solve in real numbers:

$$x^2 - 2Ax + B^2 = 0$$

$$A = \sum h_a r_a, B = r \sum (h_a + 2r_a)$$

r – inradius, r_a – exradius, h_a – height

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios –Huarmey- Peru

Resolver en números reales: $x^2 - 2Ax + B^2 = 0$

$$A = \sum h_a r_a, \qquad B = r\sum(h_a + 2r_a)$$

En un triángulo ABC tener presente lo siguiente:

$$r = 4R\,\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2} \quad (A)$$

$$\cos A + \cos B + \cos C - 1 = 4\,\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2} \quad (B)$$

$$\text{sen}^2 A + \text{sen}^2 B + \text{sen}^2 C = 2(1 + \cos A\cos B\cos C) \quad (C)$$

$$r_a + r_b + r_c = 4R + r$$

$$2\,\text{sen}\,B\,\text{sen}\,C\cos A + 2\,\text{sen}\,A\,\text{sen}\,C\cos B + 2\,\text{sen}\,A\,\text{sen}\,B\cos C = \text{sen}^2 A + \text{sen}^2 B + \text{sen}^2 C \quad (D)$$

$$A = \sum h_a r_a = \sum 2R\,\text{sen}\,B\,\text{sen}\,C\left(4R\,\text{sen}\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$A = \sum h_a r_a = \sum 2R^2\,\text{sen}\,B\,\text{sen}\,C\left(2\,\text{sen}\frac{A}{2}\right)\left(2\cos\frac{B}{2}\cos\frac{C}{2}\right)$$

$$A = \sum h_a r_a = \sum 2R^2\,\text{sen}\,B\,\text{sen}\,C\left(2\cos\left(\frac{B+C}{2}\right)\right)\left(\left(\cos\frac{B+C}{2}\right) + \cos\left(\frac{B-C}{2}\right)\right)$$

$$A = \sum h_a r_a = \sum 2R^2\,\text{sen}\,B\,\text{sen}\,C\left(2\cos^2\left(\frac{B+C}{2}\right) + \cos B + \cos C\right)$$

$$A = \sum h_a r_a = \sum 2R^2\,\text{sen}\,B\,\text{sen}\,C\,(1 + \cos(B+C) + \cos B + \cos C)$$

$$A = \sum h_a r_a = \sum 2R^2\,\text{sen}\,B\,\text{sen}\,C\,(1 - \cos A + \cos B + \cos C)$$

$$B = r\sum(h_a + 2r_a) = r\sum ha + 2r\sum ra \rightarrow B = r\sum h_a + 2r(4R + r)$$

$$B = r\sum(h_a + 2r_a) = 4R\,\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}\sum 2R\,\text{sen}\,B\,\text{sen}\,C +$$

$$+8R\left(4R\,\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}\right) + 2\left(4R\,\text{sen}\frac{A}{2}\text{sen}\frac{B}{2}\text{sen}\frac{C}{2}\right)^2$$

$$B = r\sum(h_a + 2r_a) = \left(\sum 2R^2\,\text{sen}\,B\,\text{sen}\,C + 8R^2\right)(\cos A + \cos B + \cos C - 1) +$$

$$+4R^2(1 - \cos A)(1 - \cos B)(1 - \cos C)$$

$$B = r\sum(h_a + 2r_a) = \left(\sum 2R^2\,\text{sen}\,B\,\text{sen}\,C + 8R^2\right)\left(\sum\cos A - 1\right) + 4R^2 - 4R^2\sum\cos A +$$

$$+4R^2\sum\cos A\cos B-4R^2\prod\cos A$$

$$B=r\sum(h_a+2r_a)=\sum 2R^2\operatorname{sen}B\operatorname{sen}C\left(\sum\cos A\right)-\sum 2R^2\operatorname{sen}B\operatorname{sen}C+$$

$$+4R^2\sum\cos A+4R^2\sum\cos A\cos B-4R^2\left(1+\prod\cos A\right)$$

$$B=r\sum(h_a+2r_a)=\sum 2R^2\operatorname{sen}B\operatorname{sen}C\left(\sum\cos A\right)-\sum 2R^2\operatorname{sen}B\operatorname{sen}C-$$

$$-4R^2\sum\cos(B+C)+4R^2\sum\cos A\cos B-2R^2\left(\sum\operatorname{sen}^2 A\right)$$

$$B=r\sum(h_a+2r_a)=\sum 2R^2\operatorname{sen}B\operatorname{sen}C\left(\sum\cos A\right)+\sum 2R^2\operatorname{sen}B\operatorname{sen}C-$$

$$-2R^2\left(\sum 2\operatorname{sen}B\operatorname{sen}C\cos A\right)$$

$$B=r\sum(h_a+2r_a)=\sum 2R^2\operatorname{sen}B\operatorname{sen}C\left(\sum\cos A+1-2\cos A\right)$$

$$B=r\sum(h_a+2r_a)=\sum 2R^2\operatorname{sen}B\operatorname{sen}C\,(1-\cos A+\cos B+\cos C)=\sum h_ar_a=A$$

Por la tanto, si: $A=B\to$ ***La ecuación:***

$$x^2-2Ax+B^2=0\to(x-A)^2=0\to x=A\quad \textit{(Raíz doble)}$$

$$x=A=\sum 2R^2\operatorname{sen}B\operatorname{sen}C\,(1-\cos A+\cos B+\cos C)=\sum h_ar_a$$

78. $f,g:(0,\infty)\to\mathbb{R}$

$$f(x)=\sin\left(\ln\frac{x(x+1)}{x+2}\right)+\sin\left(\ln\frac{(x+1)(x+2)}{x}\right)$$

$$g(x)=\sin\left(\ln(x(x+1)(x+2))-\sin\left(\ln\frac{x(x+2)}{x+1}\right)\right)$$

Solve the equation: $f(x)=g(x)$.

Proposed by Daniel Sitaru – Romania

Solution 1 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$\ln k(x)\Rightarrow k(x)>0$$

$$\frac{x\cdot(x+1)}{x+2}>0;\frac{(x+1)(x+2)}{x}>0;x(x+1)(x+2)>0$$

$$\frac{x(x+2)}{x+1}>0\Rightarrow D(x):x\in]-2;\ -1[U]\ 0;+\infty[\quad \textbf{\textit{(1)}}$$

1) $f(x)=f_1(x)+f_2(x)$

$$f_1(x)=\sin\left(\ln(x+1)-\ln\frac{x+2}{x}\right);\ f_2(x)=\sin\left(\ln(x+1)+\ln\frac{x+2}{x}\right)$$

$$f(x)=2\cdot\sin\left(\ln(x+1)\cdot\left(\ln\frac{x+2}{x}\right)\right)\quad (*)$$

2) $g(x)=g_1(x)+g_2(x)$

$$g_1(x)=\sin(\ln x\,(x+2)+\ln(x+1))$$

$$g_2(x)=\sin(\ln x\,(x+2)-\ln(x+1))$$

$$g(x)=2\cdot\cos\big(\ln x\,(x+2)\big)\cdot\sin(\ln(x+1))\quad (**)$$

3) (1), (*), (**) $\Rightarrow D(x):x\in\,]0,+\infty[$ ***(2)***

4) $f(x)=g(x)\Rightarrow 2\cdot\sin(\ln(x+1))\cdot\Big(\cos\big(\ln x\,(x+2)\big)\Big)-\cos\left(\ln\frac{x+2}{x}\right)$

$$=2\cdot\sin(\ln x)\cdot\sin(\ln(x+1))\cdot\sin(\ln(x+2))=0$$

$$\begin{matrix}\sin(\ln x)=0 & x=e^{\pi\cdot k}\Rightarrow & & x=e^{\pi\cdot k}\ k\in\mathbb{Z}\\ \sin(\ln(x+1))=0\Rightarrow & x=e^{\pi k}-1\Rightarrow D(x) & (2)\Rightarrow & x=e^{\pi k}-1\ k\in\mathbb{Z}^+\\ \sin(\ln(x+2))=0 & x=e^{\pi k}-2\Rightarrow & & x=e^{\pi k}-2\ \ k\in\mathbb{Z}^+\end{matrix}$$

Solution 2 by Shivam Sharma-New Delhi-India

Firstly we solve $f(x)$***, As we know,*** $\sin(C)\,|\,\sin(D)\quad 2\sin\left(\frac{C+D}{2}\right)\cos\left(\frac{C\ D}{2}\right)$

Using this we get, $f(x)=2\sin\left(\frac{2\ln(x+1)}{2}\right)\cos\left(\frac{2\ln\left(\frac{x}{x+2}\right)}{2}\right)$ ***(OR)***

$f(x)=2\sin(\ln(x|1))\cos\left(\ln\left(\frac{x}{x+2}\right)\right)$***. Then we solve*** $g(x)$***. As we know,***

$\sin(C)-\sin(D)\ \ 2\sin\left(\frac{C\ D}{2}\right)\cos\left(\frac{C+D}{2}\right)$***. Using this, we get,***

$g(x) = 2\sin(\ln(x+1))\cos(\ln(x(x+2)))$. ***Now, according to question,***

$$f(x) = g(x)$$

$$2\sin(\ln(x|1))\cos\left(\ln\left(\frac{x}{x+2}\right)\right) \quad 2\sin(\ln(x|1))\cos(\ln(x(x|2)))$$

$$\sin(\ln(x|1)) \quad 0, \cos\left(\ln\left(\frac{x}{x+2}\right)\right) - \cos(\ln(x(x|2))) \quad 0$$

$\ln(x+1) = n\pi$. ***We get,*** $x = e^{n\pi} \quad 1$. ***Now,***

$\cos\left(\ln\left(\frac{x}{x+2}\right)\right) - \cos(\ln(x(x|2))) \quad 0$. ***As we know,***

$\cos(C) - \cos(D) \;\; 2\sin\left(\frac{C+D}{2}\right)\sin\left(\frac{D \;\; C}{2}\right)$. ***Using this, we get,***

$2\sin(\ln(x))\sin(\ln(x+2)) = 0$. ***We get,***

$$\sin(\ln(x)) = 0, \sin(\ln(x+2)) = 0;\ \ln(x) = n\pi, \ln(x+2) = n\pi$$

We get, $x = e^{n\pi}, x = e^{n\pi} \quad 2$. ***So, finally we get,*** $x = e^{n\pi}$

$x = e^{n\pi} \quad 1; \vee n \leftarrow Z_{|}; x \quad e^{n\pi} - 2$. ***(Q.E.D)***

Solution 3 by Soumava Chakraborty-Kolkata-India

$f, g: (0,\infty) \to R, f(x) = \sin\left(\ln\frac{x(x+1)}{x+2}\right) + \sin\left(\ln\frac{(x+1)(x+2)}{x}\right)$ ***and***

$$g(x) = \sin(\ln(x(x+1)(x+2))) - \sin\left(\ln\frac{x(x+2)}{x+1}\right)$$

Solve the equation: $f(x) = g(x)$

$$f(x) = 2\sin\left\{\frac{\ln\frac{x(x+1)}{x+2} + \ln\frac{(x+1)(x+2)}{x}}{2}\right\}\cos\left\{\frac{\ln\frac{x(x+1)}{x+2} - \ln\frac{(x+1)(x+2)}{x}}{2}\right\} =$$

$$= 2\sin\left\{\frac{\ln((x+1)^2)}{2}\right\}\cos\left\{\frac{\ln\left(\left(\frac{x}{x+2}\right)^2\right)}{2}\right\}$$

$$= 2(\sin\ln(x+1))\left(\cos\ln\left(\frac{x}{x+2}\right)\right) \quad (\because x > 0)$$

$$g(x) = 2\cos\left\{\frac{\ln\left(x(x+1)(x+2)\right) + \ln\frac{x(x+2)}{x+1}}{2}\right\}$$

$$\sin\left\{\frac{\ln\left(x(x+1)(x+2)\right) - \ln\frac{x(x+2)}{x+1}}{2}\right\} = 2\cos\left\{\frac{\ln\{x(x+2)^2\}}{2}\right\}\sin\left\{\frac{\ln(x+1)^2}{2}\right\}$$

$$= 2\cos(\ln x(x+2))\,(\sin\ln(x+1)) \quad (\because x > 0)$$

$$\therefore f(x) = g(x) \Rightarrow \sin(\ln(x+1)) = 0 \ \textit{or}$$

$$\cos(\ln x(x+2)) = \cos\left(\ln\frac{x}{x+2}\right), \textit{or both}$$

$$\sin(\ln(x+1)) = 0 \Rightarrow \ln(x+1) = n\pi \Rightarrow x+1 = e^{n\pi}$$

$$\Rightarrow x = e^{n\pi} - 1 \because x > 0, \therefore e^{n\pi} > 1 \Rightarrow n\pi > 0 \Rightarrow n > 0 \Rightarrow n \in \mathbb{N}$$

when $\cos(\ln x(x+2)) = \cos\left(\ln\frac{x}{x+2}\right)$,

$$2\sin\left\{\frac{\ln x(x+2) + \ln\frac{x}{x+2}}{2}\right\}\sin\left\{\frac{\ln\frac{x}{x+2} - \ln x(x+2)}{2}\right\} = 0$$

$$\Rightarrow 2\sin\left(\frac{\ln x^2}{2}\right)\sin\left\{\frac{\ln\left(\frac{1}{x+2}\right)^2}{2}\right\} = 0$$

$$\Rightarrow 2(\sin\ln x)\left(\sin\ln\frac{1}{x+2}\right) = 0 \quad (\because x > 0)$$

when $\sin(\ln x) = 0, \ln x = n\pi \Rightarrow x = e^{n\pi}$

when $\sin\left(\ln\frac{1}{x+2}\right) = 0, \ln\frac{1}{x+2} = n\pi \Rightarrow \frac{1}{x+2} = e^{n\pi}$

$$\Rightarrow x+2 = e^{-n\pi} \Rightarrow x = x^{-n\pi} - 2 \therefore x > 0, \therefore e^{-n\pi} - 2 > 0 \Rightarrow e^{-n\pi} > 2$$

$$\Rightarrow -n\pi > \ln 2 \Rightarrow n\pi < -\ln 2 \Rightarrow n < \frac{-\ln 2}{\pi} \Rightarrow n \leq -1$$

$\therefore$ ***solutions are*** $\begin{pmatrix} x = e^{n\pi} - 1, & \forall n \in \mathbb{N} \\ x = e^{n\pi}, & \forall n \in Z \\ x = e^{-n\pi} - 2, & \forall n \in Z^- \end{pmatrix}$ ***where,*** $Z^- = Z - (\mathbb{N} \cup \{0\})$

The last set of solutions can also be written as: $x = e^{m\pi} - 2 \quad \forall m \in \mathbb{N}$

Solution 4 by Ravi Prakash-New Delhi-India

Let $A = \ln(x),\ B = \ln(x+1),\ C = \ln(x+2)$

$$\therefore f(x) = \sin(A + B - C) + \sin(B + C - A) = 2\sin B\cos(A - C)$$

$$g(x) = \sin(A + B + C) - \sin(A - B + C)$$

$$= \sin(B + A + C) + \sin\big(B - (A + C)\big) = 2\sin B\cos(A + C)$$

$$f(x) = g(x) \Rightarrow \sin B = 0 \text{ or } \cos(A - C) = \cos(A + C)$$

$$\Rightarrow \sin B = 0 \text{ or } \sin A\sin C = 0$$

$$\Rightarrow \sin A = 0 \text{ or } \sin B = 0 \text{ or } \sin C = 0 \Rightarrow A, B, C = n\pi, n \in \mathbb{Z}$$

$$\Rightarrow x = e^{n\pi} \text{ or } e^{n\pi} - 1 \text{ or } e^{n\pi} - 2 \text{ for same } n \in \mathbb{Z}$$

Solution 5 by Vidyamanohar Sharma Astakala-India

Let us call $\ln x = A, \ln(x+1) = B, \ln(x+2) = C$

$f(x) = g(x) = 0$ ***results***

$$\sin(A + B - C) + \sin(-A + B + C) = \sin(A + B + C) - \sin(A - B + C)$$

with trigonometric transformations. This results to

$$\sin A \cdot \sin B \cdot \sin C = 0$$

$$\Rightarrow \sin(\ln x) = 0 \text{ or } \sin(\ln(x+1)) = 0 \text{ or } \sin(\ln(x+2)) = 0$$

$$\Rightarrow x = e^{n\pi}, e^{n\pi}, e^{n\pi} - 2$$

79. In $\Delta\ ABC$ the following relationship holds:

$$abc\prod\left(\int_A^B x\sqrt[3]{\cos x}\,dx\right) \le 8Ss\prod \sin\frac{B - A}{2}$$

s – semiperimeter, S – area

Proposed by Daniel Sitaru-Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{In } \Delta\, ABC, abc\prod\left(\int_A^B x\sqrt[3]{\cos x}\,dx\right)\le 8Ss\prod\sin\frac{B-A}{2}$$

Let us first prove that, $\forall x\in\left(0,\frac{\pi}{2}\right), \sin x > x\sqrt[3]{\cos x}$

$$\sin x > x\sqrt[3]{\cos x}\Leftrightarrow \sin^3 x > x^3\cos x\Leftrightarrow \sin^2 x\tan x > x^3 \quad (1)$$

Let $f(x)=\sin^3 x\tan x - x^3$, $f(0)=0$

$$f'(x)=\sin^2 x\sec^2 x+\tan x\,(2\sin x\cos x)-3x^2$$

$$=\tan^2 x+2\sin^2 x-3x^2=g(x)\text{ (say); } g(0)=0$$

$$g'(x)=2\tan x\sec^2 x+4\sin x\cos x-6x$$

Let $\alpha(x)=\tan x\sec^2 x+2\sin x\cos x-3x$; $\alpha(0)=0$

$$\alpha'(x)=(\sec^2 x)(\sec^2 x)+\tan x\,(2\sec x)(\sec x\tan x)+2(\cos^2 x-\sin^2 x)-3$$

$$=(1+\tan^2 x)^2+2\tan^2 x\,(1+\tan^2 x)+2(2\cos^2 x-1)-3$$

$$=(1+z)^2+2z(1+z)+\frac{4}{1+z}-5\ (\text{taking } z=\tan^2 x \text{ and } \sec^2 x=1+z)$$

$$=\frac{(1+z)^3+2z(1+z)^2+4-5(1+z)}{1+z}$$

$$=\frac{1+z^3+3z+3z^2+2z+2z^3+4z^2-1-5z}{1+z}=\frac{3z^3+7z^2}{1+z}>0\ (z=\tan^2 x>0)$$

$\alpha'(x)>0$ and $\alpha(0)=0$, $\forall x\in\left(0,\frac{\pi}{2}\right), \alpha(x)>\alpha(0)$

$$\Rightarrow \alpha(x)>0\Rightarrow g'(x)=2\alpha(x)>0 \text{ and } g(0)=0,$$

$$\forall x\in\left(0,\frac{\pi}{2}\right), g(x)>g(0)=0\Rightarrow f'(x)>0$$

$f'(x)>0$ and $f(0)=0\Rightarrow f(x)>f(0)=0$, $\forall x\in\left(0,\frac{\pi}{2}\right)$

$$\forall x \in \left(0, \frac{\pi}{2}\right), \sin^2 x \tan x - x^3 > 0 \Rightarrow \sin x > x\sqrt[3]{\cos x} \text{ from (1)}$$

$$\text{For } x = \frac{\pi}{2}, \sin x > x\sqrt[3]{\cos x}$$

$$\text{and for } x \in \left(\frac{\pi}{2}, \pi\right), \sqrt[3]{\cos x} < 0, \sin x > x\sqrt[3]{\cos x}$$

$$\forall x \in (0, \pi), \sin x > x\sqrt[3]{\cos x}$$

80. **Solve in real numbers:**

$$\begin{cases} \tan x \tan y \tan z = 6 \\ \tan x \tan y + \tan x \tan z + \tan y \tan z = 11 \\ x + y + z = \pi \end{cases}$$

Proposed by Daniel Sitaru-Romania

Solution by Kevin Soto Palacios-Peru

Resolver en los números reales: $\tan x \tan y \tan z = 6 \ldots (A)$

$$\tan x \tan y + \tan y \tan z + \tan z \tan x = 1 \ldots (B)$$

Si: $x + y + z = \pi: \tan x + \tan y + \tan z = \tan x \tan y \tan z$

$$\Rightarrow \tan x + \tan y + \tan z = 6 \ldots (I)$$

Desde que tenemos la suma y el producto, se puede construir una ecuación cúbica:

$$(x - \tan x)(x - \tan y)(x - \tan z) = 0 \Leftrightarrow \textit{cuyas raices son}: \tan x, \tan y, \tan z$$

$$x^3 - x^2(\tan x + \tan y + \tan z) + x(\tan x \tan y + \tan y \tan z + \tan z \tan x) -$$

$$- \tan x \tan y \tan z = 0$$

$$x^3 - 6x^2 + 11x - 6 = 0 \to (x - 1)(x - 2)(x - 3) = 0$$

Un posible coso es cuando: $\tan x = 1, \tan y = 2 \wedge \tan z = 3$

$$\Rightarrow x = 45°, y = 63,5° \wedge z = 71,5°$$

81. SECLAMAN'S SEQUENCE

$$a_n = 1 + \frac{n(n+1)}{1+n^2} + \frac{n^2(n^2+1)}{1+n^4} + \cdots + \frac{n^n(n^n+1)}{1+n^{2n}}$$

Find:

$$\Omega = \lim_{n\to\infty}\left(\frac{a_n}{n+1}\right)^{\frac{1}{n(n+1)}}, n \in \mathbb{N}^*$$

Solution by Soumitra Moukherjee-Chandar Nagore-India

$$n \geq \mathbb{N} \text{ i.e } n \geq 1, \textit{now}$$

$$a_n = 1 + \frac{n(n+1)}{1+n^2} + \cdots + \frac{n^n(n^n+1)}{1+n^{2n}} \geq \underbrace{1+1+\cdots+1}_{n+1 \text{ times}} = n+1$$

$$\frac{a_n}{n+1} \geq 1, \textit{again } \frac{n(n+1)}{1+n^2} \leq \frac{n+1}{2}, \textit{since, } \frac{(n+1)(n-1)^2}{2(1+n^2)} \geq 0$$

$$\textit{similarly, } \frac{n^2(n^2+1)}{1+n^4} \leq \frac{1+n^2}{2}, \ldots, \frac{n^n(n^n+1)}{1+n^{2n}} \leq \frac{1+n^n}{2}$$

$$\text{so, } a_n \leq 1 + \frac{n+1}{2} + \frac{1+n^2}{2} + \cdots + \frac{1+n^n}{2} \leq 1 + n + n^2 + \cdots + n^n$$

$$\frac{a_n}{1+n} \leq \frac{1+n+n^2+\cdots+n^n}{1+n} = 1 + n^2 + n^4 + \cdots \leq \underbrace{n^n + n^n + \cdots n^n}_{n \text{ times}} = n^{n+1}$$

$$\left(\frac{a_n}{1+n}\right)^{\frac{1}{n(n+1)}} \leq \sqrt[n]{n}. \text{ So, } 1 \leq \left(\frac{a_n}{1+n}\right)^{\frac{1}{n(n+1)}} \leq \sqrt[n]{n} \Rightarrow 1 \leq \Omega \leq \lim_{n\to\infty}\sqrt[n]{n} = 1$$

$$\textit{so, by Sandwich Theorem, we have: } \Omega = \lim_{n\to\infty}\left(\frac{a_n}{1+n}\right)^{\frac{1}{n(1+n)}} = 1 \ \text{(Ans)}$$

82. SHARP'S SEQUENCE

Find:

$$\Omega = \lim_{n\to\infty}\sqrt{\frac{1}{3}\left(1-\frac{1}{3\cdot 3}+\frac{1}{3^2\cdot 5}-\frac{1}{3^5\cdot 7}+\cdots\right)}$$

Proof by Francis Fregeau – Quebec – Canada

$$\tan^{-1}(x) = \sum_0^{\infty}\frac{(-1)^k}{(2k+1)}x^{2k+1}$$

$$\Rightarrow \sqrt{x}\cdot\tan^{-1}(\sqrt{x}) = \sum_0^{\infty}\frac{(-1)^k}{(2k+1)}(\sqrt{x})^{2(k+1)} = \sum_0^{\infty}\frac{(-1)^k}{(2k+1)}x^{k+1}$$

Let $x = \frac{1}{3}$; $\sqrt{x}\cdot\tan^{-1}(\sqrt{x}) = \sum_0^{\infty}\frac{(-1)^k}{(2k+1)}\left(\frac{1}{3}\right)^{k+1} = \frac{1}{3}\cdot\left(1-\frac{1}{3\cdot 3}+\frac{1}{3^2\cdot 5}-\frac{1}{3^3\cdot 7}+\cdots\right)$

$$\Rightarrow 1-\frac{1}{3\cdot 3}+\frac{1}{3^3\cdot 5}-\frac{1}{3^3\cdot 7}+\cdots = 3\cdot\frac{1}{\sqrt{3}}\tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$\frac{1}{\sqrt{3}}\cdot\left(\frac{1}{3\cdot 3}+\frac{1}{3^2\cdot 5}-\frac{1}{3^3\cdot 7}+\cdots\right) = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right) = \frac{\pi}{6}$$

83. Solve for real numbers:

$$\begin{cases}\frac{x}{\sqrt{y}}+\frac{y}{\sqrt{x}} = xy\\ x^5+y^5 = 8xy\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Kevin Soto Palacios-Huarmey-Peru:

Resolver en números reales: $\frac{x}{\sqrt{y}}+\frac{y}{\sqrt{x}} = xy$ ***(A);*** $x^5+y^5 = 8xy$ ***(B)***

Se puede observar claramente que: $x, y > 0$. ***Aplicando:*** $MA \geq MG$ ***en (A)***

$$\frac{x}{\sqrt{y}}+\frac{y}{\sqrt{x}} \geq 2\sqrt{\frac{x}{\sqrt{y}}\frac{y}{\sqrt{x}}} \Leftrightarrow xy \geq 2\sqrt{\sqrt{xy}} \Leftrightarrow xy \geq \sqrt[3]{16} \quad \textit{(M)}$$

Aplicando: $MA \geq MG$ ***en (B)***

$$x^5 + y^5 \geq 2\sqrt{x^5y^5} \Leftrightarrow 8xy \geq 2\sqrt{x^5y^5} \Leftrightarrow \sqrt[3]{16} \geq xy \quad \textit{(N)}$$

De $(M) \wedge (N)$ ***se puede concluir que*** $xy =: \sqrt[3]{16} \Leftrightarrow x = y = \sqrt[3]{4}$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} = xy \ \ \textit{(1)};\ x^5 + y^5 = 8xy\ \textit{(2)};\ x, y > 0$$

$$\textit{(1)} \Rightarrow \frac{1}{x\sqrt{x}} + \frac{1}{y\sqrt{y}} = 1 \Rightarrow a^3 + b^3 = 1 \left(\frac{1}{\sqrt{x}} = a, \frac{1}{\sqrt{y}} = b\right)$$

$$x = \frac{1}{a^2} \text{ and } y = \frac{1}{b^2} \quad (a, b > 0)$$

$$\textit{(2)} \Rightarrow \frac{1}{a^{10}} + \frac{1}{b^{10}} = \frac{8}{a^2b^2} \Rightarrow a^{10} + b^{10} = 8a^8b^8$$

$$AM \geq GM \Rightarrow a^{10} + b^{10} \geq 2a^5b^5$$

$$\Rightarrow 8a^8b^8 \geq 2a^5b^5 \Rightarrow a^3b^3 \geq \frac{1}{4} \Rightarrow 4a^3b^3 \geq 1$$

$$\Rightarrow 4a^3b^3 \geq (a^3 + b^3)^2 \Rightarrow 0 \geq (a^3 - b^3) \Rightarrow (a^3 - b^3)^2 \leq 0$$

But $(a^3 - b^3)^2 \geq 0 \Rightarrow (a^3 - b^3)^2 = 0 \Rightarrow a^3 = b^3 \Rightarrow a = b$

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{y}} \Rightarrow x = y;\ \frac{2}{x\sqrt{x}} = 1 \Rightarrow \sqrt{x} = \sqrt[3]{2} \Rightarrow x = y = \sqrt[3]{4}$$

84. Solve for real numbers:

$$\begin{cases} \dfrac{x^2}{25} + \dfrac{y^2}{16} = 1 \\ x^2 + y^2 = \left(\dfrac{x^2}{5} + \dfrac{y^2}{4}\right)^2 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov-Maasilli-Azerbaidian

$$\frac{x^2}{25}+\frac{y^2}{16}=1;\ x^2+y^2=\left(\frac{x^2}{5}+\frac{y^2}{4}\right)^2\Rightarrow x^2=a\quad y^2=b$$

$$16a+25b=400$$

$$a+b=\frac{a^2}{25}+\frac{b^2}{16}+\frac{ab}{10}=\frac{16a^2+25b^2+40ab}{400}=\frac{(4a+5b)^2}{400}$$

$$400(a+b)=(4a+5b)^2$$

$$(16a+25b)(a+b)=(4a+5b)^2$$

$$16a^2+16ab+25ab+25b^2=16a^2+25b^2+40ab$$

$$ab=0\Rightarrow a=0\quad x=0\Rightarrow 25b=400\quad b=16\quad y=\pm 4$$

answer $(0;4)$ ***and*** $(0;\ -4)$

$$\Rightarrow b=0\quad y=0\Rightarrow\quad 16a=400\quad a=25\quad x=\pm 5$$

answer $(5;0)$ ***and*** $(-5;\ 0)$

Solution 2 by Kunihiko Chikaya-Tokyo-Japan

$$\begin{cases}\dfrac{x^2}{25}+\dfrac{y^2}{16}=1\\ x^2+y^2=\left(\dfrac{x^2}{5}+\dfrac{y^2}{4}\right)^2\end{cases}\Rightarrow (x,y)\in\mathbb{R}^2;\ \begin{cases}\dfrac{a}{25}+\dfrac{b}{16}=1\quad (*)\\ a+b=\left(\dfrac{a}{5}+\dfrac{b}{4}\right)^2\end{cases}$$

$a=x^2\geq 0;\ b=y^2\geq 0$***. Cauchy – Schwarz***

$$\left(\frac{\sqrt{a}}{5}\sqrt{a}+\frac{\sqrt{b}}{4}\sqrt{b}\right)^2\leq\left\{\left(\frac{\sqrt{a}}{5}\right)^2+\left(\frac{\sqrt{b}}{4}\right)^2\right\}\left\{\left(\sqrt{a}\right)^2+\left(\sqrt{b}\right)^2\right\}$$

$$\left(\frac{a}{5}+\frac{b}{4}\right)^2\leq 1\cdot(a+b)$$

Equality: $\begin{pmatrix}\frac{\sqrt{a}}{5}\\ \frac{\sqrt{b}}{4}\end{pmatrix}=\begin{pmatrix}\sqrt{a}\\ \sqrt{b}\end{pmatrix}\ \&\ (*)\Leftrightarrow ab=0\ \&\ (*)\Leftrightarrow (a,b)=(25,0),(0,16)$

Ans. $(x, y) = (\pm 5, 0), (0, \pm 4)$

85. **Solve in real numbers:**

$$\begin{cases} \frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} = xy \\ x^5 + y^5 = 8xy \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Soumava Chakraborty-Kolkata-India

$$\frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} = xy \ \ \textbf{(1)};\ x^5 + y^5 = 8xy\ \ \textbf{(2)};\ x, y > 0$$

$$\textbf{(1)} \Rightarrow \frac{1}{x\sqrt{x}} + \frac{1}{y\sqrt{y}} = 1 \Rightarrow a^3 + b^3 = 1, \left(\frac{1}{\sqrt{x}} = a, \frac{1}{\sqrt{y}} = b\right)$$

$$x = \frac{1}{a^2} \text{ and } y = \frac{1}{b^2}\ (a, b > 0)$$

$$\textbf{(2)} \Rightarrow \frac{1}{a^{10}} + \frac{1}{b^{10}} = \frac{8}{a^2b^2} \Rightarrow a^{10} + b^{10} = 8a^8b^8$$

$$AM \geq GM \Rightarrow a^{10} + b^{10} \geq 2a^5b^5$$

$$\Rightarrow 8a^8b^8 \geq 2a^5b^5 \Rightarrow a^3b^3 \geq \frac{1}{4} \Rightarrow 4a^3b^3 \geq 1$$

$$\Rightarrow 4a^3b^3 \geq (a^3 + b^3)^2 \Rightarrow 0 \geq (a^3 - b^3)^2 \Rightarrow (a^3 - b^3)^2 \leq 0$$

$$\text{But } (a^3 - b^3)^2 \geq 0 \Rightarrow (a^3 - b^3)^2 = 0 \Rightarrow a^3 = b^3 \Rightarrow a = b$$

$$\frac{1}{\sqrt{x}} = \frac{1}{\sqrt{y}} \Rightarrow x = y;\ \frac{2}{x\sqrt{x}} = 1 \Rightarrow \sqrt{x} = \sqrt[3]{2} \Rightarrow x = y = \sqrt[3]{4}$$

Solution 2 by Kevin Soto Palacios-Huarmey-Peru:

Resolver en números reales: $\frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} = xy$ ***(A);*** $x^5 + y^5 = 8xy$ ***(B)***

Se puede observar claramente que: $x, y > 0$

Aplicando: $MA \geq MG$ ***en (A)***

$$\frac{x}{\sqrt{y}} + \frac{y}{\sqrt{x}} \geq 2\sqrt{\frac{x}{\sqrt{y}}\frac{y}{\sqrt{x}}} \Leftrightarrow xy \geq 2\sqrt{\sqrt{xy}} \Leftrightarrow xy \geq \sqrt[3]{16} \quad \textit{(M)}$$

Aplicando: $MA \geq MG$ ***en (B)***

$$x^5 + y^5 \geq 2\sqrt{x^5y^5} \Leftrightarrow 8xy \geq 2\sqrt{x^5y^5} \Leftrightarrow \sqrt[3]{16} \geq xy \quad \textit{(N)}$$

De $(M) \wedge (N)$ ***se puede concluir que*** $xy =: \sqrt[3]{16} \Leftrightarrow x = y = \sqrt[3]{4}$

86. Find $x, y, z \in \mathbb{R}$ such that:

$$\begin{cases} x + y + z = 3 \\ 4(\max(x,y,z) - \min(x,y,z))^2 \geq 3\sum |x-y|^2 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Ravi Prakash-New Delhi-India

Let $x \geq y \geq z$***, then*** $\max\{x,y,z\} = x, \min\{x,y,z\} = z$

$$4(\max(x,y,z) - \min(x,y,z))^2 \geq 3\sum |x-y|^2$$

$$\Rightarrow 4(x-z)^2 \geq 3[(x-y)^2 + (y-z)^2 + (z-x)^2]$$

$$\Rightarrow (x-z)^2 \geq 3(x-y)^2 + 3(y-z)^2$$

$$\Rightarrow [(x-y) + (y-z)]^2 \geq 3(x-y)^2 + 3(y-z)^2$$

$$\Rightarrow (x-y)^2 + (y-z)^2 + 2(x-y)(y-z) \geq 3(x-y)^2 + 3(y-z)^2$$

$$\Rightarrow (x-y)^2 + (y-z)^2 - (x-y)(y-z) \leq 0$$

$$\Rightarrow \left[x - y - \frac{1}{2}(y-z)\right]^2 + \frac{3}{4}(y-z)^2 \leq 0 \Rightarrow x - \frac{3}{2}y + \frac{1}{2}z = 0, y = z$$

$$\Rightarrow x = y = z \therefore x = y = z = 1$$

87. Find $x, y, z \in (0, \infty)$:

$$\begin{cases} x+y+z=xyz \\ \dfrac{x}{y^3z^2}+\dfrac{y}{z^3x^2}+\dfrac{z}{x^3y^2}=\dfrac{1}{3} \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Huarmey-Peru

Hallar: $x,y,z\in<0,\infty>.\ x+y+z=xyz;\ \frac{x}{y^3z^2}+\frac{y}{z^3x^2}+\frac{z}{x^3z^2}=\frac{1}{3}$

Desde que: $x,y,z\in<0,\infty>$

$$x=\tan A,\ y=\tan B,\ z=\tan C\Leftrightarrow A+B+C=\pi$$

Reemplazando en la segunda condición:

$$\frac{\tan A}{\tan^3 B\tan^2 C}+\frac{\tan B}{\tan^3 C\tan^2 A}+\frac{\tan C}{\tan^3 A\tan^2 B}=\frac{1}{3}$$

Por desigualdades entre las medias:

$$MA\geq MG\Leftrightarrow \tan A,\tan B,\tan C>0$$

$$\frac{\tan A}{\tan^3 B\tan^2 C}+\frac{\tan B}{\tan^3 C\tan^2 A}+\frac{\tan C}{\tan^3 A\tan^2 B}\geq 3\sqrt[3]{\cot^4 A\cot^4 B\cot^4 C}$$

$$\left(\frac{1}{9}\right)^3\geq\cot^4 A\cot^4 B\cot^4 C\Rightarrow\left(\frac{1}{3}\right)^6\geq\cot^4\cot^4 B\cot^4 C,$$

$$\frac{1}{3\sqrt{3}}\geq\cot A\cot B\cot C$$

La cual es cierto en un $\Delta\ ABC$ equilátero, la igualdad se alcanza cuando:

$$x=y=z=\sqrt{3}$$

88. Find $x,y,z\in(0,\infty)$ such that:

$$\begin{cases} x+y+z=xyz \\ \dfrac{x}{y^3z^2}+\dfrac{y}{z^3x^2}+\dfrac{z}{x^3y^2}=\dfrac{1}{3} \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios – Peru

Hallar: $x, y, z \in < 0, \infty >$. $x + y + z = xyz$

$\frac{x}{y^3z^2} + \frac{y}{z^3x^2} + \frac{z}{x^3y^2} = \frac{1}{3}$. ***Desde que:*** $x, y, z \in < 0, \infty >$

$$x = \tan A, y = \tan B, z = \tan C \Leftrightarrow A + B + C = \pi$$

Reemplazando en la segunda condición:

$$\frac{\tan A}{\tan^3 B \tan^2 C} + \frac{\tan B}{\tan^3 C \tan^2 A} + \frac{\tan C}{\tan^3 \tan^2 B} = \frac{1}{3}$$

Por la desigualdades entre las medias:

$$MA \geq MG \Leftrightarrow \tan A, \tan B, \tan C > 0$$

$$\frac{\tan A}{\tan^3 B \tan^2 C} + \frac{\tan B}{\tan^3 C \tan^2 A} + \frac{\tan C}{\tan^3 A \tan^2 B} \geq 3\sqrt[3]{\cot^4 A \cot^4 B \cot^4 C}$$

$$\left(\frac{1}{9}\right)^3 \geq \cot^4 A \cot^4 B \cot^4 C \Rightarrow \left(\frac{1}{3}\right)^6 \geq \cot^4 A \cot^4 B \cot^4 C, \frac{1}{3\sqrt{3}} \geq \cot A \cot B \cot C$$

La cual es cierto en un $\Delta\, ABC$ equilátero, la igualdad se alcanza cuando:

$$x = y = z = \sqrt{3}$$

89. Solve for real numbers:

$$\begin{cases} 1 + 2\sqrt{y} = 2\sqrt{x+1} \\ \frac{2\sqrt{y}}{12y+1} + \frac{\sqrt{x+1}}{x+4} + \frac{2\sqrt{y(x+1)}}{3x+4y+3} = \frac{3}{4} \end{cases}$$

Proposed by Ngo Minh Ngoc Bao – Vietnam

Solution by Soumava Chakraborty – Kolkata – India

Solve for $x, y \in \mathbb{R}$ ***satisfying:*** $1 + 2\sqrt{y} = 2\sqrt{x+1}$ ***(1)***

$$\frac{2\sqrt{y}}{12y+1}+\frac{\sqrt{x+1}}{x+4}+\frac{2\sqrt{y(x+1)}}{3x+4y+3}=\frac{3}{4}\quad (2)$$

$$(2)\Rightarrow\frac{\sqrt{y}}{12y+1}+\frac{1+2\sqrt{y}}{13+4\sqrt{y}+4y}+\frac{2\sqrt{y}+4y}{3+12\sqrt{y}+28y}=\frac{3}{8}$$

$$\Rightarrow\frac{t}{12t^2+1}+\frac{1+2t}{13+4t+4t^2}+\frac{2t+4t^2}{3+12t+28t^2}\overset{(using\,(1))}{=}\frac{3}{8}$$

$$(t=\sqrt{y})$$

$$\Rightarrow 2496t^6-2816t^5+3440t^4-1728t^3+148t^2-160t+93=0$$

$$\Rightarrow(2t-1)^2\underbrace{(624t^4-80t^3+624t^2+212t+93)}_{e}=0$$

Now, $624t+624t^2\overset{A-G}{\geq}1248t^3>80t^3$

(when $t>0$ **and** $t=\sqrt{y}\geq 0$**)** $\Rightarrow e>0,\forall\, t>0$

For $t=0, e=93>0\Rightarrow\forall\, t\geq 0, e>0$

$$\therefore 2t=1\Rightarrow t=\frac{1}{2}\Rightarrow\sqrt{y}=\frac{1}{2}\Rightarrow y=\frac{1}{4}\Rightarrow x=0$$

$\therefore$ ***only solution is*** $(x,y)=\left(0,\frac{1}{4}\right)$***. At*** $t=\frac{1}{2}$ $f''(t)=-\frac{3}{8}<0$

$\therefore$ ***at*** $t=\frac{1}{2}$, $f(t)$ ***attains a maxima, and*** $\because f(t)$ ***never attains a minima***

$\forall\, t\geq 0,\therefore f(t)\leq f\left(\frac{1}{2}\right)=\frac{3}{8}$***. But (1)*** $\Rightarrow f(t)=\frac{3}{8}\therefore t=\frac{1}{2}\Rightarrow\sqrt{y}=\frac{1}{2}\Rightarrow y=\frac{1}{4}$

Putting, $y=\frac{1}{4}$ ***in (1),*** $\sqrt{x+1}=1\Rightarrow x=0\therefore$ ***only solution is*** $(x,y)=\left(0,\frac{1}{4}\right)$

90. Find $z\in\mathbb{C}$ such that:

$$\begin{cases}|z-7-i|=3\sqrt{2}\\|z-1-7i|\leq 3\sqrt{2}\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash - New Delhi – India

Two circles $|z-7-i| = 3\sqrt{2}$ ***and*** $|z-1-7i| = 3\sqrt{2}$

touch each other externally as $|(7+i)-(1+7i)| = 6\sqrt{2} = 3\sqrt{2}+3\sqrt{2}$

Thus, $|z-7-i| = 3\sqrt{2}$ ***(1) and*** $|z-1-7i| \le 3\sqrt{2}$

meet exactly at one point viz. the mid – point $4+4i$ ***of segment*** AB

where $A(7+i), B(1+7i)$

Solution 2 by Bedri Hajrizi-Mitrovica-Kosovo

Let $z = x + yi$***. Then:*** $\begin{cases}(x-7)^2+(y-1)^2 = 18\\(x-1)^2+(y-7)^2 \le 18\end{cases} \quad \dots \quad (\forall)$

We seek first for solutions of system:

$$\begin{cases}(x-7)^2+(y-1)^2 = 18\\(x-1)^2+(y-7)^2 = 18\end{cases} \sim \begin{cases}x^2-14x+y^2-2y = -32\\x^2-2x+y^2-14y = -32\end{cases}$$

Substracting we get: $12x-12y = 0 \Rightarrow x = y$

Substrabting in first equation, we get:

$$2x^2-16x+32 = 0 \Leftrightarrow x^2-8x+16 = 0 \Leftrightarrow x = 4.\ \textit{So}\ y = 4.$$

Obviously circles are tangentially outside, so solution of initial system is $(4,4)$***. Conclusion: Solution is*** $z = 4+4i$***.***

91. Solve in real numbers:

$$\begin{cases}3^x+3^y+3^z+3^t = 24\\ \log_z x + \log_z t = y\\ \dfrac{x}{x^4+y^2}+\dfrac{y}{x^2+y^4} = \dfrac{1}{xy}\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Redwane El Mellass – Morocco

- ***from the equation 2:*** $x, t, z \neq 1 > 0$.
- ***from the equation 3:*** $\frac{x^2y}{x^4+y^2} + \frac{y^2x}{x^2+y^4} = 1\ (\circledast)$ ***with*** $\frac{y^2x}{x^2+y^4} \overset{AM-GM}{\leq} \frac{1}{2}\frac{y^2x}{xy^2} = \frac{1}{2}$

it follows $\frac{x^2y}{x^4+y^2} = 1 - \frac{y^2x}{x^2+y^4} \geq \frac{1}{2} \Rightarrow y > 0$. ***Now we can use AM-GM in*** $(\circledast)$,

we get $\frac{x^2y}{x^4+y^2} + \frac{y^2x}{x^2+y^4} \leq \frac{1}{2}\left(\frac{x^2y}{x^2y} + \frac{xy^2}{xy^2}\right) = 1 \Rightarrow x = y$

So $(\circledast) \Rightarrow \frac{2x}{1+x^2} = 1 \Leftrightarrow (x-1)^2 = 0 \Rightarrow x = y = 1$.

- ***from the equation 2,*** $\log(t) = \log(z) = t = z$.
- ***from the equation 1,*** $2 \cdot 3^z + 6 = 24 \Rightarrow 3^z = 9 \Rightarrow z = t = 2$

Finally $(x, y, z, t) = (1, 1, 2, 2)$. ***NB: The trap of the problem is we can't know at first if*** $y < 0$ ***or*** $y > 0$***!***

Solution 2 by Ravi Prakash – New Delhi – India

Clearly, $x > 0, t > 0, z > 0, z \neq 1$. ***From 3rd equation***

$$\left(\frac{1}{2xy} - \frac{x}{x^4 + y^2}\right) + \left(\frac{1}{2xy} - \frac{y}{x^2 + y^4}\right) = 0$$

$$\Rightarrow \frac{(x^4 + y^2 - 2x^2y)}{2xy(x^4 + y^2)} + \frac{(x^2 + y^4 - 2xy^2)}{2x(x^2 + y^4)} = 0 \Rightarrow \frac{(x^2 - y)^2}{(x^4 + y^2)} + \frac{(x - y^2)^2}{x^2 + y^4} = 0$$

$$\Rightarrow x^2 - y = 0, x - y^2 = 0 \Rightarrow y^4 = x^2 = y \Rightarrow y(y^3 - 1) = 0$$

$\Rightarrow y = 1\ \ [\because y \in \mathbb{R}, y \neq 0] \therefore x = 1$. ***From 2nd equation***

$\log_z t = y = 1\ [\because \log_z x = 0] \Rightarrow t = z$. ***From first equation***

$$3 + 3 + 3^z + 3^z = 24 \Rightarrow 2(3^z) = 18 \Rightarrow z = 2 \therefore x = 1, y = 1, z = 2, t = 2$$

92. Solve the system of equations:

$$\begin{cases} \sqrt{x} + \sqrt{y} + \sqrt{z} + 1 = 4\sqrt{xyz} \\ xy + yz + zx + 3 = 2 \cdot \left(\sqrt[4]{x} + \sqrt[4]{y} + \sqrt[4]{z}\right) \end{cases} \quad \textbf{(1)}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution 1 by Hoang Le Nhat Tung – Hanoi – Vietnam

- ***We have:*** $x \geq 0, y \geq 0, z \geq 0$

- ***Put*** $\begin{cases} \sqrt{x} = a \\ \sqrt{y} = b; (a \geq 0, b \geq 0, c \geq 0) \\ \sqrt{z} = c \end{cases} \Leftrightarrow \begin{cases} x = a^2 \\ y = b^2 \\ z = c^2 \end{cases}$

- ***Therefore, (1):*** $\Leftrightarrow \begin{cases} a + b + c = 1 = 4abc \\ a^2b^2 + b^2c^2 + c^2a^2 + 3 = 2(\sqrt{a} + \sqrt{b} + \sqrt{c}) \end{cases}$ ***(2)***

**** Lemma: Let*** $a, b, c \geq 0$ ***such that:*** $a + b + c = 1 = 4abc$ ***then:***

$$ab + bc + ca \geq a + b + c \quad (3)$$

- Considering 3 terms: $a - 1; b - 1; c - 1$***. Considering Dirichle,***

Suppose: $(a - 1)(b - 1) \geq 0$

$$\Leftrightarrow ab - a - b + 1 \geq 0 \Leftrightarrow ab \geq a + b - 1 \Leftrightarrow ab + bc + ca \geq bc + ca + a + b - 1 \quad (4)$$

+ Since (3), (4), we need to prove:

$$bc + ca + a + b - 1 \geq a + b + c \Leftrightarrow bc + ca \geq c + 1 \Leftrightarrow c(a + b) \geq c + 1 \quad (5)$$

- Other: $a + b + c + 1 = 4abc \Leftrightarrow a + b + 1 = c(4ab - 1) \Leftrightarrow c = \frac{a+b+1}{4ab-1}$ ***(6)***

Since (6) and (5). Therefore inequality:

$$\Leftrightarrow \frac{(a + b + 1)(a + b)}{4ab - 1} \geq \frac{a + b + 1}{4ab - 1} + 1 \Leftrightarrow \frac{(a + b)^2 + (a + b)}{4ab - 1} \geq \frac{a + b + 1 + 4ab - 1}{4ab - 1}$$

$$\Leftrightarrow (a + b)^2 + (a + b) \geq 4ab + (a + b) \Leftrightarrow (a + b)^2 \geq 4ab \Leftrightarrow (a - b)^2 \geq 0 \quad \textit{(True)}$$

$\Rightarrow$ ***Inequality (5) True*** $\Rightarrow (3)$ ***True:*** $ab + bc + ca \geq a + b + c$ ***(7)***

$\Rightarrow$ ***Lemma is proven. * : Since inequality AM-GM for 2 real numbers:***

$$a^2b^2 + b^2c^2 + c^2a^2 + 3 = (a^2b^2 + 1) + (b^2c^2 + 1) + (c^2a^2 + 1) \geq 2ab + 2bc + 2ca =$$
$$= 2(ab + bc + ca)$$

$$\Leftrightarrow a^2b^2 + b^2c^2 + c^2a^2 + 3 \geq 2(ab + bc + ca) \quad (8)$$

- Since (7), (8) $\Rightarrow a^2b^2 + b^2c^2 + c^2a^2 + 3 \geq 2(a + b + c)$ ***(9)***

-: Since inequality AM-GM for 2 real numbers:

$$2(\sqrt{a}+\sqrt{b}+\sqrt{c})=2\sqrt{a\cdot 1}+2\sqrt{b\cdot 1}+2\sqrt{c\cdot 1}\le(a+1)+(b+1)+(c+1)=a+b+c+3\quad \textbf{(10)}$$

- Other, Since inquality AM-GM for 3 real numbers:

$$a+b+c\ge 3\cdot\sqrt[3]{abc}\Leftrightarrow abc\le\left(\frac{a+b+c}{3}\right)^3=\frac{(a+b+c)^3}{27}$$

- Since (2) $\Rightarrow a+b+c+1=4abc\le\frac{4(a+b+c)^3}{27}\Leftrightarrow 4(a+b+c)^3\ge 27(a+b+c)+27$ ***(11)***

+ Put $a+b+c=t\ge 0$. ***Therefore (11):***

$$\Leftrightarrow 4t^3\ge 27t+27\Leftrightarrow 4t^3-27t-27\ge 0$$

$$\Leftrightarrow 4t^2(t-3)+12t(t-3)+9(t-3)\ge 0\Leftrightarrow(t-3)(4t^2+12t+9)\ge 0$$

$$\Leftrightarrow(t-3)(2t+3)^2\ge 0\Leftrightarrow t-3\ge 0\Leftrightarrow t\ge 3\ (\text{Because } (2t+3)^2>0,\forall t\ge 0)$$

$$\Leftrightarrow a+b+c\ge 3\Leftrightarrow 3\le a+b+c\qquad \textbf{(12)}$$

- Since (10), (12) $\Rightarrow 2(\sqrt{a}+\sqrt{b}+\sqrt{c})\le(a+b+c)+(a+b+c)=2(a+b+c)$ ***(13)***

- Since (9), (13) $\Rightarrow a^2b^2+b^2c^2+c^2a^2+3\ge 2(\sqrt{a}+\sqrt{b}+\sqrt{c})$ ***(14)***

- Since (2), (14) $\Rightarrow$ ***Equality occurs if*** $\begin{cases}(a-1)(b-1)=0; a=b\\ ab+bc+ca=a+b+c\\ a=b=c=1; ab=bc=ca=1\\ a\ge 0; b\ge 0; c\ge 0\end{cases}\Leftrightarrow a=b=c=1$

$$\Leftrightarrow\sqrt{x}=\sqrt{y}=\sqrt{z}=1\Leftrightarrow x=y=z=1$$

So the system of equation: $(x,y,z)=(1,1,1)$

Solution 2 by Aditya Narayan Sharma-Kanchrapara-India

Set $a=x^{\frac{1}{4}}, b=y^{\frac{1}{4}}, c=z^{\frac{1}{4}}\therefore x=a^4; y=b^4; z=c^4$. ***Now,***

$$a^2+b^2+c^2+1=4a^2b^2c^2\qquad \textbf{(1)}$$

$$a^4b^4+b^4c^4+c^4a^4+3=2(a+b+c)\qquad \textbf{(2)}$$

By AM-GM: $a^4b^4+b^4c^4+c^4a^4+1\ge 4a^2b^2c^2$

$$\Rightarrow 2(a+b+c)-\frac{3}{+1}\ge a^2+b^2+c^2+1.\ \textbf{[From (1); (2)]}$$

$$\Rightarrow(a-1)^2+(b-1)^2+(c-1)^2\le 0\qquad \textbf{(3)}$$

But since they all are perfect squares,

$$(a-1)^2+(b-1)^2+(c-1)^2\geq 0 \quad \textbf{(4)}$$

So (3); (4) imply, $(a-1)^2+(b-1)^2+(c-1)^2=0$

$$\therefore a=b=c=1 \therefore x=y=z=1$$

Solution 3 by Soumava Chakraborty-Kolkata-India

Solve in $\mathbb{R}$: $\sqrt{x}+\sqrt{y}+\sqrt{z}+1=4\sqrt{xyz}$ ***(1)***

$$xy+yz+zx+3=2\left(\sqrt[4]{x}+\sqrt[4]{y}+\sqrt[4]{z}\right) \quad \textbf{(2)}$$

If $x=0$, ***(1)*** $\Rightarrow \sqrt{y}+\sqrt{z}+1=0$

But $\sqrt{y}+\sqrt{z}+1\geq 1 \Rightarrow \sqrt{y}+\sqrt{z}+1=0$ ***is impossible,*** $\Rightarrow x\neq 0$

Similalry, it can be concluded that $y,z\neq 0 \therefore x,y,z\neq 0 \therefore x,y,z>0$

Let $\sqrt[4]{x}=a, \sqrt[4]{y}=b, \sqrt[4]{z}=c; a,b,c>0$

Then (1) $\Rightarrow a^2+b^2+c^2+1=4a^2b^2c^2$ ***(3)***

(2) $\Rightarrow a^4b^4+b^4c^4+c^4a^4+3=2(a+b+c)$ ***(4)***

Now, $a^2+b^2+c^2+1\overset{A-G}{\geq} 4\sqrt[4]{a^2b^2c^2}$

$\therefore$ ***(3)*** $\Rightarrow 4a^2b^2c^2\geq 4\sqrt[4]{a^2b^2c^2}\Rightarrow a^8b^8c^8\geq a^2b^2c^2$

$$\Rightarrow a^6b^6c^6\geq 1\Rightarrow abc\geq 1$$

Now, $2(a+b+c)\leq 2abc(a+b+c)\overset{(5)}{\leq} 2(a^2b^2+b^2c^2+c^2a^2)$

($\because\ \alpha^2+\beta^2+\gamma^2\geq \alpha\beta+\beta\gamma+\gamma\alpha$, ***where*** $\alpha=ab,\beta=bc,\gamma=ca$***)***

(4), (5) $\Rightarrow a^4b^4+b^4c^4+c^4a^4+3\leq 2(a^2b^2+b^2c^2+c^2a^2)$

$$\Rightarrow 2(a^2b^2+b^2c^2+c^2a^2)\geq ((a^2b^2)^2+(b^2c^2)^2+(c^2a^2)^2)+3$$

$$\geq\left(\frac{1}{3}(a^2b^2+b^2c^2+c^2a^2)^2\right)+3$$

($\because 3(u^2+v^2+w^2)\geq(u+v+w)^2$, ***where*** $u=a^2b^2, v=b^2c^2, w=c^2a^2$***)***

$$\Rightarrow 2t \geq \frac{t^2}{3} + 3 \text{ (where } t = a^2b^2 + b^2c^2 + c^2a^2\text{)}$$

$$\Rightarrow 6t \geq t^2 + 9 \Rightarrow (t-3)^2 \leq 0$$

$$\text{But } (t-3)^2 \geq 0$$

$$\therefore (t-3)^2 = 0 \Rightarrow t = 3 \Rightarrow a^2b^2 + b^2c^2 + c^2a^2 = 3 \quad (6)$$

$$\text{But } \sum a^2b^2 \overset{A-G}{\geq} 3\sqrt[3]{a^4b^4c^4} \geq 3 \ (\because abc \geq 1),$$

$$\text{equality when } a = b = c \therefore \sum a^2b^2 = 3 \Rightarrow a = b = c \quad (7)$$

$$\therefore 3a^4 = 3 \text{ (from (6), (7))} \Rightarrow a = 1 \Rightarrow a = b = c = 1$$

$$\Rightarrow x = y = z = 1 \text{ is the only solution}$$

Solution 4 by Saptak Bhattacharya-Kolkata-India

$$\text{Clearly } x, y, z > 0$$

$$\sum \sqrt{x} + 1 = 4\sqrt{xyz} \quad (i)$$

$$2\left(\sum \sqrt[4]{x}\right) - 3\sum xy \quad (ii)$$

$$(i) - (ii) \text{ gives}$$

$$\sum\left(\sqrt{x} - 2\sqrt[4]{x} + 1\right) + 1 = 4\sqrt{x+z} - \sum xy$$

$$\Rightarrow \sum\left(\sqrt[4]{x} - 1\right)^2 + 1 = 4\sqrt{xyz} - \sum xy \quad (iii)$$

$$\text{Clearly } LHS \geq 1\text{. So: } RHS \geq 1\text{. Thus: } \sum xy + 1 \leq 4\sqrt{xyz}$$

$$\text{But by AM} \geq \text{GM: } \sum xy + 1 \geq 4\sqrt{xyz}\text{. Thus; } \sum xy + 1 = 4\sqrt{xyz}$$

$$\text{Thus; equality holds; possible if } x = y = z = k \text{ (let). Thus, from (iii)}$$

$$\sum\left(\sqrt[4]{x} - 1\right)^2 = 0 \Rightarrow x = y = z = 1$$

93. Solve for integers:

$$\begin{cases} x(y+z) = y^2 + z^2 - 6 \\ y(z+x) = z^2 + x^2 - 6 \\ z(x+y) = x^2 + y^2 - 6 \end{cases}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Serban George Florin-Romania

Ec (1) – ec (2) $z(x-y) = y^2 - x^2, z(x-y) + (x-y)(x+y) = 0, (x-y)(x+y+z) = 0$***, Case 1:***

$$x - y = 0, x = y, \begin{cases} x(x+z) = x^2 + z^2 - 6 \\ 2xz = 2x^2 - 6 \end{cases}, \begin{cases} xz = z^2 - 6 \\ xz = x^2 - 3 \end{cases}, z^2 - x^2 = 3, (z-x)(z+x) = 3$$

$$\begin{cases} z - x = 1 \\ z + x = 3 \end{cases}, z = 2, x = 1, 2 \cdot 1 = 2^2 - 6, 2 = -2 \textbf{ (F)}$$

$$\begin{cases} z - x = 3 \\ z + x = 1 \end{cases}, z = 2, x = y = -1, 2 \cdot (-1) = 2^2 - 6, -2 = -2 \textbf{ (A)}, 2 \cdot (-1) = (-1)^2 - 3 \textbf{ (A)}.$$

$$\begin{cases} z - x = -1 \\ z + x = -3 \end{cases}, z = -2, x = -1, (-2) \cdot (-1) = (-2)^2 - 6, 2 = -2 \textbf{ (F)},$$

$$\begin{cases} z - x = -3 \\ z + x = -1 \end{cases}, z = -2, x = y = 1, (-2) \cdot 1 = (-2)^2 - 6, -2 = -2 \textbf{ (A)}, (-2) \cdot 1 = (-1)^2 - 3 \textbf{ (A)}$$

Case 2:

$$x + y + z = 0, \begin{cases} x \cdot (-x) = y^2 + z^2 - 6 \\ y \cdot (-y) = x^2 + z^2 - 6, x^2 + y^2 + z^2 = 6, y^2 + z^2 = 6 - x^2 \geq 0, x \in \{-2, -1, 0, 1, 2\} \\ z \cdot (-z) = y^2 + x^2 - 6 \end{cases}$$

If $x = -2, y^2 + z^2 = 2, y + z = -x = 2, (-2, 1, 1)$ ***sol.***

If $x = 2, y^2 + z^2 = 2, y + z = -x = -2, (2, -1, -1)$ ***sol.***

If $x = -1, y^2 + z^2 = 5, y + z = -x = 1, (-1, 2 - 1), (-1, -1, 2)$ ***sol***

If $x = 1, y^2 + z^2 = 5, y + z = -x = -1, (1, -2, 1), (1, 1, -2)$ ***sol***

If $x = 0, y^2 + z^2 = 6, y + z = -x = 0$ ***(F)***

$$S = \{(-1, -1, 2), (1, 1, -2), (-2, 1, 1), (2, -1, -1), (-1, 2, -1), (1, -2, 1)\}$$

Solution 2 by Ravi Prakash-New Delhi-India

$$x(y+z) = y^2 + z^2 - 6 \quad \textbf{(1)}; \; y(z+x) = z^2 + x^2 - 6 \quad \textbf{(2)}$$

$$z(x+y) = x^2 + y^2 - 6 \quad \textbf{(3)}$$

Adding we get $2(xy + yz + zx) = 2(x^2 + y^2 + z^2 - 9)$

$\Rightarrow xy + yz + zx = x^2 + y^2 + z^2 - 9$ ***(4). From (1), (4)***

$yz = x^2 - 3 \Rightarrow x^2 - yz = 3$ ***(5). Similarly*** $y^2 - zx = 3$ ***(6)***

$z^2 - xy = 3$ ***(7)*** $\therefore x^2 - y^2 - yz + zx = 0 \Rightarrow (x-y)(x+y+z) = 0$***, etc.***

When $x + y + z \neq 0$***, we get***$\Rightarrow x = y = z$***. Not possible in view of (5)***

$\therefore x + y + z = 0$***. From (1), we get*** $-x^2 = y^2 + z^2 - 6 \Rightarrow x^2 + y^2 + z^2 = 6$

and $xy + yz + zx = -3$ ***[from (4)]***

Also, $x^2 + y^2 + (-x-y)^2 = 6 \Rightarrow x^2 + xy + y^2 = 3$

$$\Rightarrow x^2 + xy + y^2 - 3 = 0$$

$$\Rightarrow x = \frac{-y \pm \sqrt{y^2 - 4(y^2-3)}}{2} = \frac{1}{2}\left(-y \pm \sqrt{3(4-y^2)}\right)$$

Thus, $-2 \le y \le 2$***. Similarly,*** $-2 \le x \le 2, -2 \le z \le 3$

As x, y, z ***are integers***

$$(x,y,z) = (-2,1,1),(2,-1,-1),(-1,2,-1),(1,-2,1),(1,1,-2),(-1,-1,2)$$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$x(y+z) \underset{(1)}{=} y^2 + z^2 - 6;\ y(z+x) \underset{(2)}{=} z^2 + x^2 - 6;\ z(x+y) \underset{(3)}{=} x^2 + y^2 - 6$$

(1) + (2) + (3) $\Rightarrow 2\sum x^2 - 2\sum xy = 18$

$$\Rightarrow (x-y)^2 + (y-z)^2 + (z-x)^2 = 18$$

$\because x, y, z$ ***are integers,*** $\therefore$ ***possible values of***

$(x-y)^2, (y-z)^2, (z-x)^2$ ***are*** $(16,1,1,)$ ***permutating and*** $(9,9,0)$ ***and permutations, considering all*** 3 ***perfect square numbers whose sum*** $= 18$***.***

Case (1): $(x-y)^2 = 9, (y-z)^2 = 9, (z-x)^2 = 0 \therefore z = x, x - y = \pm 3$

Case (1/a): $z = x, x - y = 3 \Rightarrow y = x - 3$

$\therefore$ ***(3)*** $\Rightarrow x(2x-3) = x^2 + x^2 - 6x + 9 - 6$

$\Rightarrow 3x = 3 \Rightarrow x = 1 \Rightarrow y = -2, z = 1 \Rightarrow (x,y,z) = (1,-2,1)$ ***is a solution.***

Case (1/b): $z = x, x - y = -3 \Rightarrow y = x + 3$

$\therefore$ ***(3)*** $\Rightarrow x(2x + 3) = x^2 + x^2 + 6x + 9 - 6$

$\Rightarrow 3x = -3 \Rightarrow x = -1 \Rightarrow y = 2, z = -1$

$\Rightarrow (x, y, z) = (-1, 2, -1)$ ***is a solution.***

Case (2): $(y - z)^2 = 9, (z - x)^2 = 9, (x - y)^2 = 0 \therefore x = y, y - z = \pm 3$

Case (2/a): $x = y, y - z = 3 \Rightarrow z = y - 3$

$\therefore$ ***(1)*** $\Rightarrow y(2y - 3) = y^2 + y^2 - 6y + 9 - 6$

$\Rightarrow 3y = 3 \Rightarrow y = 1 \Rightarrow z = -2, x = 1 \Rightarrow (x, y, z) = (1, 1, -2)$ ***is a solution***

Case (2/b): $x = y, y - z = -3 \Rightarrow z = y + 3$

$\therefore$ ***(1)*** $\Rightarrow y(2y + 3) = y^2 + y^2 + 6y + 9 - 6 \Rightarrow 3y - 3 \Rightarrow y = -1$

$\Rightarrow x = -1, z = 2, \therefore (x, y, z) = (-1, -1, 2)$ ***is a solution.***

Case (3): $(z - x)^2 = 9, (x - y)^2 = 9, (y - z)^2 = 0 \therefore y = z, z - x = \pm 3$

Case (3a): $y = z, z - x = 3 \Rightarrow x = z - 3$

$\therefore$ ***(2)*** $\Rightarrow z(2z - 3) = z^2 + z^2 - 6z + 9 - 6 \Rightarrow z = 1 \Rightarrow x = -2, y = 1$

$\Rightarrow (x, y, z) = (-2, 1, 1)$ ***is a solution.***

Case (3/b): $y = z, z - x = -3 \Rightarrow x = z + 3$

$\therefore$ ***(2)*** $\Rightarrow z(2z + 3) = z^2 + z^2 + 6z + 9 - 6 \Rightarrow z = -1 \Rightarrow x = 2, y = -1$

$\Rightarrow (x, y, z) = (2, -1, -1)$ ***is a solution***

Case (4): $(x - y)^2 = 16, (y - z)^2 = 1, (z - x)^2 = 1$

$\therefore y^2 + z^2 - 2yz = z^2 + x^2 - 2zx \Rightarrow y^2 - x^2 = 2z(y - x)$

$\Rightarrow x + y = 2z$ ***(*** $\because y - x \neq 0$ ***as*** $(y - x)^2 = 16$***)***

$\therefore \left(\frac{x+y}{2}\right)(x + y) = x^2 + y^2 - 6$ ***(from (3))***

$\Rightarrow x^2 + 2xy + y^2 = 2x^2 + 2y^2 - 12 \Rightarrow (x - y)^2 = 12$

But $(x - y)^2 = 16 \Rightarrow$ ***no solution.***

Case (5): $(y-z)^2=16, (z-x)^2=1, (x-y)^2=1$

$$\therefore z^2+x^2-2zx=x^2+y^2-2xy \Rightarrow z^2-y^2=2x(z-y)$$

$$\Rightarrow z+y=2x \text{ (}\because z-y\neq 0 \text{ as } (z-y)^2=16\text{)}$$

$$\therefore \left(\frac{y+z}{2}\right)(y+z)=y^2+z^2-6 \text{ (from (1))}$$

$$\Rightarrow y^2+z^2+2yz=2y^2+2z^2-12 \Rightarrow (y-z)^2=12$$

But $(y-z)^2 \Rightarrow$ ***no solution.***

Case (6): $(z-x)^2=16, (x-y)^2=1, (y-z)^2=1$

$$\therefore x^2+y^2-2xy=y^2+z^2-2yz \Rightarrow x^2-z^2=2y(x-z)$$

$$\Rightarrow x+z=2y \text{ (}\because x-z\neq 0 \text{ as } (x-z)^2=16\text{)}$$

$$\therefore \left(\frac{x+z}{2}\right)(x+z)=x^2+z^2-6 \text{ (from (2))} \Rightarrow (x-z)^2=12$$

But $(x-z)^2=16 \Rightarrow$ ***no solution.*** $\therefore$ ***all possible integer solutions are:***

$$(x,y,z)=(1,-2,1),(-1,2,-1),(1,1,-2),(-1,-1,2),(-2,1,1),(2,-1,1)$$

94. Find $x,y,z\in(0,\infty)$ such that:

$$\begin{cases} x^3-y^3=\ln\left(\frac{y}{x}\right) \\ y^5-z^5=\ln\left(\frac{z}{y}\right) \\ 2x^y+3y^z+5z^x=10 \end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Serban George Florin-Romania

$$\textit{If } x>y \Rightarrow x^3>y^3 \Rightarrow x^3-y^3>0 \Rightarrow \ln\left(\frac{y}{x}\right)<0 \quad \textit{(F)}$$

$$\textit{If } x<y \Rightarrow x^3<y^3 \Rightarrow x^3-y^3<0 \Rightarrow \ln\left(\frac{y}{x}\right)>0 \quad \textit{(F)}$$

$$\Rightarrow x=y$$

$$\textit{If } y > z \Rightarrow y^5 > z^5 \Rightarrow y^5 - z^5 > 0 \Rightarrow \ln\left(\frac{z}{y}\right) < 0 \quad \textit{(F)}$$

$$\textit{If } y < z \Rightarrow y^5 < z^5 \Rightarrow y^5 - z^5 < 0 \Rightarrow \ln\left(\frac{z}{y}\right) > 0 \quad \textit{(F)}$$

$$\Rightarrow y = z \Rightarrow x = y = z$$

$$2x^4 + 3y^2 + 5z^x = 10 \Rightarrow 2x^x + 3x^x + 5x^x = 10$$

$$10x^x = 10 \Rightarrow x^x = 1, x \in (0,\infty) \Rightarrow x = 1 \Rightarrow x = y = z = 1$$

Solution 2 by Rovshan Pirgullyiev-Sumgait-Azerbaidian

$$x^3 - y^3 = \ln\frac{y}{x} \Rightarrow x^3 - y^3 = \ln y - \ln x \Rightarrow x^3 + \ln x = y^3 = \ln y$$

$$f(t) = t^3 + \ln t \textit{ injective} \Rightarrow x = y$$

$$y^5 - z^5 = \ln\frac{z}{y} \Rightarrow y^5 + \ln y = z^5 + \ln z$$

$$f(t) = t^5 + \ln t \;\; \textit{injective} \Rightarrow y = z$$

$$2x^4 + 3y^2 + 5z^x = 10 \overset{x=y, y=z}{\Rightarrow} 10x^x = 10 \Rightarrow x^x = 1 \Rightarrow x = 1$$

Answer $(1;1;1)$

Solution 3 by Soumava Chakraborty-Kolkata-India

$$\textit{Find } x, y, z \in (0,\infty);\; x^3 - y^3 \overset{(1)}{=} \ln\left(\frac{y}{x}\right), y^5 - z^5 \overset{(2)}{=} \ln\left(\frac{z}{y}\right)$$

$$2x^4 + 3y^z + 5z^x \overset{(3)}{=} 10$$

$$\textit{(1)} \Rightarrow x^3 + \ln x = y^3 + \ln y$$

$$\textit{Let } f(u) = u^3 + \ln u;\; f'(u) = 3u^2 + \frac{1}{u} > 0, \forall u > 0$$

$$\therefore f(u) \textit{ is increasing on } (0,\infty) \therefore f(x) = f(y) \Rightarrow x = y$$

$$\textit{(2)} \Rightarrow y^5 + \ln y = z^5 + \ln z$$

$$\textit{Let } f_0(v) = v^5 + \ln v$$

$$f_0'(v) = 5v^4 + \frac{1}{v} > 0\ \forall\ v > 0 \Rightarrow f_0(v) \text{ is increasing on } (0,\infty)$$

$$\therefore f_0(y) = f_0(z) \Rightarrow y = z \quad \therefore x = y = z$$

$\therefore$ ***(3) becomes*** $10x^x = 10 \Rightarrow x = 1 \therefore$ ***only solution is*** $(x,y,z) = (1,1,1)$

Solution 4 by Geanina Tudose-Romania

The system can be rewritten $\begin{cases} x^3 + \ln x = y^3 + \ln y \\ y^5 + \ln y = z^5 + \ln z \\ 2x^y + 3y^z + 5z^x = 10 \end{cases}$

Let $f:(0,\infty) \to R, f(x) = x^3 + \ln x$ ***a sum of strictly increasing functions,***

so f ***strictly increasing*** $\Rightarrow f$ ***one to one*** $\Rightarrow x = y$

$g:(0,\infty) \to \mathbb{R}, g(x) = x^5 + \ln x$ ***similarly we have*** $y = z$

The last equation becomes $5x^x + 5x^x = 10 \Leftrightarrow x^x = 1 \Leftrightarrow x = 1$

Note for $x > 1 \Rightarrow x^x > 1$ ***so*** $x^x = 1$ ***doesn't have solutions for***

$0 < x < 1, x = \frac{1}{a}, a > 1 \Rightarrow \left(\frac{1}{a}\right)^{\frac{1}{a}} = \frac{1}{a^{\frac{1}{a}}} < 1$***, again with no solution***

Hence $x = y = z = 1$***.***

95. **Solve in $(0, 2\pi)$:**

$$\sin(1+x) + \sin(1+2x) + \cdots + \sin(1+10x) = 0$$

Proposed by Daniel Sitaru – Romania

Solution by Kevin Soto Palacios-Huarmey-Peru

Hallar los valores de x ***en el intervalo*** $< 0{,}2\pi >$***:***

$$\text{sen}(1+x) + \text{sen}(1+2x) + \cdots + \text{sen}(1+10x) = 0$$

Sumando los extremos, se tiene que:

$$\text{sen}(1+x) + \text{sen}(1+10x) + \text{sen}(1+2x) + \text{sen}(1+9x) + \cdots$$
$$+ \text{sen}(1+5x) + \text{sen}(1+6x)$$

Aplicando transformaciones trigonómetricas:

$$2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\cos\left(\frac{9x}{2}\right)+2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\cos\left(\frac{7x}{2}\right)+\cdots+2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\cos\left(\frac{x}{2}\right)=0$$

$$2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\left[\cos\frac{9x}{2}+\cos\frac{7x}{2}+\cos\frac{5x}{2}+\cos\frac{3x}{2}+\cos\frac{x}{2}\right]=0$$

$$2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\left[\cos\frac{9x}{2}+\cos\frac{x}{2}+\cos\frac{7x}{2}+\cos\frac{3x}{2}+\cos\frac{5x}{2}\right]=0$$

$$2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\left[2\cos\frac{5x}{2}+\cos\frac{4x}{2}+2\cos\frac{5x}{2}\cos\frac{2x}{2}+\cos\frac{5x}{2}\right]=0$$

$$2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\cos\frac{5x}{2}[2\cos 2x+2\cos x+1]=0\rightarrow$$

$$\rightarrow 2\,\mathrm{sen}\left(\frac{2+11x}{2}\right)\cos\frac{5x}{2}[4\cos^2 x+2\cos x-1]=0$$

Si: $\mathrm{sen}\left(\frac{2+11x}{2}\right)=0\rightarrow\frac{2+11x}{2}=\pi k\Leftrightarrow x=\frac{2\pi k-2}{11}\rightarrow$ ***Válido para***

$$k=1,2,3,4,5,6,7,8,9,10,11\rightarrow k\in\mathbb{Z}$$

Si: $\cos\frac{5x}{2}=0\rightarrow\frac{5x}{2}=(2n+1)\frac{\pi}{2}\Leftrightarrow x=\frac{(2n+1)\pi}{5}\rightarrow$

Válido para $n=0,1,2,3,4\rightarrow n\in\mathbb{Z}$

$$\Rightarrow 4\cos^2 x+2\cos x-1=0\rightarrow\left(2\cos x+\frac{1}{2}\right)^2=\frac{5}{4}\rightarrow$$

$$\cos x=\frac{\sqrt{5}-1}{4}\ V\ \cos x=\frac{-\sqrt{5}-1}{4}$$

Si: $\cos x=\frac{\sqrt{5}-1}{4}\rightarrow x=\frac{2\pi}{5},\frac{8\pi}{5}$. ***Si:*** $\cos x=\frac{-\sqrt{5}-1}{4}\rightarrow x=\frac{4\pi}{5},\frac{6\pi}{5}$

96. Prove that $\forall\, x\in\mathbb{R}$**:**

$$\left(\sqrt[3]{2\sin 3x+3\left(\sin x+\sqrt{3}\cos x\right)}\right)^2+\left(\sqrt[3]{2\cos 3x+3\left(\sqrt{3}\sin x-\cos x\right)}\right)^2=4$$

Proposed by Maria Elena Panaitopol – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar que: $\forall\, x \in \mathbb{R}$:

$$E = \left(\sqrt[3]{2\,\text{sen}\,3x + 3\left(\text{sen}\,x + \sqrt{3}\cos x\right)}\right)^2 + \left(\sqrt[3]{2\cos 3x + 3\left(\sqrt{3}\,\text{sen}\,x - \cos x\right)}\right)^2 = 4$$

Recordar lo siguiente: Ángulo triple: $\text{sen}(3M) = 3\,\text{sen}(M) - 4\,\text{sen}^3(M)$

$$\cos(3M) = 4\cos^3(M) - 3\cos(M)$$

A) $2\,\text{sen}\,3x + 3\left(\text{sen}\,x + \sqrt{3}\cos x\right) = 2\,\text{sen}\,3x + 6\left(\frac{1}{2}\text{sen}\,x + \frac{\sqrt{3}}{2}\cos x\right) =$

$$= 2\,\text{sen}\,3x + 6\,\text{sen}(x + 60)$$

$$2\,\text{sen}(3x + 180) = 6\,\text{sen}(x + 60) - 8\,\text{sen}^3(x + 60)$$

$$\Rightarrow -2\,\text{sen}\,3x = 6\,\text{sen}(x + 60) - 8\,\text{sen}^3(x + 60) \Leftrightarrow$$

$$\Leftrightarrow 2\,\text{sen}\,3x = -6\,\text{sen}(x + 60) + 8\,\text{sen}^3(x + 60)$$

$$\Rightarrow 2\,\text{sen}\,3x + 6\,\text{sen}(x + 60) = 8\,\text{sen}^3(x + 60) \quad \textit{(A)}$$

B) $2\cos 3x + 3\left(\sqrt{3}\,\text{sen}\,x - \cos x\right) = 2\cos 3x - 6\left(-\frac{\sqrt{3}}{2}\text{sen}\,x + \frac{1}{2}\cos x\right) =$

$$= 2\cos 3x - 6\cos(x + 60)$$

$$2\cos(3x + 180) = 8\cos^3(x + 60) - 6\cos(x + 60)$$

$$\Rightarrow -2\cos 3x = 8\cos^3(x + 60) - 6\cos(x + 60) \Leftrightarrow$$

$$\Leftrightarrow 2\cos 3x = -8\cos^3(x + 60) + 6\cos(x + 60)$$

$$\Rightarrow 2\cos 3x - 6\cos(x + 60) = -8\cos^3(x + 60) \quad \textit{(B)}$$

De (A) $\wedge$ (B) se llega a lo siguiente:

$$E = 4\,\text{sen}^2(x + 60) + 4\cos^2(x + 60) = 4$$

97. **Solve for real numbers:**

$$\begin{cases}\cos 2x + \cot 3y = \tan 5z \\ \cot 3y + \cot 5z = \tan 2x \\ \cot 5z + \cot 2x = \tan 3y\end{cases}$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Seyran Ibrahimov – Maasilli – Azerbaidjan

$$\cot 2x = u \quad \cot 3y = i \quad \cot 5z = s \quad u, i, s \neq 0$$

$$\Rightarrow u + i = \frac{1}{s} \Leftrightarrow su + si = 1 \quad (1)$$

$$\Rightarrow i + s = \frac{1}{u} \Leftrightarrow ui + su = 1 \quad (2)$$

$$\Rightarrow s + u = \frac{1}{i} \Leftrightarrow si + ui = 1 \quad (3)$$

$$(1) = (2) su + si = ui + su = 1 \Leftrightarrow s = u$$
$$(1) = (3) su + si = si + ui = 1 \Leftrightarrow s = i \quad \rightarrow u = i = s$$
$$(2) = (3) ui + su = si + ui = 1 \Leftrightarrow u = i$$

$$u(i,s) + u(i,s) = \frac{1}{u(i,s)} \quad u, i, s = \pm\frac{1}{\sqrt{2}}$$

$$\cot 2x = u = \pm\frac{1}{\sqrt{2}} \quad \cot 3y = i = \pm\frac{1}{\sqrt{2}} \quad \cot 5z = s = \pm\frac{1}{\sqrt{2}}$$

$$x = \pm\frac{1}{2}\arctan\frac{1}{\sqrt{2}} + \frac{1}{2}\pi k \quad y = \pm\frac{1}{3}\arctan\frac{1}{\sqrt{2}} + \frac{1}{3}\pi k \quad z = \pm\frac{1}{5}\arctan\frac{1}{\sqrt{2}} + \frac{1}{5}\pi k$$

$$s > u > i$$

$$\begin{matrix} su + si = 1 \\ ui + su = 1 \\ si + ui = 1 \end{matrix} \quad \Rightarrow su + si > si + ui \ \textit{(no answer)}$$

answer $s = i = u = \pm\frac{1}{\sqrt{2}}$

Solution 2 by Soumava Chakraborty – Kolkata – India

Let $\cot 2x = a, \cot 3y = b, \cot 5z = c$

$\tan 2x$, $\tan 3y$, $\tan 5z$ **are defined,** $\cos 2x, \cos 3y, \cos 5z \neq 0 \Rightarrow a, b, c \neq 0$

$$a+b=\frac{1}{c}\quad (1)$$

$$b+c=\frac{1}{a}\quad (2)$$

$$c+a=\frac{1}{b}\quad (3)$$

(1) – (2) $\Rightarrow a-c=\frac{1}{c}-\frac{1}{a}\Rightarrow (a-c)\left(1-\frac{1}{ac}\right)=0$ *(4)*

(2) – (3) $\Rightarrow b-a=\frac{1}{a}-\frac{1}{b}\Rightarrow (b-a)\left(1-\frac{1}{ab}\right)=0$ *(5)*

(3) – (1) $\Rightarrow c-b=\frac{1}{b}-\frac{1}{c}\Rightarrow (c-b)\left(1-\frac{1}{bc}\right)=0$ *(6)*

If $1=\frac{1}{ac}$*, then* $\frac{1}{c}=a\Rightarrow a+b=a$ *(form (1))* $\Rightarrow b=0$

If $1=\frac{1}{ab}$*, then* $\frac{1}{b}=a\Rightarrow c+a=a$ *(from (3))* $\Rightarrow c=0$

If $1=\frac{1}{bc}$*, then* $\frac{1}{c}=b\Rightarrow a+b=b$ *(from (1))* $\Rightarrow a=0$

But $a,b,c\neq 0,\ \ 1\neq\frac{1}{ac},1\neq\frac{1}{ab},1\neq\frac{1}{bc}$

(4), (5), (6) $\Rightarrow a=b=c$

Putting $b=c=a$ *in (1),* $a+a=\frac{1}{a}\Rightarrow a^2=\frac{1}{2}\Rightarrow a=\pm\frac{1}{\sqrt{2}}$

$(a,b,c)=\left(\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}},\frac{1}{\sqrt{2}}\right)$ *or* $\left(-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}},-\frac{1}{\sqrt{2}}\right)$

$$\cot 2x=\cot 3y=\cot 5z=\frac{1}{\sqrt{2}}\Rightarrow \tan 2x=\tan 3y=\tan 5z=\sqrt{2}$$

$$\Rightarrow 2x=\tan^{-1}(\sqrt{2})+n\pi\Rightarrow x=\frac{1}{2}(\tan^{-1}\sqrt{2})+\frac{n\pi}{2}$$

$$y=\frac{1}{3}\tan^{-1}(\sqrt{2})+\frac{n'\pi}{3};\ z=\frac{1}{5}\tan^{-1}(\sqrt{2})+\frac{n''\pi}{5}$$

Similarly, $\cot 2x=\cot 3y=\cot 5z=-\frac{1}{\sqrt{2}}\Rightarrow x=-\frac{1}{2}\tan^{-1}(\sqrt{2})+\frac{n\pi}{2}$

$$y=-\frac{1}{3}\tan^{-1}(\sqrt{2})+\frac{n'\pi}{3};\ z=-\frac{1}{5}\tan^{-1}(\sqrt{2})+\frac{n''\pi}{5}$$

$$\textit{solutions are:}\begin{cases} x = \pm\frac{1}{2}\tan^{-1}(\sqrt{2}) + \frac{n\pi}{2} \\ y = \pm\frac{1}{3}\tan^{-1}(\sqrt{2}) + \frac{n'\pi}{3} \\ z = \pm\frac{1}{5}\tan^{-1}(\sqrt{2}) + \frac{n''\pi}{5} \end{cases}$$

98. Prove that without using a computer:

$$\pi^{\pi+2} > (\pi+1)^{\pi+1}$$

Proposed by Regragui El Khammal-Morocco

Solution by Daniel Sitaru-Romania

$$f:(0,\infty)\to\mathbb{R};\ f(x) = (x+2)\ln x - (x+1)\ln(x+1)$$

$$f'(x) = \ln x - \ln(x+1) + \frac{2}{x};\ f''(x) = \frac{1}{x(x+1)} + \frac{2}{x} > 0 \Rightarrow f' \textit{ increasing}$$

$$\frac{x-1}{x} \le \ln x \Rightarrow -\frac{1}{x} \le \ln\left(\frac{x}{x+1}\right) \Rightarrow \ln\left(\frac{x}{x+1}\right) + \frac{2}{x} \ge \frac{1}{x} > 0$$

$$f'(x) > 0 \Rightarrow f \textit{ increasing } f(\pi) > f(2)$$

$$(\pi+2)\ln\pi - (\pi+1)\ln(\pi+1) > 0$$

$$\ln\pi^{\pi+2} > \ln(\pi+1)^{\pi+1};\ \pi^{\pi+2} > (\pi+1)^{\pi+1}$$

99. Prove that:

$$\frac{1}{\ln 4} + \frac{1}{\ln 9} + \frac{1}{\ln 25} \ge \frac{1}{\ln 6} + \frac{1}{\ln 15} + \frac{1}{\ln 10}$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren – Mongolia

$$\begin{matrix} \ln 4 = 2\cdot\ln 2 & \ln 6 = \ln 2 + \ln 3 \\ \ln 9 = 2\cdot\ln 3 & \ln 15 = \ln 3 + \ln 5 \\ \ln 25 = 2\cdot\ln 5 & \ln 10 = \ln 5 + \ln 2 \end{matrix}$$

$$\left.\begin{matrix} \ln 2 = x \\ \ln 3 = y \\ \ln 5 = z \end{matrix}\right\}\ \frac{1}{2}\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right) \ge \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x}$$

$$\sum_{x;y;z}\frac{1}{x+y}=\frac{1}{4}\sum_{x,y,z}\frac{4}{x+y}=\frac{1}{4}\cdot\sum\frac{(1+1)^2}{x+y}\overset{CBS}{\leq}\frac{1}{4}\sum\left(\frac{1}{x}+\frac{1}{y}\right)=$$

$$=\frac{1}{4}\cdot\left(\left(\frac{1}{x}+\frac{1}{y}\right)+\left(\frac{1}{y}+\frac{1}{z}\right)+\left(\frac{1}{z}+\frac{1}{x}\right)\right)=\frac{1}{2}\cdot\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right).$$ ***True***

100. Without Softs! Just Think!-2

Prove that:

$$\begin{vmatrix}\ln\frac{2}{15} & \ln 4 & \ln 4\\ \ln 9 & \ln\frac{3}{10} & \ln 9\\ \ln 25 & \ln 25 & \ln\frac{5}{6}\end{vmatrix}>\ln 8\ln 27\ln 125$$

Proposed by Daniel Sitaru – Romania

Solution by Myagmarsuren Yadamsuren – Mongolia

$$\begin{vmatrix}\ln\frac{2}{15} & \ln 4 & \ln 4\\ \ln 9 & \ln\frac{3}{10} & \ln 9\\ \ln 25 & \ln 25 & \ln\frac{5}{9}\end{vmatrix}>\underbrace{\ln 8\cdot\ln 27\cdot\ln 125}_{27xyz}$$

$$x=\ln 2;\ y=\ln 3;\ z=\ln 5$$

$$\begin{vmatrix}x-(y+z) & 2x & 2x\\ 2y & y-(x+z) & 2y\\ 2z & 2z & z-(x+y)\end{vmatrix}=$$

$$=4xyz+4xy\cdot(y+x)+4yz\cdot(y+z)+$$

$$+4zx\cdot(z+x)-(x+y-z)\cdot(y+z-x)\cdot(z+x-y)\overset{\text{Bergstrom}}{\geq}$$

$$\geq 3xyz + 4xyz \cdot \left(\left(\frac{x}{y} + \frac{y}{x}\right) + \left(\frac{y}{z} + \frac{z}{y}\right) + \left(\frac{z}{x} + \frac{x}{z}\right)\right) \overset{\text{Cauchy}}{\overbrace{\geq}} 27xyz$$

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To be continued!

Daniel Sitaru