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INEQUALITY IN TRIANGLE 339
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1. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\ge\frac{3}{2Rr}.$$

Proposed by Rovsen Pirguliev - Sumgait - Azerbaidian

Proof.

We have $AI=\frac{r}{\sin\frac{A}{2}}$. *We obtain* $\sum\frac{1}{AI^2}=\sum\frac{\sin^2\frac{A}{2}}{r^2}=\frac{1}{r^2}\sum\sin^2\frac{A}{2}=$

$$=\frac{1}{r^2}\cdot\frac{2R-r}{2R}=\frac{2R-r}{2Rr^2}.$$

The inequality we have to prove can be written

$\frac{2R-r}{2Rr^2}\ge\frac{3}{2Rr}\Leftrightarrow R\ge 2r$ *(Euler's inequality).*

The equality holds if and only if the triangle is equilateral.

□

Remark.

The inequality can be strengthened:

2. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\ge\frac{3}{4r^2}.$$

Proposed by Marin Chirciu - Romania

Proof.

We use the identity $\sum\frac{1}{AI^2}=\frac{2R-r}{2Rr^2}$, *the inequality can be written*

$\frac{2R-r}{2Rr^2}\ge\frac{3}{4r^2}\Leftrightarrow R\ge 2r$, *obviously from Euler's inequality.*

The equality holds if and only if the triangle is equilateral.

□

Remark.

*Inequality **2.** is stronger than inequality **1.***

3. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\ge\frac{3}{4r^2}\ge\frac{3}{2Rr}.$$

Proof.

*See inequality **2** and Euler's inequality.*

The equality holds if and only if the triangle is equilateral.

□

Remark.

The following inequality holds:

4. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\geq\frac{3}{4r^2}\geq\frac{3}{2Rr}\geq\frac{81}{4p^2}\geq\frac{3}{R^2}.$$

Proof.

*The first inequality is **2**, the second follows from Euler's inequality, the third is equivalent with $2p^2\geq 27Rr$, which follows from Gerretsen's inequality $p^2\geq 16Rr-5r^2$ and Euler's inequality $R\geq 2r$ and the forth is Mitrinović's ineuality $p^2\leq\frac{27R^2}{4}$.*

□

Now, let's find an inequality having an opposite sense.

5. In ΔABC

$$\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\leq\frac{3R}{8r^3}.$$

Proof.

Using the identity $\sum\frac{1}{AI^2}=\frac{2R-r}{2Rr^2}$, the inequality can be written

$$\frac{2R-r}{2Rr^2}\leq\frac{3R}{8r^3}\Leftrightarrow 4r(2R-r)\leq 3R^2\Leftrightarrow 3R^2-8Rr+4r^2\geq 0\Leftrightarrow(R-2r)(3R-2r)\geq 0,$$

obviously from Euler's inequality $R\geq 2r$.

The equality holds if and only if the triangle is equilateral.

□

The following double inequality can be written:

6. In ΔABC

$$\frac{3}{4r^2}\leq\frac{1}{AI^2}+\frac{1}{BI^2}+\frac{1}{CI^2}\leq\frac{3R}{8r^3}.$$

Proposed by Marin Chirciu - Romania

Proof.

*See inequalities **2** and **5**.*

The equality holds if and only if the triangle is equilateral.

□

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