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134 INEQUALITY IN TRIANGLE
MATH ADVENTURES ON CUTTHEKNOT MATH 101-150

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1. Prove that in any acute triangle the following relationship holds

$$\frac{a^2}{\tan B+\tan C}+\frac{b^2}{\tan C+\tan A}+\frac{c^2}{\tan A+\tan B}\le pR.$$

Proposed by Daniel Sitaru - Romania

Proof.

We have

$$\sum\frac{a^2}{\tan B+\tan C}=\sum\frac{a^2}{\frac{\sin(B+C)}{\cos B\cos C}}=\sum\frac{a^2\cos B\cos C}{\sin A}=\sum\frac{a^2\cos B\cos C}{\frac{a}{2R}}=$$

$$=2R\sum a\cos B\cos C=2R\cdot\frac{pr}{R}=2pr\overbrace{\le}^{Euler}pR.$$

From the above proof it follows that the relationship holds for any non-right angled triangle.

The equality holds if and only if the triangle is equilateral.

□

Remark.

In the same way we can propose the following:

2. Prove that in any triangle the following relationship holds:

$$\frac{a^2}{\cot B+\cot C}+\frac{b^2}{\cot C+\cot A}+\frac{c^2}{\cot A+\cot B}\le 3pR$$

Proposed by Marin Chirciu - Romania

Proof.

We have $\sum\frac{a^2}{\cot B+\cot C}=\sum\frac{a^2}{\frac{\sin(B+C)}{\sin B\sin C}}=\sum\frac{a^2\sin B\sin C}{\sin A}=\sum\frac{a^2\cdot\frac{a}{2R}\cdot\frac{b}{2R}}{\frac{a}{2R}}=$

$$=\frac{1}{2R}\sum abc=\frac{1}{2R}\cdot 3abc=\frac{1}{2R}\cdot 12pRr=6pr\overbrace{\le}^{Euler}3pR.$$

The equality holds if and only if the triangle is equilateral.

□

Remark.

Adding the two inequalities we obtain:

3. Prove that in any triangle the following relationship holds:

$$\sum \frac{a^2 \cos(B-C)}{\sin A} \leq 4pR.$$

Proposed by Daniel Sitaru - Romania and Marin Chirciu - Romania

Proof 1.

We have $\dfrac{a^2}{\tan B + \tan C} + \dfrac{a^2}{\cot B + \cot C} = \dfrac{a^2 \cos B \cos C}{\sin A} + \dfrac{a^2 \sin B \sin C}{\sin A} = \dfrac{a^2 \cos(B-C)}{\sin A}.$

Then $\sum \dfrac{a^2}{\tan B + \tan C} = 2S$ *and* $\sum \dfrac{a^2}{\cot B + \cot C} = 6S.$

It follows $\sum \dfrac{a^2 \cos(B-C)}{\sin A} = 8S.$

We write the inequality $8S \leq 4pR \Leftrightarrow 8pr \leq 4pR \Leftrightarrow 2r \leq R$ *(Euler's Inequality).*

The equality holds if and only if the triangle is equilateral.

□

Proof 2.

We have $\sum \dfrac{a^2 \cos(B-C)}{\sin A} = \sum \dfrac{a^2 \cos(B-C)}{\frac{a}{2R}} = 2R \sum \cos(B-C) = 2R \cdot \dfrac{4pr}{R} =$

$= 8pr \leq 4pR.$

*From the above proof it follows that the inequality from **3.** is true in any triangle.*

□

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