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SOLUTIONS

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JP.061. Let a, b, c be positive real numbers such that $a + b + c = 3$.

Prove that

$$\frac{1}{a^3 + b^3 + c^3} + \frac{8}{ab + bc + ca} \geq 3$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números R^+ de tal manera que $a + b + c = 3$. Probar que

$$\frac{1}{a^3 + b^3 + c^3} + \frac{8}{ab + bc + ca} \geq 3. \text{ Realizamos los siguientes cambios de variables}$$

$x = a^3 + b^3 + c^3 > 0, y = ab + bc + ca > 0$. La desigualdad propuesta es equivalente

$$\frac{1}{x} + \frac{8}{y} \geq 3. \text{ Aplicando la desigualdad de Cacuchy}$$

$$\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{y} + \frac{1}{y} + \frac{1}{y} + \frac{1}{y} + \frac{1}{y} + \frac{1}{y} + \frac{1}{y}\right)(x + y + y + y + y + y + y + y + y) \geq 9^2 = 81$$

Es suficiente probar

$$x + 8y \leq 27 \Leftrightarrow a^3 + b^3 + c^3 + 8(ab + bc + ca) \leq (a + b + c)^3 \Leftrightarrow$$

$$\Leftrightarrow 8(ab + bc + ca) \leq 3(a + b)(b + c)(c + a). \text{ Lo cual es cierto ya que}$$

$$3(a + b)(b + c)(c + a) \geq 3 \cdot \frac{8}{9}(a + b + c)(ab + bc + ca) = 8(ab + bc + ca)$$

$$\text{Por lo tanto} \rightarrow \frac{1}{x} + \frac{8}{y} \geq \frac{81}{x+8y} \geq 3 \Leftrightarrow \frac{1}{a^3 + b^3 + c^3} + \frac{8}{ab + bc + ca} \geq 3$$

Solution 2 by Nguyen Ngoc Tu-Ha Giang-Vietnam

$$\text{We have: } (a + b)(b + c)(c + a) \geq \frac{8}{9}(a + b + c)(ab + bc + ca)$$

$$\Rightarrow ab + bc + ca \leq \frac{3}{8}(a + b)(b + c)(c + a). \text{ Hence}$$

$$\frac{1}{\sum a^3} + \frac{8}{\sum ab} \geq \frac{1}{\sum a^3} + \frac{64}{3(a + b)(b + c)(c + a)} \geq \frac{(1 + 8)^2}{\sum a^3 + 3 \prod(a + b)} = \frac{81}{(a + b + c)^3} = 3$$

Solution 3 by Boris Colakovic-Belgrade-Serbia

$$\frac{1}{a^3 + b^3 + c^3} + \frac{8}{ab + bc + ca} \geq 3 \mid \cdot (a^3 + b^3 + c^3) \Leftrightarrow$$

$$\Leftrightarrow 1 + \frac{8(a^3 + b^3 + c^3)}{ab + bc + ca} \geq 3(a^3 + b^3 + c^3) \quad (1)$$

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$$\begin{aligned} \frac{1}{ab+bc+ca} &\geq \frac{3}{(a+b+c)^2} = \frac{1}{3}. \text{ From (1)} \Rightarrow \\ \Rightarrow 1 + \frac{8}{3}(a^3 + b^3 + c^3) &\geq 3(a^3 + b^3 + c^3) \Leftrightarrow \\ \Leftrightarrow 3 + 8(a^3 + b^3 + c^3) &\geq 9(a^3 + b^3 + c^3) \Leftrightarrow a^3 + b^3 + c^3 \leq 3 \quad (2) \\ a + b + c &\stackrel{AM-GM}{\geq} 3\sqrt[3]{abc} \Rightarrow abc \leq 1 \\ a^3 + b^3 + c^3 &\stackrel{AM-GM}{\geq} 3\sqrt[3]{(abc)^3} \Rightarrow 3abc \leq a^3 + b^3 + c^3 \Rightarrow abc \leq \frac{a^3+b^3+c^3}{3} \leq 1 \Rightarrow \quad (2) \end{aligned}$$

Equality holds for $a = b = c = 1$

Solution 4 by Truong Nguyen-Phan Rang-Vietnam

$$\begin{aligned} (a + b + c)(ab + bc + ca) &\leq \frac{9}{8}(a + b)(b + c)(c + a) \\ \Leftrightarrow a(b - c)^2 + b(c - a)^2 + c(a - b)^2 &\geq 0 \\ \text{So } \frac{1}{a^3+b^3+c^3} + \frac{8}{ab+bc+ca} &= \frac{1}{a^3+b^3+c^3} + \frac{24}{(a+b+c)(ab+bc+ca)} \\ \geq \frac{1}{a^3 + b^3 + c^3} + \frac{8^2}{3(a + b)(b + c)(c + a)} &\geq \frac{(1 + 8)^2}{a^3 + b^3 + c^3 + 3(a + b)(b + c)(c + a)} = \\ &= \frac{9^2}{(a + b + c)^3} = \frac{9^2}{3^3} = 3 \end{aligned}$$

JP.062. Prove that if: $a, b, c, d \in [1, \infty)$ then:

$$3a + 3b + 2c + d \leq 6 + ab(1 + c + cd)$$

Proposed by Daniel Sitaru-Romania

Solution by proposer

We prove by induction by n that $x_1, x_2, \dots, x_n \in [1, \infty)$; $n \in \mathbb{N}^$ implies:*

$$x_1 + x_2 + \dots + x_n \leq n - 1 + x_1 x_2 \dots x_n. \text{ Checking:}$$

$$n = 1; \quad x_1 \leq 1 - 1 + x_1 \Leftrightarrow x_1 \leq x_1$$

$$n = 2; \quad x_1 + x_2 \leq 1 + x_1 x_2 \Leftrightarrow (x_1 - 1)(x_2 - 1) \geq 0 \text{ which it's true.}$$

$$P(k): x_1 + x_2 + \dots + x_k \leq k - 1 + x_1 x_2 \dots x_k. \text{ Suppose that it's true.}$$

$$P(k + 1): x_1 + x_2 + \dots + x_k + x_{k+1} \leq k + x_1 x_2 \dots + x_k x_{k+1}. \text{ To proved.}$$

$$x_1 + x_2 + \dots + x_k + x_{k+1} \leq k - 1 + x_1 x_2 \dots x_k + x_{k+1}$$

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Remains to prove that: $k - 1 + x_1 x_2 \dots x_k + x_{k+1} \leq k + x_1 x_2 \dots x_k x_{k+1}$

$$0 \leq x_1 x_2 \dots x_k (x_{k+1} - 1) - (x_{k+1} - 1)$$

$$(x_1 x_2 \dots x_k - 1)(x_{k+1} - 1) \geq 0$$

$$P(k) \rightarrow P(k+1)$$

For $x_1 = a; x_2 = b; x_3 = c; x_4 = d$

$$a + b \leq 1 + ab$$

$$a + b + c \leq 2 + abc$$

$$a + b + c + d \leq 3 + abcd$$

By adding: $3a + 3b + 2c + d \leq 6 + ab(1 + c + cd)$

JP.063. Let x, y, z **be non-negative real numbers satisfying**

$\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 3$. **Find the minimum possible value of**

$$\sqrt[3]{x + 2y + 5z} + \sqrt[3]{y + 2z + 5x} + \sqrt[3]{z + 2x + 5y}.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo x, y, z **números reales no negativos de tal manera que** $\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z} = 3$

Hallar el mínimo valor $E = \sqrt[3]{x + 2y + 5z} + \sqrt[3]{y + 2z + 5x} + \sqrt[3]{z + 2x + 5y}$

Como $x, y, z \geq 0$. **Aplicando la desigualdad de Holder**

$$\sqrt[3]{\frac{(x+y+y+z+z+z+z+z)(1+1+1+1+1+1+1+1)(1+1+1+1+1+1+1+1)}{64}} \geq \frac{\sqrt[3]{(3\sqrt[3]{x}+2\sqrt[3]{y}+5\sqrt[3]{z})^3}}{4}$$

$$\Rightarrow \sqrt[3]{x + 2y + 5z} \geq \frac{3\sqrt[3]{x} + 2\sqrt[3]{y} + 5\sqrt[3]{z}}{4} \quad (A)$$

Análogamente para los siguientes términos se cumplirá

$$\sqrt[3]{y + 2z + 5x} \geq \frac{3\sqrt[3]{y} + 2\sqrt[3]{z} + 5\sqrt[3]{x}}{4} \quad (B),$$

$$\sqrt[3]{z + 2x + 5y} \geq \frac{3\sqrt[3]{z} + 2\sqrt[3]{x} + 5\sqrt[3]{y}}{4} \quad (C)$$

Sumando (A) + (B) + (C)

$$E = \sqrt[3]{x + 2y + 5z} + \sqrt[3]{y + 2z + 5x} + \sqrt[3]{z + 2x + 5y} \geq \frac{8(\sqrt[3]{x} + \sqrt[3]{y} + \sqrt[3]{z})}{4} = \frac{8 \cdot 3}{4} = 6$$

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La igualdad se alcanza cuando $x = y = z = 1$

(LQOD)

Solution 2 by Nguyen Thanh Nho-Tra Vinh-Vietnam

$$\text{Let } a = \sqrt[3]{x}, b = \sqrt[3]{y}, c = \sqrt[3]{z} \Rightarrow a + b + c = 3$$

$$P = \sqrt[3]{a^3 + 2b^3 + 5c^3} + \sqrt[3]{b^3 + 2c^3 + 5a^3} + \sqrt[3]{c^3 + 2a^3 + 5b^3}$$

$$\min(P) = ?$$

$$\left(\frac{a^3 + 2b^3 + 5c^3}{1+2+5}\right)^{\frac{1}{3}} \geq \frac{a+2b+5c}{1+2+5} \quad (\text{Radon})$$

$$\Rightarrow \sqrt[3]{a^3 + 2b^3 + 5c^3} \geq \frac{1}{4}(a + 2b + 5c) \quad (1)$$

$$\text{Simillary } \sqrt[3]{b^3 + 2c^3 + 5a^3} \geq \frac{1}{4}(b + 2c + 5a) \quad (2)$$

$$\sqrt[3]{c^3 + 2a^3 + 5b^3} \geq \frac{1}{4}(c + 2a + 5b) \quad (3)$$

$$(1)+(2)+(3) \Rightarrow P \geq 2(a + b + c) = 6$$

$$\Rightarrow \min(P) = 6 \Leftrightarrow a = b = c = 1 \Leftrightarrow x = y = z = 1$$

JP.064. Let a, b, c be non-negative real numbers. Prove that

$$\sqrt[3]{1+a^3} + \sqrt[3]{1+b^3} + \sqrt[3]{1+c^3} \geq \frac{\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}}{\sqrt[6]{2}}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números reales no negativos. Probar que

$$\sqrt[3]{1+a^3} + \sqrt[3]{1+b^3} + \sqrt[3]{1+c^3} \geq \frac{\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}}{\sqrt[6]{2}}$$

Como $a, b, c \geq 0$. Aplicando la desigualdad de $MA \geq MG$ y Holder

$$\sqrt[3]{a^3 + 1} + \sqrt[3]{1 + b^3} \geq 2 \sqrt[6]{\frac{(a^3+1)(1+b^3)(1+1)}{2}} = 2 \sqrt[6]{\frac{(a+b)^3}{2}} = \frac{2\sqrt{a+b}}{\sqrt[6]{2}} \quad (A)$$

Análogamente para los siguientes términos se cumplirá

$$\sqrt[3]{b^3 + 1} + \sqrt[3]{1 + c^3} \geq \frac{2\sqrt{b+c}}{\sqrt[6]{2}} \quad (B),$$

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$$\sqrt[3]{c^3 + 1} + \sqrt[3]{1 + a^3} \geq \frac{2\sqrt{c+a}}{\sqrt[6]{2}} \quad (C)$$

Sumando (A) + (B) + (C)

$$\begin{aligned} 2\sqrt[3]{1 + a^3} + 2\sqrt[3]{1 + b^3} + 2\sqrt[3]{1 + c^3} &\geq \frac{2\sqrt{a+b} + 2\sqrt{b+c} + 2\sqrt{c+a}}{\sqrt[6]{2}} \\ \Rightarrow \sqrt[3]{1 + a^3} + \sqrt[3]{1 + b^3} + \sqrt[3]{1 + c^3} &\geq \frac{\sqrt{a+b} + \sqrt{b+c} + \sqrt{c+a}}{\sqrt[6]{2}} \end{aligned}$$

Solution 2 by Sanong Hauerai-Nakonpathom-Thailand

$$\begin{aligned} \sqrt[3]{(1 + a^3)} + \sqrt[3]{(1 + b^3)} + \sqrt[3]{(1 + c^3)} &= (1 + a^3)^{\frac{1}{3}} + (1 + b^3)^{\frac{1}{3}} + (1 + c^3)^{\frac{1}{3}} \\ &= (1 + a^3)^{\frac{2}{6}} + (1 + b^3)^{\frac{2}{6}} + (1 + c^3)^{\frac{2}{6}} = ((1 + a^3)^2)^{\frac{1}{6}} + ((1 + b^3)^2)^{\frac{1}{6}} + ((1 + c^3)^2)^{\frac{1}{6}} \\ &\geq \left(((1 + a^3)^2)^{\frac{1}{6}} ((1 + b^3)^2)^{\frac{1}{6}} \right)^{\frac{1}{2}} + \dots + \left(((1 + c^3)^2)^{\frac{1}{6}} ((1 + a^3)^2)^{\frac{1}{6}} \right)^{\frac{1}{2}} \\ &= \left(((1 + a^3)^2)^{\frac{1}{2}} ((1 + b^3)^2)^{\frac{1}{2}} \right)^{\frac{1}{6}} + \dots + \left(((1 + c^3)^2)^{\frac{1}{2}} ((1 + a^3)^2)^{\frac{1}{2}} \right)^{\frac{1}{6}} \\ &= \left((1 + a^3)(1 + b^3) \right)^{\frac{1}{6}} + \dots + \left((1 + c^3)(1 + a^3) \right)^{\frac{1}{6}} \\ &= \left(\frac{2}{2} (1 + a^3)(1 + b^3) \right)^{\frac{1}{6}} + \dots + \left(\frac{2}{2} (1 + c^3)(1 + a^3) \right)^{\frac{1}{6}} \\ &\geq \frac{(a+b)^{\frac{1}{2}} + (b+c)^{\frac{1}{2}} + (c+a)^{\frac{1}{2}}}{\frac{1}{2^{\frac{1}{6}}}}. \text{ Because } \left(\frac{2}{2} (1 + a^3)(1 + b^3) \right)^{\frac{1}{6}} \geq \frac{(a+b)^{\frac{1}{2}}}{2^{\frac{1}{6}}} \end{aligned}$$

$$\text{If } \left(2(1 + a^3)(1 + b^3) \right)^{\frac{1}{6}} \geq (a+b)^{\frac{1}{2}}. \text{ If } 2(1 + a^3)(1 + b^3) \geq (a+b)^3$$

$$\text{If } 2 + a^3 + b^3 + a^3b^3 \geq 3a^2b + 3b^2a.$$

$$\text{If } (1 + a^2 + a^3b^3) + c + b^3 + a^3b^3 \geq 3(a^2b + b^2a). \text{ Is to be true}$$

Solution 3 by Sladjan Stankovik-Macedonia

$$a, b, c \geq 0. \text{ Prove that: } \sum_{cyc} \sqrt[3]{1 + a^3} \geq \frac{1}{\sqrt[6]{2}} \cdot \sum_{cyc} \sqrt{a+b} \Leftrightarrow \sum_{cyc} \sqrt[3]{\frac{1+a^3}{2}} \geq \sum_{cyc} \sqrt{\frac{a+b}{2}}$$

$$\text{LHS} = \sum_{cyc} \sqrt[3]{\frac{1+a^3}{2}} \geq \sum_{cyc} \frac{a+1}{2} = \frac{a+b+c+3}{2} \geq \sqrt{3(a+b+c)} \geq \sum_{cyc} \sqrt{\frac{a+b}{2}} =$$

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$$\frac{1+a^3}{2} \geq \left(\frac{a+1}{2}\right)^3 \Leftrightarrow 3(a-1)^2 \cdot (a+1) \geq 0$$

$$p+3 \geq 2\sqrt{3p} \Leftrightarrow (\sqrt{p}-\sqrt{3})^2 \geq 0; \quad 3(x^2+y^2+z^2) \geq (x+y+z)^2 \text{ where:}$$

$$x = \sqrt{\frac{a+b}{2}}; \quad y = \sqrt{\frac{b+c}{2}}; \quad z = \sqrt{\frac{c+a}{2}}.$$

JP.065. Let ABC be an equilateral triangle inscribed in the circle (O) whose radius R . Prove that for an arbitrary point P lies on (O) ,

$$6\sqrt{2} < \frac{PA^3 + PB^3 + PC^3}{R^3} < 3\sqrt[4]{216}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Ravi Prakash-New Delhi-India

$$\text{Let's take circle to be } x^2 + y^2 = R^2 \quad (1)$$

If $A(R \cos \theta, R \sin \theta)$ and $B(R \cos \phi, R \sin \phi)$, are two points on (1), then

$$\begin{aligned} AB^2 &= R^2(\cos \theta - \cos \phi)^2 + R^2(\sin \theta - \sin \phi)^2 \\ &= R^2[\cos^2 \theta + \cos^2 \phi - 2 \cos \theta \cos \phi + \sin^2 \theta + \sin^2 \phi - 2 \sin \theta \sin \phi] \\ &= R^2[2 - 2 \cos(\theta - \phi)] = 4R^2 \sin^2 \left(\frac{\theta - \phi}{2}\right) \Rightarrow AB = 2R \left| \sin \left(\frac{\theta - \phi}{2}\right) \right| \end{aligned}$$

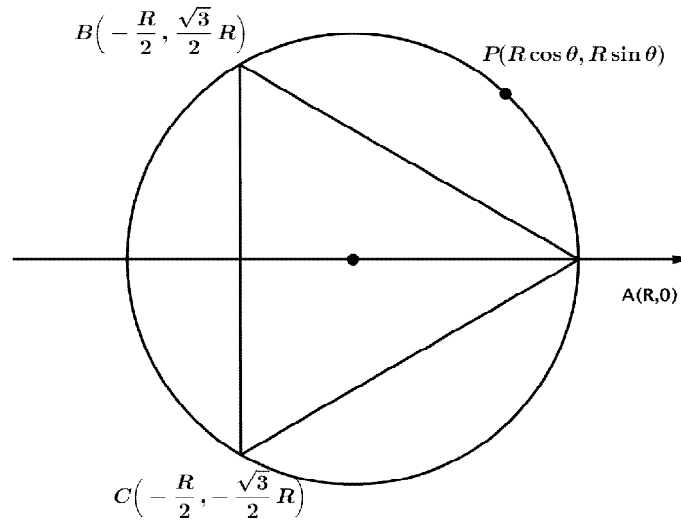
Let's take ΔABC with vertices as $A(R, 0)$, $B\left(-\frac{R}{2}, \frac{\sqrt{3}}{2}R\right)$, $C\left(-\frac{R}{2}, -\frac{\sqrt{3}}{2}R\right)$

In view of the symmetry we take point P as $(R \cos \theta, R \sin \theta)$ where $0 \leq \theta \leq \pi$

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$$\text{Now, } PA = 2R \left| \sin \frac{\theta}{2} \right| \quad [\text{For } A, \phi = 0]$$

$$= 2R \sin \frac{\theta}{2} \quad [\because 0 \leq \phi \leq \pi]$$

$$PB = 2R \left| \sin \left(\frac{\theta - \frac{2\pi}{3}}{2} \right) \right|$$

$$[\text{For } B, \phi = \frac{2\pi}{3}]$$

$$= 2R \left| \sin \left(\frac{\theta}{2} - \frac{\pi}{3} \right) \right|$$

$$\text{Similarly, } PC = 2R \left| \sin \left(\frac{\theta}{2} + \frac{\pi}{3} \right) \right| = 2R \sin \left(\frac{\theta}{2} + \frac{\pi}{3} \right)$$

$$E(\theta) = 6\sqrt{3} \cos \left(\frac{\theta}{2} \right) + 8 \sin^3 \left(\frac{\theta}{2} \right)$$

$$0 \leq \theta \leq \frac{2\pi}{3}$$

$$\text{For } \frac{2\pi}{3} < \theta < \pi$$

$$E(\theta) = 2 \left[\begin{aligned} &3 \sin \left(\frac{\theta}{2} \right) - \sin \left(\frac{3\theta}{2} \right) - 3 \sin \left(\frac{\pi}{3} - \frac{\theta}{2} \right) + \sin \left(\pi - \frac{3\theta}{2} \right) + \\ &+ 3 \sin \left(\frac{\pi}{3} + \frac{\theta}{2} \right) - \sin \left(\pi + \frac{3\theta}{2} \right) \end{aligned} \right]$$

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$$\begin{aligned}
 &= 2 \left[3 \sin \left(\frac{\theta}{2} \right) + 3(2) \cos \left(\frac{\pi}{3} \right) \sin \left(\frac{\theta}{2} \right) + 2 \sin \left(\frac{3\theta}{2} \right) \right] \\
 &= 2 \sin \left(\frac{3\theta}{2} \right) + 12 \sin \left(\frac{\theta}{2} \right) = 18 \sin \left(\frac{\theta}{2} \right) - 8 \sin^3 \frac{\theta}{2}
 \end{aligned}$$

Not that $E(\theta)$ is continuous at $\theta = \frac{2\pi}{3}$ and hence on $[0, \pi]$

Using $4 \sin^3 A = 3 \sin A - \sin(3A)$, we get

$$\text{For } 0 \leq \theta \leq \frac{2\pi}{3}$$

$$E(\theta) = 2 \left[3 \sin \left(\frac{\theta}{2} \right) - \sin \left(\frac{3\theta}{2} \right) + 3 \sin \left(\frac{\pi}{3} - \frac{\theta}{2} \right) - \sin \left(\pi - \frac{3\theta}{2} \right) + \right. \\
 \left. + 3 \sin \left(\frac{\pi}{3} + \frac{\theta}{2} \right) - \sin \left(\pi - \frac{3\theta}{2} \right) \right]$$

$$\text{if } 0 \leq \theta \leq \frac{2\pi}{3}$$

$$\begin{aligned}
 &= 6 \left[\sin \left(\frac{\theta}{2} \right) + 2 \sin \left(\frac{\pi}{3} \right) \cos \left(\frac{\theta}{2} \right) \right] - 2 \sin \left(\frac{3\theta}{2} \right) \\
 &= 6 \sin \left(\frac{\theta}{2} \right) + 6\sqrt{3} \cos \left(\frac{\theta}{2} \right) - 6 \sin \left(\frac{\theta}{2} \right) + 8 \sin^3 \left(\frac{\theta}{2} \right)
 \end{aligned}$$

$$E'(\theta) = \begin{cases} -3\sqrt{3} \sin \left(\frac{\theta}{2} \right) + 12 \sin^2 \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right); & 0 < \theta < \frac{2\pi}{3} \\ 9 \cos \left(\frac{\theta}{2} \right) - 12 \sin^2 \left(\frac{\theta}{2} \right) \cos \left(\frac{\theta}{2} \right); & \frac{2\pi}{3} < \theta < \pi \end{cases}$$

$$= \begin{cases} 6 \sin \left(\frac{\theta}{2} \right) \left[\sin \theta - \frac{\sqrt{3}}{2} \right]; & 0 < \theta < \frac{2\pi}{3} \\ 12 \cos \left(\frac{\theta}{2} \right) \left(\frac{\sqrt{3}}{2} - \sin \theta \right) \left(\frac{\sqrt{3}}{2} + \sin \theta \right); & \frac{2\pi}{3} < \theta < \pi \end{cases}$$

Note that $E'(\theta) = 0$ for $\frac{\pi}{3}$. Also, at $\frac{2\pi}{3}$, $E'(\theta) = 0$. Now, $E(0) = 8 \left[2 \sin^3 \left(\frac{\pi}{3} \right) \right] = 6\sqrt{3}$

$$E \left(\frac{\pi}{3} \right) = 8 \left[\sin^3 \left(\frac{\pi}{6} \right) + \sin^3 \left(\frac{\pi}{6} \right) + \sin^2 \left(\frac{\pi}{2} \right) \right] = 8 \left(\frac{1}{8} + \frac{1}{8} + 1 \right) = 10$$

$$E \left(\frac{2\pi}{3} \right) = 8 \left[\frac{3\sqrt{3}}{8} + 0 + \frac{3\sqrt{3}}{8} \right] = 6\sqrt{3}. \text{ Also, } E(\pi) = 10. \text{ As } 10 < 6\sqrt{3},$$

$\min E(\theta) = 10\sqrt{100} > \sqrt{72} = 6\sqrt{2}$ and $\max E(\theta) = 6\sqrt{3} = 3(2\sqrt{3}) = 3(2^4 \cdot 3^2)^{\frac{1}{4}}$

$$= 3(144)^{\frac{1}{4}} < 3 \left(\frac{2}{6} \right)^{\frac{1}{4}}$$

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JP.066. Prove that in any triangle

$$\sum \frac{m_a^2}{h_b h_c} \geq n \cdot \frac{R}{r} + (3 - 2n) \cdot \frac{s}{3r\sqrt{3}}, n \leq \frac{3}{2}$$

Proposed by Marin Chirciu – Romania

Solution by Kevin Soto Palacios – Huarmey – Peru

Probar en un triángulo ABC $\sum \frac{m_a^2}{h_b h_c} \geq n \cdot \frac{R}{r} + (3 - 2n) \cdot \frac{s}{3\sqrt{3}r}$, *donde* $n \leq \frac{3}{2}$

Tener en cuenta lo siguiente

$$abc = 4RS, a + b + c = 2s = \frac{2S}{r} \Leftrightarrow (a + b + c)abc = 8S^2 \cdot \frac{R}{r}$$

Ahora bien

$$\frac{R}{r} - \frac{2s}{3\sqrt{3}r} = \frac{r(3\sqrt{3}R - 2s)}{3\sqrt{3}r} \geq 0 \Leftrightarrow \text{Lo cual es cierto ya que en un}$$

$$\Delta ABC \rightarrow 3\sqrt{3}R \geq 2s. \text{ Como } n \leq \frac{3}{2}$$

$$\rightarrow n \cdot \frac{R}{r} + (3 - 2n) \cdot \frac{s}{3\sqrt{3}r} = n \cdot \left(\frac{R}{r} - \frac{2s}{3\sqrt{3}r} \right) + \frac{3s}{3\sqrt{3}r} \leq \frac{3}{2} \left(\frac{R}{r} - \frac{2s}{3\sqrt{3}r} \right) + \frac{s}{\sqrt{3}r} = \frac{3R}{2r}$$

Es suficiente probar $\sum \frac{4m_a^2}{4h_b h_c} \geq \frac{3R}{2r}$

$$\Leftrightarrow \frac{(2b^2 + 2c^2 - a^2)bc}{16S^2} + \frac{(2c^2 + 2a^2 - b^2)ca}{16S^2} + \frac{(2a^2 + 2b^2 - c^2)ab}{16S^2} \geq \frac{3R}{2r}$$

$$\Leftrightarrow 2bc \sum (b^2 + c^2) - abc(a + b + c) \geq 16S^2 \cdot \frac{3R}{2r} = 3abc(a + b + c)$$

Aplicando MA ≥ MG

$$2bc \sum (b^2 + c^2) - abc(a + b + c) \geq 4 \sum b^2 c^2 - abc(a + b + c) \geq \\ \geq 4abc(a + b + c) - abc(a + b + c) = 3abc(a + b + c)$$

JP.067. Prove that in any triangle

$$n \cdot \frac{s^2 + r^2}{Rr} + \sqrt[k]{\frac{2r}{R}} \geq 14n + 1, n \geq \frac{1}{2}, k \in \mathbb{N}, k \geq 2.$$

Proposed by Marin Chirciu – Romania

Solution by Kevin Soto Palacios Peru – Huarmey – Peru

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Probar en un triángulo ABC : $n \cdot \frac{s^2+r^2}{Rr} + \sqrt[k]{\frac{2r}{R}} \geq 14n + 1, n \geq \frac{1}{2}, k \in N, k \geq 2$

Tener en cuenta lo siguiente

1) Siendo a, b, c los lados de un triángulo se cumple la siguiente desigualdad

$$(ab + bc + ca) \left(\frac{1}{ba} + \frac{1}{bc} + \frac{1}{ca} \right) \geq 9 \Leftrightarrow \frac{(s^2 + r^2 + 4Rr)}{2Rr} \geq 9 \Leftrightarrow \\ \Leftrightarrow s^2 + r^2 - 14Rr \geq 0$$

2) Para todo $k \geq 2$ it checks $\left(\frac{2r}{R}\right)^{k-1} \leq 1 \Leftrightarrow \left(\frac{2r}{R}\right)^k \leq \frac{2r}{R} \Leftrightarrow \frac{2r}{R} \leq \sqrt[k]{\frac{2r}{R}}$

La desigualdad propuesta s equivalente

$$n \cdot \left(\frac{s^2 + r^2}{Rr} - 14 \right) + \sqrt[k]{\frac{2r}{R}} \geq 1 \Leftrightarrow n \left(\frac{s^2 + r^2 - 14Rr}{Rr} \right) + \sqrt[k]{\frac{2r}{R}} \geq 1$$

$$\text{Ahora bien } n \left(\frac{s^2+r^2-14Rr}{Rr} \right) + \sqrt[k]{\frac{2r}{R}} \geq \frac{1}{2} \left(\frac{s^2+r^2-14Rr}{Rr} \right) + \frac{2r}{R}$$

Es suficiente probar

$$\frac{1}{2} \left(\frac{s^2 + r^2 - 14Rr}{Rr} \right) + \frac{2r}{R} \geq 1 \Leftrightarrow \frac{s^2 + r^2}{2Rr} + \frac{4r^2}{2Rr} \geq 8 \Leftrightarrow s^2 \geq 16Rr - 5r^2$$

(Válido por desigualdad de Gerretsen)

JP.068. Let a, b and c be the side lengths of a triangle ABC , with circumradius R and inradius r . Prove that

$$\frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{a + b} + \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{b + c} + \frac{\tan \frac{C}{2} + \tan \frac{A}{2}}{c + a} \leq \frac{1}{r} - \frac{1}{R}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

$$\frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{a + b} + \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{b + c} + \frac{\tan \frac{C}{2} + \tan \frac{A}{2}}{c + a} \leq \frac{1}{r} - \frac{1}{R}$$

Teniendo en cuenta las siguientes notaciones y desigualdades en un ΔABC

$$\tan \frac{A}{2} = \frac{(s-b)(s-c)}{s}, \tan \frac{B}{2} = \frac{(s-c)(s-a)}{s}, \tan \frac{C}{2} = \frac{(s-a)(s-b)}{s}$$

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$$\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b} \geq \frac{a+b+c}{2} \quad (\text{Nesbit}), R \geq 2r \quad (\text{Euler})$$

La desigualdad propuesta es equivalente

$$\frac{(s-b)(s-c) + (s-c)(s-a)}{S(a+b)} + \frac{(s-c)(s-a) + (s-a)(s-b)}{S(b+c)} +$$

$$+ \frac{(s-a)(s-b) + (s-b)(s-c)}{S(c+a)} \leq \frac{1}{r} - \frac{1}{R}$$

$$\frac{(s-c)(s-b+s-a)}{S(a+b)} + \frac{(s-a)(s-c+s-b)}{S(b+c)} + \frac{(s-b)(s-c+s-a)}{S(c+a)} \leq \frac{1}{r} - \frac{1}{R}$$

$$\frac{(a+b-c)c}{2S(a+b)} + \frac{(b+c-a)a}{2S(b+c)} + \frac{(a+c-b)b}{2S(c+a)} \leq \frac{1}{r} - \frac{1}{R}$$

$$\sum \frac{(a+b)c}{2S(a+b)} - \sum \frac{c^2}{2S(a+b)} \leq \sum \frac{c}{2S} - \frac{a+b+c}{4S} =$$

$$= \frac{a+b+c}{2S} - \frac{a+b+c}{4S} = \frac{a+b+c}{4S} = \frac{2p}{4pr} = \frac{1}{2r} \leq \frac{1}{r} - \frac{1}{R} \quad (\text{LQOD})$$

Lo cual es cierto ya que $\rightarrow \frac{1}{2r} \leq \frac{1}{r} - \frac{1}{R} \Leftrightarrow \frac{1}{R} \leq \frac{1}{2r} \Leftrightarrow R \geq 2r$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$s \tan \frac{A}{2} = s \cdot \sqrt{\frac{(s-b)(s-c)}{s(s-a)}} = \frac{\sqrt{s(s-a)(s-b)(s-c)}}{s-a} = \frac{\Delta}{s-a} = \frac{rs}{s-a}$$

$$\Rightarrow \tan \frac{A}{2} = \frac{r}{s-a}. \text{ Similarly, } \tan \frac{B}{2} = \frac{r}{s-b} \text{ \& } \tan \frac{C}{2} = \frac{r}{s-c}. \text{ Now, } \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{a+b} = \frac{r}{a+b} \left(\frac{1}{s-a} + \frac{1}{s-b} \right)$$

$$= \frac{r(s-b+s-a)}{(a+b)(s-a)(s-b)} \stackrel{(1)}{=} \frac{rc}{(s-a)(s-b)(a+b)}. \text{ Similarly, } \frac{\tan \frac{B}{2} + \tan \frac{C}{2}}{b+c} \stackrel{(2)}{=} \frac{ra}{(s-b)(s-c)(b+c)},$$

$$\text{and, } \frac{\tan \frac{C}{2} + \tan \frac{A}{2}}{c+a} \stackrel{(3)}{=} \frac{rb}{(s-c)(s-a)(c+a)}$$

(1)+(2)+(3) \Rightarrow LHS

$$= r \left\{ \frac{c}{(s-a)(s-b)(a+b)} + \frac{a}{(s-b)(s-c)(b+c)} + \frac{b}{(s-c)(s-a)(c+a)} \right\}$$

$$\stackrel{(4)}{=} \frac{r \sum \{c(b+c)(c+a)(s-c)\}}{\{\prod (s-a)\}\{\prod (a+b)\}}. \text{ Now, } \sum \{c(b+c)(c+a)(s-c)\} = \sum \{(cs - c^2)(\sum ab + c^2)\}$$

$$= \sum (cs - c^2) (\sum ab) + s \sum a^3 - \sum a^4$$

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$$\begin{aligned}
 &= \left(\sum ab\right) \cdot \left(s \sum a - \sum a^2\right) + s \left\{3abc + 2s \left(\sum a^2 - \sum ab\right)\right\} - \\
 &\quad - \left\{\left(\sum a^2\right)^2 - 2 \sum a^2 b^2\right\} \\
 &= 2s^2 \left(\sum ab\right) - \left(\sum ab\right) \left(\sum a^2\right) + 3sabc + 2s^2 \left(\sum a^2\right) - 2s^2 \left(\sum ab\right) - \\
 &\quad - \left(\sum a^2\right)^2 + 2 \left\{\left(\sum ab\right)^2 - 2abc(2s)\right\} \\
 &= \left(\sum a^2\right) \left(2s^2 - \sum ab\right) + 2 \left(\sum ab\right)^2 - \left(\sum a^2\right)^2 - 5sabc \\
 &= \left(\sum a^2\right) \left(s^2 - 4Rr - r^2\right) - \left(\sum a^2\right)^2 + 2 \left(\sum ab\right)^2 - 5sabc \\
 &= \frac{\left(\sum a^2\right) \left(\sum a^2\right)}{2} - \left(\sum a^2\right)^2 + 2 \left(\sum ab\right)^2 - 5sabc \\
 &= \frac{4 \left(\sum ab\right)^2 - \left(\sum a^2\right)^2 - 10s(4Rrs)}{2} \\
 &= \frac{\left(2 \sum ab + \sum a^2\right) \left(2 \sum ab - \sum a^2\right) - 40Rrs^2}{2} \\
 &= \frac{2 \left(s^2 + 4Rr + r^2 + s^2 - 4Rr - r^2\right) \left(16Rr + 4r^2\right) - 40Rrs^2}{2}
 \end{aligned}$$

$$= 2s^2(16Rr + 4r^2) - 20Rrs^2 = 2s^2(6Rr + 4r^2) \stackrel{(5)}{=} 4s^2r(3R + 2r)$$

$$\text{Again, } \prod(a + b) = 2abc + \sum ab(2s - c)$$

$$= 2abc + 2s \left(\sum ab\right) - 3abc = 2s \left(\sum ab\right) - 4Rrs \stackrel{(6)}{=} 2s(s^2 + 2Rr + r^2)$$

$$\text{Also, } \prod(s - a) = \frac{s \cdot \prod(s - a)}{s} = \frac{r^2 s^2}{s} \stackrel{(7)}{=} r^2 s$$

(4),(5),(6),(7) ⇒ LHS

$$= \frac{4s^2 r^2 (3R + 2r)}{(r^2 s) \{2s(s^2 + 2Rr + r^2)\}} = \frac{2(3R + 2r)}{s^2 + 2Rr + r^2} \stackrel{?}{\leq} \frac{1}{r} - \frac{1}{R} = \frac{R - r}{Rr}$$

$$\Leftrightarrow (s^2 + 2Rr + r^2)(R - r) \stackrel{?}{\geq} 2Rr(3r + 2r) \quad (8)$$

$$\text{LHS of (8)} \geq (18Rr - 4r^2)(R - r) \quad (\text{Gerretsen})$$

$$\stackrel{?}{\geq} 2Rr(3R + 2r)$$

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$$\Leftrightarrow (9R - 2r)(R - r) \stackrel{?}{\geq} 3R^2 + 2Rr$$

$$\Leftrightarrow 6R^2 - 13Rr + 2r^2 \stackrel{?}{\geq} 0 \Leftrightarrow (R - 2r)(6R - r) \stackrel{?}{\geq} 0 \rightarrow \text{true} \because R \geq 2r \text{ (Euler) (Proved)}$$

Solution 3 by Myagmarsuren Yadamsuren-Darkhan-Mongolia

$$1) \sum \frac{a}{b+c} \leq 2 - \frac{r}{R} \text{ (ASSURE)}$$

$$\sum \frac{a}{b+c} = \frac{2 \cdot (p^2 - r^2 - Rr)}{p^2 + r^2 + 2Rr} \leq 2 - \frac{r}{R} \Leftrightarrow$$

$$\Leftrightarrow \frac{2 \cdot (p^2 - r^2 - Rr)}{p^2 + r^2 + 2Rr} + \frac{r}{R} \leq 2 \Leftrightarrow \frac{2R \cdot (p^2 - r^2 - Rr) + r(p^2 + r^2 + 2Rr)}{R \cdot (p^2 + r^2 + 2Rr)} \leq 2$$

$$\Leftrightarrow p^2 \leq 6R^2 + 2Rr - r^2 \stackrel{\text{Gerretsen}}{\Rightarrow} 4R^2 + 4Rr + 3r^2 \leq 6R^2 + 2Rr - r^2$$

$$\Leftrightarrow 2R^2 - 2Rr - 4r^2 \geq 0 \Leftrightarrow (R - 2r) \cdot (R + r) \geq 0 \text{ Euler}$$

True

$$2) -\left(\frac{a^2}{b+c} + \frac{b^2}{c+a} + \frac{c^2}{a+b}\right) \stackrel{\text{Schwarz}}{\leq} -\frac{(a+b+c)^2}{2 \cdot (a+b+c)} = -\frac{a+b+c}{2} = -p$$

$$\begin{aligned} 3) \sum \frac{\tan \frac{A}{2} + \tan \frac{B}{2}}{a+b} &= \sum \frac{\sqrt{\frac{p-c}{p}} \cdot \left(\sqrt{\frac{p-b}{p-a}} + \sqrt{\frac{p-a}{p-b}}\right)}{a+b} = \\ &= \sum \frac{\sqrt{\frac{p-c}{p}} \cdot \left(\frac{p-b+p-a}{\sqrt{(p-a) \cdot (p-b)}}\right)}{a+b} = \sum \frac{\sqrt{\frac{p-c}{p}} \cdot \left(\frac{c}{\sqrt{(p-a)(p-b)}}\right)}{a+b} = \\ &= \sum \frac{(p-c) \cdot c}{\frac{S}{a+b}} = \frac{1}{S} \cdot \sum \frac{(p-c) \cdot c}{a+b} = \frac{1}{S} \cdot \left(p \cdot \sum \frac{c}{a+b} - \frac{c^2}{a+b}\right) \stackrel{(1)(2)}{\leq} \\ &\leq \frac{1}{S} \cdot \left(p \cdot \left(2 - \frac{r}{R}\right) - p\right) = \frac{1}{r} \cdot \left(2 - \frac{r}{R} - 1\right) = \frac{1}{r} - \frac{1}{R} \end{aligned}$$

JP.069. Let a, b be positive real numbers such that $a^2 + ab + b^2 = 9$. Find the maximal value of expression:

$$(a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

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Siendo a, b, c números R^+ de tal manera que $a^2 + ab + b^2 = 9$. Hallar el máximo valor $E = (a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$. Como $a, b, c > 0$.

Aplicando $MA \geq MG$

$$9 = a^2 + ab + b^2 \geq 3\sqrt[3]{a^3b^3} = 3ab \Leftrightarrow 3 \geq ab$$

$$(a + b)^2 = (a^2 + ab + b^2) + ab = 9 + ab \leq 12. \text{ Por lo tanto}$$

$$E = (a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17 \leq 12^3 + 3^5 + 2 \cdot 3^3 + 3^2 - 17 = \\ = 1728 + 243 + 54 + 9 - 17 = 2017; E \leq 2017$$

La igualdad se alcanza cuando $a = b = \sqrt{3}$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaijan

$$a, b, c > 0$$

$$a^2 + ab + b^2 = 9 \Rightarrow ab \leq 3 \text{ from (AM-GM)}$$

$$(a + b)^6 = ((a + b)^2)^3 = (a^2 + ab + ab + b^2)^3 = (9 + ab)^3 \leq 1728$$

$$(ab)^5 \leq 24^3; 2(ab)^3 \leq 54; (ab)^2 \leq 9. \text{ Problem: maximum values:}$$

$$(a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17 \leq 2017$$

Solution 3 by Ravi Prakash-New Delhi-India

$$\text{Let } E = (a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$= (a^2 + b^2 + ab + ab)^3 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$= (9 + ab)^3 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$= 9^3 + 243ab + 27(ab)^2 + (ab)^3 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$E = 712 + 243ab + 28(ab)^2 + 3(ab)^3 + (ab)^5. \text{ As } a^2 + ab + b^2 = 9$$

$$4a^2 + 4b^2 + 4ab = 36 \Rightarrow 2\{(a + b)^2 + (a - b)^2\} + (a + b)^2 - (a - b)^2 = 36$$

$$\Rightarrow \frac{(a+b)^2}{12} + \frac{(a-b)^2}{36} = 1. \text{ Let } a + b = 2\sqrt{3} \cos \theta$$

$$a - b = 6 \sin \theta \Rightarrow a = \sqrt{3} \cos \theta + 3 \sin \theta \text{ and } b = \sqrt{3} \cos \theta - 3 \sin \theta$$

$$\therefore ab = 3 \cos^2 \theta - 9 \sin^2 \theta = \frac{3}{2}(1 + \cos 2\theta) - \frac{9}{2}(1 - \cos 2\theta) = 6 \cos 2\theta - 3 \leq 3$$

Thus, maximum possible value of E is $712 + (243)(3) + (28)(3^2) + 3(3^3) + 3^5$

$$= 712 + 729 + 252 + 81 + 243 = 2017$$

Solution 4 by SK Rejuan-West Bengal-India

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$$a, b \in \mathbb{R}^+ \text{ and } a^2 + ab + b^2 = 9 \quad (i)$$

By $AM \geq GM$ we get, $a^2 + ab + b^2 \geq 3ab \Rightarrow 9 \geq 3ab$ [from (i)] $\Rightarrow ab \leq 3$ (ii)

$$\text{Again, } a^2 + ab + b^2 = 9 \Rightarrow (a + b)^2 - ab = 9$$

$$\Rightarrow (a + b)^2 = 9 + ab \leq 9 + 3 \text{ [from (ii)]} \Rightarrow (a + b)^2 \leq 12 \quad (iii)$$

\Rightarrow LHS

$$= (a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

$$\leq (12)^3 + (3)^5 + 2 \cdot (3)^3 + (3)^2 - 17 \text{ [from (iii) \& (ii)]}$$

$$= 1728 + 243 + 54 + 9 - 17$$

$$= 2017$$

$$\Rightarrow (a + b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17 \leq 2017$$

\therefore Maximum value is 2017

JP.070. Let a, b be positive real numbers with $a^2 + ab + b^2 = k^2, k > 0$.

Prove that

$$\sqrt{a + b} + \sqrt[4]{ab} \leq \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \sqrt{k}.$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c números \mathbb{R}^+ de tal manera que $a^2 + ab + b^2 = k^2, k > 0$

$$\sqrt{a + b} + \sqrt[4]{ab} \leq \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \sqrt{k}. \text{ Ahora bien}$$

$$k^2 = a^2 + ab + b^2 \geq \frac{3}{4}(a + b)^2 + \frac{1}{4}(a - b)^2 \geq \frac{3}{4}(a + b)^2 \Leftrightarrow \sqrt{k} \geq \frac{\sqrt[4]{3} \cdot \sqrt{a + b}}{\sqrt{2}}$$

Aplicando $MA \geq MG$

$$\Rightarrow \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \sqrt{k} \geq \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \frac{\sqrt[4]{3} \cdot \sqrt{a + b}}{\sqrt{2}} = \sqrt{a + b} + \sqrt[4]{ab}$$

Solution 2 by Seyran Ibrahimov-Maasilli-Azerbaijan

$$k^2 \geq \frac{3}{4}(a + b)^2 \Rightarrow \text{if } \sqrt{a + b} \Rightarrow a + b = x^2; \sqrt[4]{ab} \leq \sqrt{\frac{a+b}{2}}; x + \frac{x}{\sqrt{2}} = \frac{\sqrt{2} + 1}{\sqrt[4]{3}} \cdot \frac{\sqrt[4]{3}}{\sqrt{2}} \cdot x$$

Solution 3 by Nguyen Ngoc Tu-Ha Giang-Vietnam

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Fact

$$1) \text{ We have: } a^2 + b^2 \geq \frac{1}{2}(a + b)^2$$

$$\begin{aligned} \Rightarrow a^2 + b^2 + 2k^2 &\geq \frac{1}{2}(a + b)^2 + \frac{2k^2}{3} + \frac{2k^2}{3} + \frac{2k^2}{3} \stackrel{AM-GM}{\geq} 4\sqrt[4]{4 \cdot (a + b)^2 \cdot \frac{k^6}{27}} \\ &= \frac{4\sqrt{2}}{\sqrt[4]{3}} \cdot k \cdot \sqrt{\frac{k}{3}} \cdot \sqrt{a + b} \end{aligned}$$

Fact

$$2) ab + \frac{k^2}{3} + \frac{k^2}{3} + \frac{k^2}{3} \stackrel{AM-GM}{\geq} 4\sqrt[4]{\frac{ab \cdot k^6}{27}} = \frac{4}{\sqrt[4]{3}} \cdot k \cdot \sqrt{\frac{k}{3}} \cdot \sqrt[4]{ab}$$

$$\Rightarrow a^2 + b^2 + ab + 3k^2 \geq \left(4 \cdot k \cdot \sqrt{\frac{k}{3}} \cdot \frac{1}{\sqrt[4]{3}}\right) (\sqrt{2} \cdot \sqrt{a + b} + \sqrt[4]{ab})$$

$$\Leftrightarrow 4k^2 \geq \left(4k \cdot \sqrt{\frac{k}{3}} \cdot \frac{1}{\sqrt[4]{3}}\right) (\sqrt{2} \cdot \sqrt{a + b} + \sqrt[4]{ab}) \quad (*)$$

Fact

$$3) a + b \geq 2\sqrt{ab} \Rightarrow \sqrt{a + b} \geq \sqrt{2} \cdot \sqrt[4]{ab}$$

Fact

$$\begin{aligned} 4) \sqrt{2} \cdot \sqrt{a + b} + \sqrt[4]{ab} &= (\sqrt{2} - 3 + 2\sqrt{2})(\sqrt{a + b}) + (3 - 2\sqrt{2})\sqrt{a + b} + \sqrt[4]{ab} \\ &\stackrel{(3)}{\geq} 3(\sqrt{2} - 1) \cdot \sqrt{a + b} + (3 - 2\sqrt{2})\sqrt{2} \cdot \sqrt[4]{ab} + \sqrt[4]{ab} \\ &= 3(\sqrt{2} - 1)(\sqrt{a + b} + \sqrt[4]{ab}) \end{aligned}$$

Hence, (*) becomes:

$$4k^2 \geq \left(4k \cdot \sqrt{\frac{k}{3}} \cdot \frac{1}{\sqrt[4]{3}}\right) 3(\sqrt{2} - 1) \cdot \sqrt{a + b} + \sqrt[4]{ab}$$

$$\Rightarrow \sqrt{a + b} + \sqrt[4]{ab} \leq \frac{4k^2}{4k \cdot \sqrt{\frac{k}{3}} \cdot \frac{1}{\sqrt[4]{3}} \cdot 3(\sqrt{2} - 1)} = \frac{\sqrt{2} + 1}{4\sqrt{3}} \sqrt{k}$$

Solution 4 by SK Rejuan-West Bengal-India

$$a, b \in \mathbb{R}^+ \text{ and } a^2 + ab + b^2 = k^2 \quad (i)$$

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By $AM \geq GM$ we get, $a^2 + ab + b^2 \geq 3ab \Rightarrow k^2 \geq 3ab$ [$ask^2 = a^2 + ab + b^2$]

$$\Rightarrow ab \leq \frac{k^2}{3} \quad (ii)$$

$$\Rightarrow \sqrt[4]{ab} \leq \frac{\sqrt{k}}{\sqrt[4]{3}} \quad (iii)$$

$$\text{Again, } a^2 + ab + b^2 = k^2 \Rightarrow (a + b)^2 - 2ab + ab = k^2$$

$$\Rightarrow (a + b)^2 = k^2 + ab \leq k^2 + \frac{k^2}{3} \quad [\text{from (ii)}] \Rightarrow (a + b)^2 \leq \frac{4}{3}k^2$$

$$\Rightarrow \sqrt{a + b} \leq \frac{\sqrt{2}}{\sqrt[4]{3}}\sqrt{k} \quad (iv)$$

$$\text{Adding (iv) \& (iii) we get } \sqrt{a + b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}}{\sqrt[4]{3}}\sqrt{k} + \frac{\sqrt{k}}{\sqrt[4]{3}} \Rightarrow \sqrt{a + b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}+1}{\sqrt[4]{3}}\sqrt{k}$$

JP.071. Let ABC be a triangle with circumradius R and inradius r , and let w_a, w_b, w_c be the lengths of the internal bisectors of the angle opposite of the sides of lengths a, b, c , respectively.

Prove that

$$\left(\frac{w_a}{a}\right)^2 \cdot \tan \frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \cdot \tan \frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan \frac{C}{2} \leq \frac{3\sqrt{3}}{8} \cdot \frac{R}{r}$$

Proposed by George Apostolopoulos – Messolonghi – Greece

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo ABC un triángulo con circunradio R e inradio r , además w_a, w_b, w_c son los ángulos internos bisectors, opuestos a los lados a, b, c respectivamente.

$$\text{Probar que } \left(\frac{w_a}{a}\right)^2 \cdot \tan \frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \cdot \tan \frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan \frac{C}{2} \leq \frac{3\sqrt{3}}{8} \cdot \frac{R}{r} \quad (A)$$

Teniendo en cuenta la siguientes identidades y desigualdades en un triángulo ABC

$$\tan \frac{A}{2} = \frac{(s-b)(s-c)}{s} = \frac{(a+c-b)(a+b-c)}{4s} = \frac{a^2 - (b-c)^2}{4s} \leq \frac{a^2}{4s}$$

$$\tan \frac{B}{2} \leq \frac{b^2}{4s}, \tan \frac{C}{2} \leq \frac{c^2}{4s}$$

$$w_a \leq \sqrt{s(s-a)}, w_b \leq \sqrt{s(s-b)}, w_c \leq \sqrt{s(s-c)}, s \leq \frac{3\sqrt{3}R}{2}$$

Utilizando las desigualdades previas en (A)

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$$\begin{aligned} \left(\frac{w_a}{a}\right)^2 \tan \frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \tan \frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \tan \frac{C}{2} &\leq \frac{w_a^2}{4S} + \frac{w_b^2}{4S} + \frac{w_c^2}{4S} \leq \\ &\leq \frac{s(s-a) + s(s-b) + s(s-c)}{4S} = \frac{s^2}{4sr} = \frac{s}{4r} \leq \frac{3\sqrt{3}R}{8r} \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\begin{aligned} LHS &\leq \sum \left[\frac{s(s-a)}{a^2} \tan \frac{A}{2} \right] \quad (\because w_a \leq \sqrt{s(s-a)}) = \sum \left(s \tan \frac{A}{2} \cdot \frac{s-a}{a^2} \right) \\ &= \sum \left(r_a \cdot \frac{s-a}{a^2} \right) = \sum \left(\frac{\Delta}{s-a} \cdot \frac{s-a}{a^2} \right) \\ &= \Delta \left(\sum \frac{1}{a^2} \right) = \Delta \cdot \frac{\sum a^2 b^2}{16R^2 r^2 s^2} \stackrel{\text{Goldstone}}{\leq} \frac{\Delta \cdot 4R^2 S^2}{16R^2 r^2 s^2} \\ &= \frac{rs \cdot 4R^2 s^2}{16R^2 r^2 s^2} = \frac{s}{4r} \stackrel{\text{Mitrinovic}}{\leq} \frac{3\sqrt{3}R}{2 \cdot 4r} = \frac{3\sqrt{3}}{8} \cdot \frac{R}{r} \end{aligned}$$

Solution 3 by Marin Chirciu – Romania

We prove the following helpful result: Lemma: In ΔABC

$$\left(\frac{w_a}{a}\right)^2 \tan \frac{A}{2} + \left(\frac{w_b}{b}\right)^2 \tan \frac{A}{2} + \left(\frac{w_c}{c}\right)^2 \tan \frac{A}{2} \leq \frac{p}{4r}.$$

Proof

$$\text{We use } w_a^2 \leq p(p-a) \text{ and } \tan \frac{A}{2} = \frac{(p-b)(p-c)}{s}$$

$$\text{It follows } \left(\frac{w_a}{a}\right)^2 \tan \frac{A}{2} \leq \frac{p(p-a)}{a^2} \cdot \frac{(p-b)(p-c)}{s} = \frac{s}{a^2}$$

$$\text{We obtain } \sum \left(\frac{w_a}{a}\right)^2 \tan \frac{A}{2} \leq S \sum \frac{1}{a^2} \leq rp \cdot \frac{1}{4r^2} = \frac{p}{4r}.$$

The equality holds if and only if the triangle is equilateral.

Let's pass to solving the problem from the enuntiation.

Using the Lemma and Mitrinovic's inequality $p \leq \frac{3R\sqrt{3}}{2}$ we obtain the conclusion.

The equality holds if and only if the triangle is equilateral.

JP.072. Let a, b and c denote, as usual, the lengths of the sides $BC, CA,$ and $AB,$ respectively, in $\Delta ABC.$ Let R be the circumradius, r the inradius of $\Delta ABC,$ and r_a, r_b and r_c the exradii to A, B and $C,$ respectively.

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Prove that

$$(a) \frac{r_a}{a^3} + \frac{r_b}{b^3} + \frac{r_c}{c^3} \leq \frac{\sqrt{3}}{8r^2}, \quad (b) \frac{3R}{2r} \geq \sqrt{\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a}} + 6$$

Proposed by George Apostolopoulos – Messolonghi - Greece

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\text{Probar en un triángulo } ABC: \frac{r_a}{a^3} + \frac{r_b}{b^3} + \frac{r_c}{c^3} \leq \frac{\sqrt{3}}{8r^2} \quad (A)$$

Teniendo en cuenta las siguientes identidades y desigualdades en triángulo ABC

$$r_a = p \tan \frac{A}{2}, r_b = p \tan \frac{B}{2}, r_c = p \tan \frac{C}{2}, S = pr, \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \leq \frac{\sqrt{3}}{2r}$$

$$\tan \frac{A}{2} = \frac{(s-b)(s-c)}{S} = \frac{(a+c-b)(a+b-c)}{4S} = \frac{a^2 - (b-c)^2}{4S} \leq \frac{a^2}{4S}$$

$$\tan \frac{B}{2} \leq \frac{b^2}{4S}, \tan \frac{C}{2} \leq \frac{c^2}{4S}. \text{ Utilizando las desigualdades previas en (A)}$$

$$\begin{aligned} \frac{r_a}{a^3} + \frac{r_b}{b^3} + \frac{r_c}{c^3} &= \frac{p \tan \frac{A}{2}}{a^3} + \frac{p \tan \frac{B}{2}}{b^3} + \frac{p \tan \frac{C}{2}}{c^3} \leq \frac{p}{4Sa} + \frac{p}{4Sb} + \frac{p}{4Sc} = \\ &= \frac{1}{4r} \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \leq \frac{1}{4r} \cdot \frac{\sqrt{3}}{2r} = \frac{\sqrt{3}}{8r^2} \quad (LOQD) \end{aligned}$$

$$\text{Probar en un triángulo } ABC: \left(\frac{3R}{2r} \right)^2 \geq \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6$$

1) Teniendo en cuenta las siguientes identidades en un triángulo ABC

$$r_a = \frac{S}{s-a}, r_b = \frac{S}{s-b}, r_c = \frac{S}{s-c}, \frac{R}{2r} = \frac{abc}{8(s-a)(s-b)(s-c)}$$

Realizando los siguientes cambios de variables

$$x = s-a, y = s-b, z = s-c, x+y = b, y+z = a, z+x = c$$

2) Siendo $x, y, z > 0$, se cumple la siguiente desigualdad

$$9(x+y)(y+z)(z+x) \geq 8(x+y+z)(xy + yz + zx)$$

Solo es necesario demostrar lo siguiente

$$\frac{9R}{2r} \geq \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6 \Leftrightarrow \frac{9(x+y)(y+z)(z+x)}{8xyz} \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 6$$

Es suficiente probar

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$$\frac{(x+y+z)(xy+yz+zx)}{xyz} = (x+y+z) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right) \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 6 \Leftrightarrow$$

$$\Leftrightarrow 3 + \frac{x}{y} + \frac{y}{z} + \frac{z}{x} + \frac{y}{x} + \frac{z}{y} + \frac{x}{z} \geq \frac{y}{x} + \frac{z}{y} + \frac{x}{z} + 6 \quad (MA \geq MG)$$

$$\text{Se concluye que} \rightarrow \left(\frac{3R}{2r} \right)^2 \geq \frac{9R}{2r} \geq \frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6. \quad (LQOD)$$

JP.073. If $a, b, c > 0; n \geq 1$ then:

$$\frac{3n(a^4 + b^4 + c^4)}{(a^2 + b^2 + c^2)^2} + \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq n + 1$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c , de tal manera que $n \geq 1$. Probar que

$$\frac{3n(a^4 + b^4 + c^4)}{(a^2 + b^2 + c^2)^2} + \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq n + 1$$

$$n \left(\frac{3(a^4 + b^4 + c^4)}{(a^2 + b^2 + c^2)^2} - 1 \right) + \frac{ab + bc + ca}{a^2 + b^2 + c^2} - 1 \geq 0$$

$$\Rightarrow n \left(\frac{2(a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2)}{(a^2 + b^2 + c^2)^2} \right) + \frac{(ab + bc + ca - a^2 - b^2 - c^2)(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} \geq 0$$

Como $n \geq 1$

Es suficiente demostrar

$$\frac{2(a^4 + b^4 + c^4 - a^2b^2 - b^2c^2 - c^2a^2)}{(a^2 + b^2 + c^2)^2} + \frac{(ab + bc + ca - a^2 - b^2 - c^2)(a^2 + b^2 + c^2)}{(a^2 + b^2 + c^2)^2} \geq 0$$

$$\Leftrightarrow 2(a^4 + b^4 + c^4) - 2(a^2b^2 + b^2c^2 + c^2a^2) + (ab + bc + ca)(a^2 + b^2 + c^2) - (a^2 + b^2 + c^2)^2 \geq 0$$

$$\Leftrightarrow a^4 + b^4 + c^4 - 4(a^2b^2 + b^2c^2 + c^2a^2) + ab(a^2 + b^2) + bc(b^2 + c^2) + ca(c^2 + a^2) \geq 0$$

$$\Leftrightarrow abc(a + b + c) + ab(a - b)^2 + bc(b - c)^2 + ca(c - a)^2 \geq$$

$$\geq 2(a^2b^2 + b^2c^2 + c^2a^2) - a^4 - b^4 - c^4$$

$$\Leftrightarrow abc(a + b + c) + ab(a - b)^2 + bc(b - c)^2 + ca(c - a)^2 \geq$$

$$\geq (a + b + c)(a + b - c)(b + c - a)(a + c - b)$$

$$\Leftrightarrow (a + b + c)(abc - (b + c - a)(a + c - b)(b + a - c)) + ab(a - b)^2 + bc(b - c)^2 + ca(c - a)^2 \geq 0$$

$$\text{Es suficiente demostrar: } abc \geq (a + b - c)(b + c - a)(c + a - b)$$

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$$\begin{aligned}
 abc &\geq (b^2 - (c - a)^2)(c + a - b) \\
 abc &\geq (b^2 - c^2 - a^2 + 2ac)(c + a - b) \\
 abc &\geq b^2c + b^2a - b^3 - c^3 - c^2a + c^2b - a^2c - a^3 + a^2b + 2c^2a + 2a^2c - 2abc \\
 a^3 + b^3 + c^3 + 3abc &\geq ab(a + b) + ca(c + a) + bc(b + c)
 \end{aligned}$$

Válido por (Shur Inequality)

Solution 2 by Nguyen Minh Tri-Vietnam

$$\begin{aligned}
 &\frac{3n(a^4 + b^4 + c^4)}{(a^2 + b^2 + c^2)^2} + \frac{ab + bc + ca}{a^2 + b^2 + c^2} \geq n + 1 \\
 \Leftrightarrow &3n(a^4 + b^4 + c^4) + (a^2 + b^2 + c^2)(ab + bc + ac) \geq (n + 1)(a^2 + b^2 + c^2)^2 \\
 \Leftrightarrow &n[3(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2] + \sum ab(a^2 + b^2) + \\
 &+ abc(a + b + c) \geq (a^2 + b^2 + c^2)^2 \quad (**) \\
 &\text{We have } n \geq 1; 3(a^4 + b^4 + c^4) \geq (a^2 + b^2 + c^2)^2 \\
 \Rightarrow &(n - 1)[3(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2] \geq 0 \\
 \Leftrightarrow &n[3(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2] \geq 3(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2 \\
 &\text{So we have to prove that: } 3(a^4 + b^4 + c^4) - (a^2 + b^2 + c^2)^2 + \sum ab(a^2 + b^2) + \\
 &+ abc(a + b + c) \geq (a^2 + b^2 + c^2)^2 \quad (*) \\
 \Leftrightarrow &3(a^4 + b^4 + c^4) + \sum ab(a^2 + b^2) + abc(a + b + c) \geq 2(a^2 + b^2 + c^2)^2 \\
 \Leftrightarrow &a^4 + b^4 + c^4 + abc(a + b + c) + \sum ab(a^2 + b^2) \geq 4(a^2b^2 + b^2c^2 + a^2c^2)
 \end{aligned}$$

Use Schur inequality exponent two and we have:

$$a^4 + b^4 + c^4 + abc(a + b + c) \geq \sum ab(a^2 + b^2). \text{ So we need to prove:}$$

$$\sum ab(a^2 + b^2) \geq 2(\sum a^2b^2) \quad (1)$$

$$\text{Have: } a^2 + b^2 \geq 2ab \Leftrightarrow ab(a^2 + b^2) \geq 2a^2b^2. \text{ Similarly: } bc(b^2 + c^2) \geq 2b^2c^2$$

$$ac(a^2 + c^2) \geq 2a^2c^2 \Rightarrow \sum ab(a^2 + b^2) \geq 2(\sum a^2b^2)$$

$$\Rightarrow (1) \text{ true} \Rightarrow (*) \text{ true} \Rightarrow (**) \text{ true} \Rightarrow \text{Q.E.D.}$$

JP.074. If $a, b, c, n > 0; n(ab + bc + ca) + 2abc = n^3$ then:

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$$\frac{1}{a+b+2n} + \frac{1}{b+c+2n} + \frac{1}{c+a+2n} \leq \frac{1}{n}$$

Proposed by Marin Chirciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $a, b, c > 0$, de tal manera que $n(ab + bc + ca) + 2abc = n^3$. Probar que

$$\frac{1}{a+b+2n} + \frac{1}{b+c+2n} + \frac{1}{c+a+2n} \leq \frac{1}{n}$$

La desigualdad propuesta es equivalente

$$\begin{aligned} n((b+c+2n)(c+a+2n) + (c+a+2n)(a+b+2n) + (a+b+2n)(b+c+2n)) &\leq \\ &\leq (a+b+2n)(b+c+2n)(c+a+2n) \end{aligned}$$

Ahora bien:

$$E = (b+c+2n)(c+a+2n) + (c+a+2n)(a+b+2n) + (a+b+2n)(b+c+2n)$$

$$E = 12n^2 + 2n(2(b+c) + 2(c+a) + 2(a+b)) + (b+c)(c+a) + (c+a)(a+b) + (a+b)(b+c)$$

$$E = 12n^2 + 8n(a+b+c) + a^2 + b^2 + c^2 + 3(ab+bc+ca)$$

$$F = (a+b+2n)(b+c+2n)(c+a+2n)$$

$$F = 8n^3 + 4n^2 \cdot 2(a+b+c) + 2n((a+b)(b+c) + (b+c)(c+a) + (c+a)(a+b)) + (a+b)(b+c)(c+a)$$

$$F = 8n^3 + 8n^2(a+b+c) + 2n(a^2 + b^2 + c^2) + 6n(ab+bc+ca) + (a+b)(b+c)(c+a)$$

$$\Leftrightarrow n \cdot E \leq F \Leftrightarrow n \cdot E - F = 4n^3 - n(a^2 + b^2 + c^2) - 3n(ab+bc+ca) - (a+b)(b+c)(c+a) \leq 0 \Leftrightarrow 4n(ab+bc+ca) + 8abc \leq$$

$$\leq n(a^2 + b^2 + c^2 + 3ab + 3bc + 3ca) + (a+b)(b+c)(c+a)$$

$$\Leftrightarrow \text{Lo cual es cierto ya que } a^2 + b^2 + c^2 \geq ab + bc + ca \wedge (a+b)(b+c)(c+a) \geq 8abc$$

(Válido por $MA \geq MG$)

Solution 2 by Ravi Prakash-New Delhi-India

$$n(ab + bc + ca) + 2abc = n^3$$

$$\Leftrightarrow \left(\frac{a}{n}\right)\left(\frac{b}{n}\right) + \left(\frac{b}{n}\right)\left(\frac{c}{n}\right) + \left(\frac{c}{n}\right)\left(\frac{a}{n}\right) + 2\left(\frac{a}{n}\right)\left(\frac{b}{n}\right)\left(\frac{c}{n}\right) = 1 \quad (1)$$

$$\text{Let } \frac{a}{n} + 1 = x, \frac{b}{n} + 1 = y, \frac{c}{n} + 1 = z$$

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(1) now becomes

$$(x-1)(y-1) + (y-1)(z-1) + (z-1)(x-1) + 2(x-1)(y-1)(z-1) = 1$$

$$\Leftrightarrow xy - x - y + 1 + yz - y - z + 1 + zx - x - z + 1 + 2(xyz - xy - yz - zx + x + y + z - 1) = 1$$

$$\Leftrightarrow 2xyz = xy + yz + zx \Leftrightarrow \frac{1}{x} + \frac{1}{y} + \frac{1}{z} = 2. \text{ Let } \frac{1}{x} = \alpha, \frac{1}{y} = \beta, \frac{1}{z} = \gamma$$

So that $\alpha + \beta + \gamma = 2$. Now, to prove $\frac{1}{a+b+2n} + \frac{1}{b+c+2n} + \frac{1}{c+a+2n} \leq \frac{1}{n}$

$$\Leftrightarrow \frac{1}{\left(\frac{a}{n} + 1\right) + \left(\frac{b}{n} + 1\right)} + \frac{1}{\left(\frac{b}{n} + 1\right) + \left(\frac{c}{n} + 1\right)} + \frac{1}{\left(\frac{c}{n} + 1\right) + \left(\frac{a}{n} + 1\right)} \leq 1$$

$$\Leftrightarrow \frac{1}{x+y} + \frac{1}{y+z} + \frac{1}{z+x} \leq 1$$

$$\Leftrightarrow \frac{\alpha\beta}{\alpha+\beta} + \frac{\beta\gamma}{\beta+\gamma} + \frac{\gamma\alpha}{\alpha+\gamma} \leq 1 \quad (2)$$

LHS of (2)

$$\frac{1}{2} \left\{ \frac{2\alpha\beta}{\alpha+\beta} + \frac{2\beta\gamma}{\beta+\gamma} + \frac{2\gamma\alpha}{\gamma+\alpha} \right\} \leq \frac{1}{2} \left\{ \frac{\alpha+\beta}{2} + \frac{\beta+\gamma}{2} + \frac{\gamma+\alpha}{2} \right\} = \frac{\alpha+\beta+\gamma}{2} = \frac{2}{2} = 1$$

$\therefore (2)$ is true

JP.075. Let R and r be the circumradius and the inradius of a triangle ABC respectively.

Prove that

$$\csc A + \csc B + \csc C \geq 3\sqrt{3} \frac{R}{R+r}$$

Proposed by Martin Lukarevski – Stip – Macedonia

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo R y r circunradio e inradio de un triángulo ABC . Probar que

$$\csc A + \csc B + \csc C \geq \frac{3\sqrt{3}R}{R+r}$$

Tener en cuenta las siguientes identidades y desigualdades en un ΔABC

$$\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} = \frac{1}{2Rr}, R \geq 2r \quad (\text{Euler})$$

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La desigualdad propuesta es equivalente $2R \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq \frac{3\sqrt{3}R}{R+r} \Leftrightarrow \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \frac{3\sqrt{3}}{2(R+r)}$

$$\text{Luego } \frac{1}{a} + \frac{1}{b} + \frac{1}{c} \geq \sqrt{3 \left(\frac{1}{ab} + \frac{1}{bc} + \frac{1}{ca} \right)} = \sqrt{\frac{3}{2Rr}} \geq \frac{3\sqrt{3}}{2(R+r)}$$

Es necesario demostrar lo siguiente

$$\begin{aligned} \sqrt{\frac{3}{2Rr}} \geq \frac{3\sqrt{3}}{2(R+r)} &\Leftrightarrow \sqrt{2}(R+r) \geq 3\sqrt{Rr} \Leftrightarrow 2(R+r)^2 \geq 9Rr \Leftrightarrow \\ &\Leftrightarrow 2(R+r)^2 - 9Rr = (R-2r)(2R-r) \geq 0 \quad (R \geq 2r) \end{aligned}$$

Solution 2 by Soumava Chakraborty-Kolkata-India

$$\sum \csc A = 2R \left(\sum \frac{1}{a} \right) = \frac{2R(\sum ab)}{4Rrs} = \frac{\sum ab}{2rs}$$

$$\therefore \text{it suffices to prove: } \frac{(\sum ab)^2}{4r^2s^2} \geq \frac{27R^2}{(R+r)^2} \Leftrightarrow (s^2 + 4Rr + r^2)^2 (R+r)^2 \geq 108R^2r^2s^2$$

$$\Leftrightarrow (s^4 + (4Rr + r^2)^2 + 2s^2(4Rr + r^2))(R+r)^2 \geq 108R^2r^2s^2 \quad (1)$$

$$\text{Now, Trucht} \Rightarrow 4R + r \geq s\sqrt{3} \Rightarrow (4R + r)^2 \geq 3s^2$$

$$\Leftrightarrow r^2(4R + r)^2 \geq 3s^2r^2 \Leftrightarrow (4Rr + r^2)^2 \geq 3s^2r^2 \quad (2)$$

$$(2) \Rightarrow \text{LHS of (1)} \geq \{s^4 + 3s^2r^2 + 2s^2(4Rr + r^2)\}(R+r)^2 \stackrel{?}{\geq} 108R^2r^2 \quad (3)$$

$$\text{LHS of (3)} \stackrel{\text{Gerretsen}}{\geq} (16Rr - 5r^2 + 3r^2 + 8Rr + 2r^2)(R+r)^2$$

$$= 24Rr(R+r)^2 \stackrel{?}{\geq} 108R^2r^2 \Leftrightarrow 2(R+r)^2 \stackrel{?}{\geq} 9Rr \Leftrightarrow 2R^2 - 5Rr + 2r^2 \stackrel{?}{\geq} 0$$

$$\Leftrightarrow (R-2r)(2R-r) \stackrel{?}{\geq} 0 \rightarrow \text{true, } \therefore R \geq 2r \text{ (Euler) (Proved)}$$

Solution 3 by Marin Chirciu - Romania

$$\text{We have: } \sum \csc A = \sum \frac{1}{\sin A} = \sum \frac{2R}{a} = 2R \sum \frac{1}{a} = 2R \cdot \frac{\sum bc}{abc} = 2R \cdot \frac{p^2+r^2+4Rr}{4pRr} = \frac{p^2+r^2+4Rr}{2rp}$$

The inequality can be written:

$$\frac{p^2 + r^2 + 4Rr}{2rp} \geq 3\sqrt{3} \cdot \frac{R}{R+r} \Leftrightarrow \left(\frac{p^2 + r^2 + 4Rr}{2rp} \right)^2 \geq \frac{27R^2}{(R+r)^2} \Leftrightarrow$$

$$\Leftrightarrow (R+r)^2(p^2 + r^2 + 4Rr)^2 \geq 108R^2r^2p^2 \Leftrightarrow$$

$$\Leftrightarrow p^4(R+r)^2 + 2p^2[(R+r)^2(r^2 + 4Rr) - 54R^2r^2] + r^2(4R+r)^2(R+r)^2 \geq 0 \Leftrightarrow$$

$$\Leftrightarrow p^2[p^2(R+r)^2 + 2(R+r)^2(r^2 + 4Rr) - 54R^2r^2] + r^2(4R+r)^2(R+r)^2 \geq 0$$

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We distinguish the following cases:

1) If $p^2(R+r)^2 + 2(R+r)^2(r^2 + 4Rr) - 54R^2r^2 \geq 0$, the inequality is obvious.

2) If $p^2(R+r)^2 + 2(R+r)^2(r^2 + 4Rr) - 54R^2r^2 < 0$, we rewrite the inequality:

$$p^2[54R^2r^2 - 2(R+r)^2(r^2 + 4Rr) - p^2(R+r)^2] \leq r^2(4Rr+r)^2(R+r)^2$$

which follows from Gerretsen's inequality: $16Rr - 5r^2 \leq p^2 \leq 4R^2 + 4Rr + 3r^2$

It remains to prove that:

$$(4R^2 + 4Rr + 3r^2)[54R^2r^2 - 2(R+r)^2(r^2 + 4Rr) - (16Rr - 5r^2)(R+r)^2] \leq r^2(4R+r)^2(R+r)^2$$

$$\Leftrightarrow 24R^5 - 35R^4r - 17R^3r^2 - 24R^2r^3 + 13Rr^3 - 2r^5 \leq 0 \Leftrightarrow$$

$\Leftrightarrow (R-2r)(24R^4 + 13R^3r + 9R^2r^2 - 6Rr^3 + r^4) \geq 0$, obviously from Euler's inequality $R \geq 2r$. The equality holds if and only if the triangle is equilateral.

SP.061. Let x_1, x_2, \dots, x_n be non-negative real numbers satisfying

$$\frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \dots + \frac{nx_n}{1+x_n} = 1$$

Find the maximum possible value of

$$P = x_1 x_1^2 \dots x_n^n.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by proposer

From the given condition and by the AM-GM inequality, we obtain

$$\begin{aligned} \frac{1}{1+x_1} &= \frac{2x_1}{1+x_2} + \frac{3x_3}{1+x_3} + \dots + \frac{nx_n}{1+x_n} \\ &\geq (2+3+\dots+n)^{2+3+\dots+n} \sqrt{\left(\frac{x_2}{1+x_2}\right)^2 \left(\frac{x_3}{1+x_3}\right)^3 \dots \left(\frac{x_n}{1+x_n}\right)^n} \\ \frac{1}{1+x_2} &= \frac{x_1}{1+x_1} + \frac{x_2}{1+x_2} + \frac{3x_3}{1+x_3} + \dots + \frac{nx_n}{1+x_n} \\ &\geq (2+3+\dots+n)^{2+3+\dots+n} \sqrt{\left(\frac{x_1}{1+x_1}\right) \left(\frac{x_2}{1+x_2}\right) \left(\frac{x_3}{1+x_3}\right)^3 \dots \left(\frac{x_n}{1+x_n}\right)^n} \end{aligned}$$

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$$\begin{aligned} \frac{1}{1+x_3} &= \frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \frac{2x_3}{1+x_3} + \dots + \frac{nx_n}{1+x_n} \\ &\geq (2+3+\dots+n)^{2+3+\dots+n} \sqrt{\left(\frac{x_1}{1+x_1}\right) \left(\frac{x_2}{1+x_2}\right)^2 \left(\frac{x_3}{1+x_3}\right)^2 \dots \left(\frac{x_n}{1+x_n}\right)^n} \\ &\dots \dots \dots \\ \frac{1}{1+x_n} &= \frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \dots + \frac{(n-1)x_{n-1}}{1+x_{n-1}} + \frac{(n-1)x_n}{1+x_n} \\ &\geq (2+3+\dots+n)^{2+3+\dots+n} \sqrt{\left(\frac{x_1}{1+x_1}\right) \left(\frac{x_2}{1+x_2}\right)^2 \dots \left(\frac{x_{n-1}}{1+x_{n-1}}\right)^{n-1} \left(\frac{x_n}{1+x_n}\right)^{n-1}} \end{aligned}$$

From these relations above, we infer that

$$\begin{aligned} \frac{1}{1+x_1} \cdot \frac{1}{(1+x_2)^2} \cdot \frac{1}{(1+x_3)^3} \dots \frac{1}{(1+x_n)^n} &\geq \\ (2+3+\dots+n)^{1+2+\dots+n} \left(\frac{x_1}{1+x_1}\right) \left(\frac{x_2}{1+x_2}\right)^2 \left(\frac{x_3}{1+x_3}\right)^3 \dots \left(\frac{x_n}{1+x_n}\right)^n & \end{aligned}$$

Which implies that $x_1 x_2^2 \dots x_n^n \leq \frac{1}{(2+3+\dots+n)^{1+2+\dots+n}}$

The equality holds if and only if: $x_1 = x_2 = \dots = x_n = \frac{1}{2+3+\dots+n}$

$$\text{Thus } \max P = \frac{1}{(2+3+\dots+n)^{1+2+\dots+n}}$$

SP.062. If $a, b, c \in \mathbb{C}$ then:

$$|a^3 + b^3 + c^3 - 3abc| \leq |a + b + c|(|a| + |b| + |c|)^2$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\begin{aligned} |a^3 + b^3 + c^3 - 3abc| &= |(a+b+c)(a+b\varepsilon+c\varepsilon^2)(a+b\varepsilon+c\varepsilon)| \leq \\ &\leq |a+b+c|(|a|+|b\varepsilon|+|c\varepsilon^2|)(|a|+|b\varepsilon|+|c\varepsilon|) = \\ &= |a+b+c|(|a|+|b|+|c|)(|a|+|b|+|c|) = |a+b+c|(|a|+|b|+|c|)^2 \text{ when:} \\ &\varepsilon = \cos \frac{2\pi}{3} + i \sin \frac{2\pi}{3} \end{aligned}$$

SP.063. Prove that for any triangle ABC ,

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$$\begin{aligned} & \cos \frac{A}{2} \cos \frac{B}{2} + \cos \frac{B}{2} \cos \frac{C}{2} + \cos \frac{C}{2} \cos \frac{A}{2} \geq \\ & \geq \sqrt{3} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \frac{1}{2} \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right). \end{aligned}$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

$$\begin{aligned} & \cos \frac{A}{2} \cos \frac{B}{2} + \cos \frac{B}{2} \cos \frac{C}{2} + \cos \frac{C}{2} \cos \frac{A}{2} \geq \\ & \geq \sqrt{3} \cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2} + \frac{1}{2} \left(\cos^2 \frac{A}{2} + \cos^2 \frac{B}{2} + \cos^2 \frac{C}{2} \right) \end{aligned}$$

$$\text{En un } \Delta ABC \rightarrow \cos \frac{A}{2}, \cos \frac{B}{2}, \cos \frac{C}{2} > 0$$

Dividiendo (\div) $\cos \frac{A}{2} \cos \frac{B}{2} \cos \frac{C}{2}$ a la desigualdad propuesta

$$\Leftrightarrow \sec \frac{C}{2} + \sec \frac{A}{2} + \sec \frac{B}{2} \geq \sqrt{3} + \frac{1}{2} \left(\frac{\cos \frac{A}{2}}{\cos \frac{B}{2} \cos \frac{C}{2}} + \frac{\cos \frac{B}{2}}{\cos \frac{C}{2} \cos \frac{A}{2}} + \frac{\cos \frac{C}{2}}{\cos \frac{A}{2} \cos \frac{B}{2}} \right)$$

$$\Leftrightarrow \sec \frac{C}{2} + \sec \frac{A}{2} + \sec \frac{B}{2} \geq \sqrt{3} + \frac{1}{2} \left(\frac{\sin(\frac{B+C}{2})}{\cos \frac{B}{2} \cos \frac{C}{2}} + \frac{\sin(\frac{C+A}{2})}{\cos \frac{C}{2} \cos \frac{A}{2}} + \frac{\sin(\frac{A+B}{2})}{\cos \frac{A}{2} \cos \frac{B}{2}} \right)$$

$$\Leftrightarrow \sec \frac{C}{2} + \sec \frac{A}{2} + \sec \frac{B}{2} \geq \sqrt{3} + \frac{1}{2} \left(\left(\tan \frac{B}{2} + \tan \frac{C}{2} \right) + \left(\tan \frac{C}{2} + \tan \frac{A}{2} \right) + \left(\tan \frac{A}{2} + \tan \frac{B}{2} \right) \right)$$

$$\Leftrightarrow \sec \frac{A}{2} - \tan \frac{A}{2} + \sec \frac{B}{2} - \tan \frac{B}{2} + \sec \frac{C}{2} - \tan \frac{C}{2} \geq \sqrt{3}$$

Calculamos la primera y segunda derivada

$$\text{Sea } f(x) = \sec x - \tan x, x \in \left(0, \frac{\pi}{2} \right), f'(x) = \sec x \tan x - \sec^2 x,$$

$$f''(x) = \sec x (\sec x - \tan x)^2 > 0$$

Como $f''(x) > 0$, entonces $f(x)$ es estrictamente convexo en $\left(0, \frac{\pi}{2} \right)$

Dado que $\frac{A}{2}, \frac{B}{2}, \frac{C}{2} \in \left(0, \frac{\pi}{2} \right)$ de tal manera que $A + B + C = \pi$.

Aplicamos la desigualdad de Jensen

$$\sec \frac{A}{2} - \tan \frac{A}{2} + \sec \frac{B}{2} - \tan \frac{B}{2} + \sec \frac{C}{2} - \tan \frac{C}{2} = f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \geq$$

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$$\geq 3f\left(\frac{\frac{A}{2} + \frac{B}{2} + \frac{C}{2}}{3}\right) = 3f\left(\frac{\pi}{6}\right)$$

$$\Leftrightarrow f\left(\frac{A}{2}\right) + f\left(\frac{B}{2}\right) + f\left(\frac{C}{2}\right) \geq 3f\left(\frac{\pi}{6}\right) = 3\left(\sec\frac{\pi}{6} - \tan\frac{\pi}{6}\right) = 3\left(\frac{2}{\sqrt{3}} - \frac{1}{\sqrt{3}}\right) = \sqrt{3}$$

$$\Rightarrow \sec\frac{A}{2} - \tan\frac{A}{2} + \sec\frac{B}{2} - \tan\frac{B}{2} + \sec\frac{C}{2} - \tan\frac{C}{2} \geq \sqrt{3}$$

(LQOD)

SP.064. If $x, y, z \in (0, 1)$ then:

$$\left(\frac{yz}{1-x^2}\right)^{2n} + \left(\frac{zx}{1-y^2}\right)^{2n} + \left(\frac{xy}{1-z^2}\right)^{2n} \geq \frac{3^{3n+1}}{4^n} (xyz)^{2n}$$

for all $n \in \mathbb{N}$.

Proposed by Mihály Bencze – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\text{Let } f(x) = x - x^3 \text{ for all } x \in (0, 1), f'(x) = 1 - 3x^2, f''(x) = -6x$$

$$\because f'(x_0) = 0 \text{ where } x_0 \in (0, 1) \Rightarrow x_0 = \pm \frac{1}{\sqrt{3}} \text{ choosing } x_0 = \frac{1}{\sqrt{3}}$$

$$\therefore f''\left(\frac{1}{\sqrt{3}}\right) < 0 \text{ hence } f \text{ attains maximum at } x = \frac{1}{\sqrt{3}} \Rightarrow f(x) \leq f\left(\frac{1}{\sqrt{3}}\right)$$

$$\therefore \sum_{\text{cyc}} \left(\frac{yz}{1-x^2}\right)^{2n} = (xyz)^{2n} \sum_{\text{cyc}} \frac{1}{(x-x^3)^{2n}} \geq (xyz)^{2n} \frac{3}{\left(\frac{1}{\sqrt{3}} - \frac{1}{3\sqrt{3}}\right)^{2n}}$$

$$= \frac{3^{3n+1}}{4^n} (xyz)^{2n} \text{ (proved)}$$

SP.065. If $a, b, c > 0$ and $n \in \mathbb{N}^*$ then:

$$2^n(a^n + b^n + c^n) \geq (a+b)^{n-1}(a+c) + (b+c)^{n-1}(b+a) + (c+a)^{n-1}(c+b)$$

Proposed by Mihály Bencze – Romania

Solution by Sanong Huayrerai-Nakon Pathom-Thailand

$$\text{Prove that } x^n + y^n + z^n \geq x^{n-1}z + z^{n-1}y + y^{n-1}x; x, y, z > 0, n \in \mathbb{N}$$

$$\text{Since } x^n + x^n + x^n + \dots + x^n (n-1) \text{ term} + 2^n \geq nx^{n-1}z$$

$$z^n + z^n + z^n + \dots + z^n (n-1) \text{ term} + y^n \geq nz^{n-1}y$$

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$$y^n + y^n + y^n + \dots + y^n (n-1) \text{ term} + x^n \geq ny^{n-1}x$$

$$\text{Hence } n(x^n + y^n + z^n) \geq n(x^{n-1}z + z^{n-1}y + y^{n-1}x)$$

$$x^n + y^n + z^n \geq x^{n-1} + z^{n-1}y + y^{n-1}x$$

Solution

For $a, b, c > 0$ and $n \in \mathbb{N}$, we have

$$\frac{a^n + b^n}{2} \geq \left(\frac{a+b}{2}\right)^n, \frac{b^n + c^n}{2} \geq \left(\frac{b+c}{2}\right)^n, \frac{c^n + a^n}{2} \geq \left(\frac{c+a}{2}\right)^n$$

$$\text{Hence } a^n + b^n + c^n \geq \left(\frac{a+b}{2}\right)^n + \left(\frac{b+c}{2}\right)^n + \left(\frac{c+a}{2}\right)^n \Rightarrow$$

$$\Rightarrow 2^n(a^n + b^n + c^n) \geq (a+b)^n + (b+c)^n + (c+a)^n \geq$$

$$\geq (a+b)^{n-1}(c+a) + (b+c)^{n-1}(b+a) + (c+a)^{n-1}(c+b)$$

Therefore it is to be true.

SP.066. Let $t \in \mathbb{R}^+$ and $f: \mathbb{R}_+^* \rightarrow \mathbb{R}_+^*$ such that

$$\lim_{x \rightarrow \infty} \frac{f(x+1)}{x^t f(x)} \in \mathbb{R}_+ \text{ and } \lim_{x \rightarrow \infty} (f(x))^{\frac{1}{x}} \cdot \frac{1}{x^t} \in \mathbb{R}_+^*.$$

Prove that

$$\lim_{x \rightarrow \infty} (f(x))^{\frac{1}{x}} \cdot \frac{1}{x^t} = \frac{1}{e^t} \cdot \lim_{x \rightarrow \infty} \frac{f(x+1)}{x^t f(x)}$$

Proposed by D.M. Băţineţu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(f(x))^{\frac{1}{x}}}{x^t} &= \lim_{\substack{x \rightarrow \infty \\ n \in \mathbb{N}}} \frac{\sqrt[n]{f(n)}}{n^t} = \lim_{x \rightarrow \infty} \sqrt[n]{\frac{f(n)}{n^{nt}}} \stackrel{\text{CESARO-STOLZ}}{\cong} \lim_{x \rightarrow \infty} \frac{f(n+1)}{(n+1)^{(n+1)t}} \cdot \frac{n^{nt}}{f(n)} \\ &= \lim_{x \rightarrow \infty} \left(\frac{f(n+1)}{n^t f(n)} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^{nt}} \cdot \left(\frac{n}{n+1}\right)^t \right) = \frac{1}{e^t} \cdot \lim_{x \rightarrow \infty} \frac{f(n+1)}{n^t f(n)} = \frac{1}{e^t} \cdot \lim_{x \rightarrow \infty} \frac{f(x+1)}{x^t f(x)} \end{aligned}$$

SP.067. If $x, y, z > 0$ then:

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$$\frac{1}{(x^2 + yz)(3x^2 + 2y^2 + z^2)} + \frac{1}{(y^2 + zx)(3y^2 + 2z^2 + x^2)} + \frac{1}{(z^2 + xy)(3z^2 + 2x^2 + y^2)} \leq \frac{x^2 + y^2 + z^2 + xy + yz + zx}{24x^2y^2z^2}$$

Proposed by Mihály Bencze – Romania

Solution by proposer

For all $x, y, z > 0$ we have: $\frac{x}{3x^2+2y^2+z^2} \geq \frac{1}{18} \left(\frac{2}{y} + \frac{1}{z} \right) \Leftrightarrow$

$$\Leftrightarrow \frac{3x^2y + 6x^2z + 2y^3 + 2z^3 + 4y^2z + yz^2}{18} \geq \frac{18\sqrt{(x^2y)^3(x^2z)(y^3)^2(z^3)^2(y^2z)^4yz^2}}{18} =$$

$$= xyz \text{ and } \frac{x}{x^2+yz} \leq \frac{1}{4} \left(\frac{1}{y} + \frac{1}{z} \right) \Leftrightarrow y(x-z)^2 + z(x-y)^2 \geq 0 \Rightarrow$$

$$\frac{x^2}{(x^2 + yz)(3x^2 + 2y^2 + z^2)} \leq \frac{1}{72} \left(\frac{2}{y} + \frac{1}{z} \right) \left(\frac{1}{y} + \frac{1}{z} \right) = \frac{(y+z)(y+2z)}{72y^2z^2} \Rightarrow$$

$$\sum_{\text{cyclic}} \frac{1}{(x^2 + yz)(3x^2 + 2y^2 + z^2)} \leq \sum_{\text{cyclic}} \frac{(y+z)(y+2z)}{72x^2y^2z^2} =$$

$$= \frac{3 \sum x^2 + 3 \sum xy}{72x^2y^2z^2} = \frac{\sum x^2 + \sum xy}{24x^2y^2z^2}$$

SP.068. Find:

$$\Omega = \lim_{n \rightarrow \infty} \left({}^{mn+m}\sqrt{(2n+1)!!} - {}^{mn}\sqrt{(2n-1)!!} \right) \cdot n^{\frac{m-1}{m}}$$

Proposed by D. M. Băţineţu – Giurgiu; Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\Omega = \lim_{n \rightarrow \infty} \left({}^{m(n+1)}\sqrt{(2n+1)!!} - {}^{mn}\sqrt{(2n-1)!!} \right) n^{\frac{m-1}{m}}$$

$$\lim_{n \rightarrow \infty} \left(\frac{{}^{mn}\sqrt{(2n-1)!!}}{m\sqrt{(n-1)}} \cdot m \sqrt{\left(1 - \frac{1}{n}\right) \frac{u_{m-1}}{\ln u_n}} \cdot \ln u_n^n \right) \text{ where } u_n = \frac{{}^{m(n+1)}\sqrt{(2n+1)!!}}{{}^{mn}\sqrt{(2n-1)!!}} \text{ for all } n \in \mathbb{N}$$

$$\text{Now, } \lim_{n \rightarrow \infty} m \sqrt[n]{\frac{{}^{m(n+1)}\sqrt{(2n+1)!!}}{(n-1)^m}} \stackrel{D'ALEMBERT}{=} \lim_{n \rightarrow \infty} m \sqrt{\frac{(2n+1)!!}{n^{n+1}} \cdot \frac{(n-1)^n}{(2n-1)!!}}$$

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$$= \lim_{n \rightarrow \infty}^m \sqrt{\frac{\frac{(2n+1)!}{2^{m n!}}}{\frac{(2n-1)!}{2^{m-1}(n-1)!}}} \cdot \left(1 - \frac{1}{n}\right)^n \cdot \frac{1}{n} = \lim_{n \rightarrow \infty}^m \sqrt{\frac{2n(2n-1)}{2n(n-1)}} \cdot \left(1 - \frac{1}{n}\right)^n = \sqrt{\frac{2}{e}}$$

$$\lim_{n \rightarrow \infty} u_m = \lim_{n \rightarrow \infty} \frac{\frac{m(n+1)\sqrt{(2n+1)!!}}{m\sqrt{n}}}{\frac{m\sqrt{(2n-1)!!}}{m\sqrt{n-1}}} \cdot \sqrt{\frac{n}{n-1}} = 1. \text{ Hence } \lim_{n \rightarrow \infty} \frac{u_m - 1}{\ln u_m} = 1$$

$$\text{Now, } \lim_{n \rightarrow \infty} u_n^n = \lim_{n \rightarrow \infty} \frac{m\sqrt{(2n+1)!!}}{\sqrt{(2n-1)!!}} \cdot \frac{1}{m(n+1)\sqrt{(2n+1)!!}}$$

$$= \lim_{n \rightarrow \infty}^m \sqrt{\frac{\frac{(2n+1)!!}{n^n}}{\frac{(2n-1)!!}{(n-1)^{n-1}}} \cdot \frac{1}{\left(1 - \frac{1}{n}\right)^m}} \cdot \sqrt{1 - \frac{1}{n}} \cdot \frac{1}{\frac{m(n+1)\sqrt{(2n+1)!!}}{m\sqrt{n}}} = \sqrt[2]{e}$$

$$\therefore \Omega = \sqrt{\frac{2}{e}} \cdot 1 \cdot \ln \sqrt[2]{e} = \frac{1}{m} \sqrt{\frac{2}{e}} \quad (\text{proved})$$

Solution 2 by Shivam Sharma-New Delhi-India

As we know, $(2n+1)!! = \frac{(2n+1)!}{2^n n!}$ & $(2n-1)!! = \frac{(2n)!}{2^n n!}$. Using this, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\left(\frac{(2n+1)!}{2^{2n} n!} \right)^{\frac{1}{mn+m}} - \left(\frac{(2n)!}{2^{2n} n!} \right)^{\frac{1}{mn}} \right] n^{\frac{m-1}{m}}. \text{ As we know, the Stirling's formula,}$$

$$n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}. \text{ Using this, we get,}$$

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\frac{\left(\left(\frac{2n+1}{e} \right)^{2n+1} \sqrt{2\pi(2n+1)} \right)^{\frac{1}{mn+m}}}{\left(2^n \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \right)^{\frac{1}{mn+m}}} - \frac{\left(2 \left(\frac{2n}{e} \right)^{2n} \sqrt{\pi n} \right)^{\frac{1}{mn}}}{\left(2^n \left(\frac{n}{e} \right)^n \sqrt{2\pi n} \right)^{\frac{1}{mn}}} \right] \cdot n^{\frac{m-1}{m}}$$

As we can see many terms are cancelling now, applying ratio test,

$\frac{2n+2}{2n+1} \rightarrow 1$, as $n \rightarrow \infty$ & $\frac{2n+1}{2n} \rightarrow 1$, as $n \rightarrow \infty$. so, our limit is now, equal to,

$$\Omega = \frac{1}{m} \left(\frac{2}{e} \right)^{\frac{1}{m}}$$

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SP.069. If $x, y, z > 0$ and $b \geq a > 0$ then:

$$\ln \frac{b^2 + xy}{a^2 + xy} + \ln \frac{b^2 + yz}{a^2 + yz} + \ln \frac{b^2 + zx}{a^2 + zx} \leq (b - a) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

For all $y, z, t > 0$ we have: $\frac{t}{t^2 + yz} \leq \frac{1}{y} \left(\frac{1}{y} + \frac{1}{z} \right) \Leftrightarrow y(t - z)^2 + z(y - t)^2 \geq 0$

therefore $\int_a^b \frac{t dt}{t^2 + yz} \leq \int_a^b \frac{1}{4} \left(\frac{1}{y} + \frac{1}{z} \right) dt \Rightarrow \frac{1}{2} \ln \frac{b^2 + yz}{a^2 + yz} \leq \frac{1}{4} \left(\frac{1}{y} + \frac{1}{z} \right) (b - a) \Rightarrow$

$$\sum_{cyclic} \ln \frac{b^2 + yz}{a^2 + yz} \leq \frac{1}{2} \sum_{cyclic} \left(\frac{1}{y} + \frac{1}{z} \right) (b - a) = (b - a) \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$$

SP.070. Prove that if $a, b, c \in \mathbb{R}$ then:

$$(2 - a - b - c + abc)^2 \leq (a^2 + 2)(b^2 + 2)(c^2 + 2)$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Nguyen Phuc Tang-Hanoi-Vietnam

We have

$$(a^2 + 2)(b^2 + 2) = (a + b + 1)^2 + (a - 1)^2 + (b - 1)^2 + (ab - 1)^2 \geq$$

$$\geq (1 - a^2) + (1 - b)^2 + (ab - 1)^2. \text{ By Cauchy - Schwarz}$$

$$(a^2 + 2)(b^2 + 2)(c^2 + 2) \geq [(1 - a)^2 + (1 - b)^2 + (ab - 1)^2](1 + 1 + c^2) \geq$$

$$\geq (2 - a - b - c + abc)^2$$

Equality holds if $\begin{cases} a + b + 1 = 0 \\ 1 - a = 1 - b = \frac{ab - 1}{c} \Leftrightarrow a = b = c = -\frac{1}{2} \end{cases}$

Solution 2 by Sladjan Stankovik-Macedonia

$$a, b, c \in \mathbb{R}$$

$$(2 - a - b - c + abc)^2 \leq (a^2 + 2)(b^2 + 2)(c^2 + 2)$$

$$(2 - p + c)^2 \leq c^2 + 2(q^2 - 2pc) + 4(p^2 - 2q) + p$$

$$4 + p^2 + c^2 - 4p + 4c - 2pc \leq c^2 + 2q^2 - 4pc + 4p^2 - 8q + 8$$

$$c \cdot (4 + 2p) + (-4 - 3p^2 - 4p - 2q^2 + 8q) \leq 0$$

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If we fix p and $q \Rightarrow c$ has min or MAX \Leftrightarrow two variables are equal.

$$\text{Let } b = c \Rightarrow \begin{cases} p = a + 2b \\ q = ab + b^2 \\ c = ab^2 \end{cases}$$

$$ab^2 \cdot (4 + 2a + 4b) \leq 4 + 3(a + 2b)^2 + 4(a + 2b) + 2(2ab + b^2)^2 - 8(2ab + b^2)$$

$$\Leftrightarrow a^2(-6b^2 - 3) + a \cdot (-4b^3 + 4b^2 + 4b - 4) - (2b^4 + 4b^2 + 8b + 4) \leq 0$$

$$\begin{aligned} D_a &= (-4b^3 + 4b^2 + 4b - 4)^2 - 4(6b^2 + 3)(2b^4 + 4b^2 + 8b + 4) = \\ &= -8(4b^6 + 4b^5 + 17b^4 + 16b^3 + 20b^2 + 16b + 4) \leq 0 \end{aligned}$$

$$D_a \leq 0$$

$$k_a = -6b^2 - 3 < 0 \Leftrightarrow -8(2b + 1)^2(b^2 + 2)^2 \leq 0; b = -\frac{1}{2} = a = c$$

SP.071. Prove that if $a, b, c \in \mathbb{R}$ then:

$$(3abc - a^3 - b^3 - c^3)^2 \leq (a^2 + b^2 + c^2)^3$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$3abc - a^3 - b^3 - c^3 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \Rightarrow (3abc - a^3 - b^3 - c^3)^2 = \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} \begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix}$$

$$= \begin{vmatrix} x & y & y \\ y & x & y \\ y & y & x \end{vmatrix} = x^3 + 2y^3 - 3xy^2 \text{ where } x = a^2 + b^2 + c^2$$

$$\begin{aligned} y &= bc + ca + ab = x^3 - y^2(2x - 2y) - xy^2 \\ &= (a^2 + b^2 + c^2)^3 - (ab + bc + ca)^2(a^2 + b^2 + c^2) - \\ &-(ab + bc + ca)^2\{(a - b)^2 + (b - c)^2 + (c - a)^2\} \leq (a^2 + b^2 + c^2)^3 \end{aligned}$$

Solution 2 by Nguyen Phuc Tang-Hanoi-Vietnam

$$\text{Let } \begin{cases} x = a^2 + b^2 + c^2 \\ y = ab + bc + ca \end{cases} \Rightarrow x \geq y. \text{ We have}$$

$$(a^3 + b^3 + c^3 - 3abc)^2 = (a + b + c)(a^2 + b^2 + c^2 - ab - bc - ca)$$

The inequality is equivalent to $(x + 2y)(x - y)^2 \leq x^3$ (*)

Case $x = 0 \Rightarrow a = b = c = 0 \Rightarrow (*)$ is true

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Case $x > 0$

$$(*) \Leftrightarrow (1 + 2t)(1 - t)^2 \leq 1 \text{ (with } t = \frac{y}{x} \leq 1) \Leftrightarrow t^2(2t - 3) \leq 0 \text{ which is true}$$

Equality holds if and only if $t = 0 \Leftrightarrow a = b = c = 0$

SP.072. If $a, b, c > 0; n \in \mathbb{N}^*$ then:

$$\left(\frac{2na}{b + (2n-1)c}\right)^{\frac{2}{3}} + \left(\frac{2nb}{c + (2n-1)a}\right)^{\frac{2}{3}} + \left(\frac{2nc}{a + (2n-1)b}\right)^{\frac{2}{3}} \geq 3$$

Proposed by Marin Chirciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\begin{aligned} \sum_{cyc} \left(\frac{2na}{b + (2n-1)c}\right)^{\frac{2}{3}} &= \sum_{cyc} \frac{2na}{\sqrt[3]{2na(b + (2n-1)c)^2}} \stackrel{AM \geq GM}{\geq} \sum_{cyc} \frac{6na}{2na + 2b + 2(2n-1)c} \\ &= 3n \sum_{cyc} \frac{a}{na + b + (2n-1)c} = 3n \sum_{cyc} \frac{a^2}{na^2 + ab + (2n-1)ca} \geq \\ &\geq 3n \frac{(a+b+c)^2}{n \sum_{cyc} a^2 + \sum_{cyc} ab + (2n-1) \sum_{cyc} ab} = 3 \text{ (proved)} \end{aligned}$$

SP.073. If $x, y, z > 0$ then:

$$\log\left(1 + \frac{1}{x}\right) + \log\left(1 + \frac{1}{y}\right) + \log\left(1 + \frac{1}{z}\right) \geq 3 \log\left(1 + \frac{3}{x+y+z}\right)$$

Proposed by Marin Chirciu – Romania

Solution 1 by Tran Hong-Vietnam

Let $f(t) = \log\left(1 + \frac{1}{t}\right)$ for $t > 0$

$$\Rightarrow f'(t) = \frac{\left(1 + \frac{1}{t}\right)'}{\left(1 + \frac{1}{t}\right) \ln 10} = -\frac{1}{t(t+1) \ln 10} \Rightarrow f''(t) = \frac{1}{\ln 10} \cdot \frac{2t+1}{[t(t+1)]^2} > 0 \forall t > 0$$

\Rightarrow using Jensen's inequality we have

$$LHS = f(x) + f(y) + f(z) \geq 3f\left(\frac{x+y+z}{3}\right) = 3 \log\left(1 + \frac{3}{x+y+z}\right)$$

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Proved. Equality $y \Leftrightarrow x = y = z$.

Solution 2 by Sanong Huayrerai-Nakon Pathom-Thailand

For $x, y, z > 0$, we have

$$\begin{aligned} \log\left(1 + \frac{1}{x}\right) + \log\left(1 + \frac{1}{y}\right) + \log\left(1 + \frac{1}{z}\right) &= \log\left(1 + \frac{1}{x}\right)\left(1 + \frac{1}{y}\right)\left(1 + \frac{1}{z}\right) \\ &\geq \log\left(1 + \frac{1}{\sqrt[3]{xyz}}\right)^3 = 3\log\left(1 + \frac{1}{\sqrt[3]{xyz}}\right) \geq 3\log\left(1 + \frac{3}{x+y+z}\right) \end{aligned}$$

Therefore it is to be true.

SP.074. Let x, y, z be positive real numbers such that: $x + y + z = 3$. Find the minimum value of:

$$P = \frac{x^4}{y^4 \cdot \sqrt[3]{4z(x^5 + 1)}} + \frac{y^4}{z^4 \cdot \sqrt[3]{4x(y^5 + 1)}} + \frac{z^4}{x^4 \cdot \sqrt[3]{4y(z^5 + 1)}}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by proposer

** By AM-GM inequality we have:*

$$\begin{aligned} \frac{x^4}{y^4 \cdot \sqrt[3]{4z(x^5 + 1)}} &= \frac{x^4}{y^4 \cdot \sqrt[3]{4z(x+1)(x^4 - x^3 + x^2 - x + 1)}} = \\ &= \frac{x^4}{y^4 \cdot \sqrt[3]{2(zx + z)(2x^4 - 2x^3 + 2x^2 - 2x + 2)}} \geq \\ &\geq \frac{x^4}{y^4 \left(\frac{2 + zx + z + 2x^4 - 2x^3 + 2x^2 - 2x + 2}{3}\right)} = \\ &= \frac{3x^4}{y^4(2x^4 - 2x^3 + 2x^2 + zx - 2x + z + 4)} \Rightarrow \\ \Rightarrow \frac{x^4}{y^4 \cdot \sqrt[3]{4z(x^5 + 1)}} &\geq \frac{3x^4}{y^4(2x^4 - 2x^3 + 2x^2 + zx - 2x + z + 4)} \\ &+ \text{ Similar: } \frac{y^4}{z^4 \cdot \sqrt[3]{4x(y^5 + 1)}} \geq \frac{3y^4}{z^4(2y^4 - 2y^3 + 2y^2 + xy - 2y + x + 4)} \end{aligned}$$

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$$\frac{z^4}{x^4 \cdot \sqrt[3]{4y(z^5+1)}} \geq \frac{3z^4}{x^4(2z^4 - 2z^3 + 2z^2 + yz - 2z + y + 4)}$$

- Hence: $\Rightarrow \frac{P}{3} = \frac{1}{3} \left(\frac{x^4}{y^4 \cdot \sqrt[3]{4z(x^5+1)}} + \frac{y^4}{z^4 \cdot \sqrt[3]{4x(y^5+1)}} + \frac{z^4}{x^4 \cdot \sqrt[3]{4y(z^5+1)}} \right) \geq$

$$\geq \frac{x^4}{y^4(2x^4 - 2x^3 + 2x^2 + zx - 2x + z + 4)} + \frac{y^4}{z^4(2y^4 - 2y^3 + 2y^2 + xy - 2y + x + 4)} + \frac{z^4}{x^4(2z^4 - 2z^3 + 2z^2 + yz - 2z + y + 4)} \quad (1)$$

- Other, by Cauchy Schwarz inequality we have:

$$\frac{x^4}{y^4(2x^4 - 2x^3 + 2x^2 + zx - 2x + z + 4)} + \frac{y^4}{z^4(2y^4 - 2y^3 + xy - 2y + x + 4)} +$$

$$+ \frac{z^4}{x^4(2z^4 - 2z^3 + 2z^2 + yz - 2z + y + 4)} =$$

$$= \frac{\left(\frac{x^2}{y^2}\right)^2}{2x^4 - 2x^3 + 2x^2 + zx - 2x + z + 4} + \frac{\left(\frac{y^2}{z^2}\right)^2}{2y^4 - 2y^3 + 2y^2 + xy - 2y + x + 4} +$$

$$+ \frac{\left(\frac{z^2}{x^2}\right)^2}{2z^4 - 2z^3 + 2z^2 + yz - 2z + y + 4} \geq$$

$$\geq \frac{\left(\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}\right)^2}{(2x^4 - 2x^3 + 2x^2 + zx - 2x + z + 4) + (2y^4 - 2y^3 + 2y^2 + xy - 2y + x + 4) + (2z^4 - 2z^3 + 2z^2 + yz - 2z + y + 4)} \quad (2)$$

- Let (1), (2):

$$\Rightarrow \frac{P}{3} \geq \frac{\left(\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}\right)^2}{(2x^4 - 2x^3 + 2x^2 + zx - 2x + z + 4) + (2y^4 - 2y^3 + 2y^2 + xy - 2y + x + 4) + (2z^4 - 2z^3 + 2z^2 + yz - 2z + y + 4)} \quad (3)$$

- By AM-GM inequality and $x + y + z = 3$. We have:

$$\frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} = \frac{\frac{x^2}{y^2} + \frac{x^2}{y^2} + \frac{y^2}{z^2}}{3} + \frac{\frac{y^2}{z^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2}}{3} + \frac{\frac{z^2}{x^2} + \frac{z^2}{x^2} + \frac{x^2}{y^2}}{3} \geq$$

$$\geq \frac{3 \cdot \sqrt[3]{\frac{x^2}{y^2} \cdot \frac{x^2}{y^2} \cdot \frac{y^2}{z^2}}}{3} + \frac{3 \cdot \sqrt[3]{\frac{y^2}{z^2} \cdot \frac{y^2}{z^2} \cdot \frac{z^2}{x^2}}}{3} + \frac{3 \cdot \sqrt[3]{\frac{z^2}{x^2} \cdot \frac{z^2}{x^2} \cdot \frac{x^2}{y^2}}}{3} = \sqrt[3]{\frac{x^4}{y^2 z^2}} + \sqrt[3]{\frac{y^4}{z^2 x^2}} + \sqrt[3]{\frac{z^4}{x^2 y^2}} = \frac{x^2 + y^2 + z^2}{\sqrt[3]{x^2 y^2 z^2}}$$

$$\Rightarrow \frac{x^2}{y^2} + \frac{y^2}{z^2} + \frac{z^2}{x^2} \geq \frac{x^2 + y^2 + z^2}{\sqrt[3]{x^2 y^2 z^2}} \geq \frac{x^2 + y^2 + z^2}{\left(\frac{x+y+z}{3}\right)^2} = \frac{x^2 + y^2 + z^2}{\left(\frac{3}{3}\right)^2} = x^2 + y^2 + z^2 \quad (4)$$

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- Let (3), (4): $\Rightarrow \frac{P}{3} \geq \frac{(x^2+y^2+z^2)^2}{2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12}$ (5)

We will prove:

$$\frac{(x^2+y^2+z^2)^2}{2(x^4+y^4+z^4)-2(x^3+y^3+z^3)+2(x^2+y^2+z^2)+(xy+yz+zx)-(x+y+z)+12} \geq \frac{1}{2} \quad (6)$$

$$\begin{aligned} &\Leftrightarrow 2(x^2 + y^2 + z^2)^2 \geq 2(x^4 + y^4 + z^4) - 2(x^3 + y^3 + z^3) + \\ &+ 2(x^2 + y^2 + z^2) + 2(x^2 + y^2 + z^2) + (xy + yz + zx) - (x + y + z) + 12 \\ &\Leftrightarrow 2(x^3 + y^3 + z^3) + 4(x^2y^2 + y^2z^2 + z^2x^2) \geq 2(x^2 + y^2 + z^2) + xy + yz + zx - 3 + 12 \\ &\quad (x + y + z = 3) \\ &\Leftrightarrow 2(x^3 + y^3 + z^3) + 4(x^2y^2 + y^2z^2 + z^2x^2) \geq 2(x^2 + y^2 + z^2) + xy + yz + zx + 9 \\ &\Leftrightarrow 18(x^3 + y^3 + z^3) + 36(x^2y^2 + y^2z^2 + z^2x^2) \geq 18(x^2 + y^2 + z^2) + 9(xy + yz + zx) + 81 \\ &\quad \Leftrightarrow 6(x + y + z)(x^3 + y^3 + z^3) + 36(x^2y^2 + y^2z^2 + z^2x^2) \geq \\ &\quad \geq 2(x + y + z)^2(x^2 + y^2 + z^2) + (x + y + z)^2(xy + yz + zx) + (x + y + z)^4 \\ &\text{(because } x + y + z = 3 \text{ then: } 18 = 6(x + y + z); 18 = 2(x + y + z)^2; 81 = (x + y + z)^4) \\ &\Leftrightarrow 6(x^4 + y^4 + z^4) + 6(xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)) + 36(x^2y^2 + y^2z^2 + z^2x^2) \geq \\ &\quad \geq 3(x + y + z)^2(x^2 + y^2 + z^2 + xy + yz + zx) \\ &\Leftrightarrow 2(x^4 + y^4 + z^4) + 2(xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)) + 12(x^2y^2 + y^2z^2 + z^2x^2) \geq \\ &\quad \geq (x^2 + y^2 + z^2 + 2xy + 2yz + 2zx)(x^2 + y^2 + z^2 + xy + yz + zx) \\ &\Leftrightarrow 2(x^4 + y^4 + z^4) + 2(xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)) + 12(x^2y^2 + y^2z^2 + z^2x^2) \geq \\ &\geq x^4 + y^4 + z^4 + 3(xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)) + 4(x^2y^2 + y^2z^2 + z^2x^2) + 7xyz(x + y + z) \\ &\Leftrightarrow x^4 + y^4 + z^4 + 8(x^2y^2 + y^2z^2 + z^2x^2) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) + 7xyz(x + y + z) \quad (7) \end{aligned}$$

- We have:

$$\begin{aligned} (x^2 + y^2 + z^2 - xy - yz - zx)^2 \geq 0 &\Leftrightarrow (x^2 + y^2 + z^2) + (xy + yz + zx)^2 \geq 2(x^2 + y^2 + z^2)(xy + yz + zx) \\ &\Leftrightarrow x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) \geq 2(xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2)) \quad (8) \end{aligned}$$

- By AM-GM inequality:

$$\begin{aligned} xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) &\geq xy \cdot 2xy + yz \cdot 2yz + zx \cdot 2zx = 2(x^2y^2 + y^2z^2 + z^2x^2) \quad (9) \\ \text{- Let (8), (9): } &\Rightarrow x^4 + y^4 + z^4 + 3(x^2y^2 + y^2z^2 + z^2x^2) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) + 2(x^2y^2 + y^2z^2 + z^2x^2) \\ &\Leftrightarrow x^4 + y^4 + z^4 + (x^2y^2 + y^2z^2 + z^2x^2) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) \quad (10) \end{aligned}$$

$$+ \text{ Other: } x^2y^2 + y^2z^2 + z^2x^2 = \frac{x^2(y^2+z^2)}{2} + \frac{y^2(z^2+x^2)}{2} + \frac{z^2(x^2+y^2)}{2} \geq \frac{x^2 \cdot 2yz}{2} + \frac{y^2 \cdot 2zx}{2} + \frac{z^2 \cdot 2xy}{2}$$

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$$\Rightarrow x^2y^2 + y^2z^2 + z^2x^2 \geq xyz(x + y + z) \Leftrightarrow xyz(x + y + z) \Leftrightarrow 7(x^2y^2 + y^2z^2 + z^2x^2) \geq 7xyz(x + y + z) \quad (11)$$

– Hence (10), (11):

$$\Rightarrow x^4 + y^4 + z^4 + 8(x^2y^2 + y^2z^2 + z^2x^2) \geq xy(x^2 + y^2) + yz(y^2 + z^2) + zx(z^2 + x^2) + 7xyz(x + y + z)$$

\Rightarrow Inequality (7) true \Rightarrow (6) true.

– Let (5), (6): $\Rightarrow \frac{P}{3} \geq \frac{1}{2} \Rightarrow P \geq \frac{3}{2} \Rightarrow P_{Min} = \frac{3}{2}$. Equality occurs if:

$$\begin{cases} x, y, z > 0; x + y + z = 3 \\ x = y = z \end{cases} \Leftrightarrow x = y = z = 1$$

SP.075. Let x, y, z be positive real numbers such that: $xyz = 1$. Find the minimum of expression:

$$P = 2(x + y + z) + \frac{x}{y^3 + z^3 + 1} + \frac{y}{z^3 + x^3 + 1} + \frac{z}{x^3 + y^3 + 1}$$

Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo x, y, z números R^+ de tal manera que $xyz = 1$. Hallar el mínimo valor

$$P = 2(x + y + z) + \frac{x}{y^3 + z^3 + 1} + \frac{y}{z^3 + x^3 + 1} + \frac{z}{x^3 + y^3 + 1}. \text{ Aplicando la desigualdad de Cauchy}$$

$$\begin{aligned} & \frac{x^2}{x(y^3 + z^3) + x} + \frac{y^2}{y(z^3 + x^3) + y} + \frac{z^2}{z(x^3 + y^3) + z} \geq \\ & \geq \frac{(\sum x)^2}{\sum xy(x^2 + y^2) + \sum x} = \frac{(\sum x)^2}{\sum xy(x^2 + y^2) + (\sum x)xyz} = \frac{(\sum x)^2}{(\sum x^2)(\sum xy)} \end{aligned}$$

Camo $x, y, z > 0$. Aplicando $MA \geq MG$

$$\frac{(\sum x)^2(\sum xy)}{(\sum x^2)(\sum xy)(\sum xy)} \geq \frac{3(\sum x)^2}{\left(\frac{\sum x^2 + \sum xy + \sum xy}{3}\right)^3} = \frac{81(\sum x)^2}{(\sum x)^6} = \frac{81}{(x + y + z)^4}$$

Luego

$$\begin{aligned} P &= 2(x + y + z) + \frac{x}{y^3 + z^3 + 1} + \frac{y}{z^3 + x^3 + 1} + \frac{z}{x^3 + y^3 + 1} \geq \\ &\geq 2(x + y + z) + \frac{81}{(x + y + z)^4} \end{aligned}$$

Nuevamente por $MA \geq MG$

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$$2(x + y + z) + \frac{81}{(x+y+z)^4} = \left(\frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{x+y+z}{3} + \frac{81}{(x+y+z)^4} \right) + \frac{2(x+y+z)}{3} \geq 5 + 2 = 7$$

Por transitividad $\rightarrow P = 2(x + y + z) + \frac{x}{y^3+z^3+1} + \frac{y}{z^3+x^2+1} + \frac{z}{x^3+y^3+1} \geq 7$

La igualdad se alcanza cuando $x = y = z = 1$

UP.061. Prove that in all triangle ABC with usual notations holds the following inequalities:

(a) $\frac{\tan^3 \frac{A}{2}}{m \cdot \tan \frac{B}{2} + n \cdot \tan \frac{C}{2}} + \frac{\tan^3 \frac{B}{2}}{m \cdot \tan \frac{C}{2} + n \cdot \tan \frac{A}{2}} + \frac{\tan^3 \frac{C}{2}}{m \cdot \tan \frac{A}{2} + n \cdot \tan \frac{B}{2}} \geq \frac{((4R+r)^2 - 2s^2)^2}{(m+n)s^4}$;

(b) $\frac{\tan \frac{A}{2}}{m+n \cdot \tan \frac{B}{2} \cdot \tan \frac{C}{2}} + \frac{\tan \frac{B}{2}}{m+n \cdot \tan \frac{C}{2} \cdot \tan \frac{A}{2}} + \frac{\tan \frac{C}{2}}{m+n \cdot \tan \frac{A}{2} \cdot \tan \frac{B}{2}} \geq \frac{(4R+r)^2}{s(m(4R+r)+3nr)}$;

(c) $\frac{\tan^3 \frac{A}{2}}{m \cdot \cot \frac{B}{2} + n \cdot \cot \frac{C}{2}} + \frac{\tan^3 \frac{B}{2}}{m \cdot \cot \frac{C}{2} + n \cdot \cot \frac{A}{2}} + \frac{\tan^3 \frac{C}{2}}{m \cdot \cot \frac{A}{2} + n \cdot \cot \frac{B}{2}} \geq \frac{(4R+r)r}{(m+n)s^2}$;

(d) $\frac{\tan \frac{A}{2}}{(x+y \cdot \tan \frac{B}{2} \cdot \tan \frac{C}{2})^m} + \frac{\tan \frac{B}{2}}{(x+y \cdot \tan \frac{C}{2} \cdot \tan \frac{A}{2})^m} + \frac{\tan \frac{C}{2}}{(x+y \cdot \tan \frac{A}{2} \cdot \tan \frac{B}{2})^m} \geq \frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$

for any positive real numbers m, n, x, y

Proposed by D.M. Băţineţu – Giurgiu, Neculai Stanciu – Romania

Solution 1 by Kevin Soto Palacios – Huarmey – Peru

Siendo $m, n, x, y > 0$. Probar en un triángulo ABC las siguientes desigualdades

a) $\frac{\tan^3 \frac{A}{2}}{m \tan \frac{B}{2} + n \tan \frac{C}{2}} + \frac{\tan^3 \frac{B}{2}}{m \tan \frac{C}{2} + n \tan \frac{A}{2}} + \frac{\tan^3 \frac{C}{2}}{m \tan \frac{A}{2} + n \tan \frac{B}{2}} \geq \frac{((4R+r)^2 - 2s^2)^2}{(m+n)s^4}$

Tener en cuenta las siguientes identidades en un ΔABC

$$\tan \frac{A}{2} \tan \frac{B}{2} + \tan \frac{B}{2} \tan \frac{C}{2} + \tan \frac{C}{2} \tan \frac{A}{2} = 1, \cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} = \frac{s}{r}$$

$$\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} = \frac{4R+r}{s},$$

$$\cot \frac{A}{2} \cot \frac{B}{2} + \cot \frac{B}{2} \cot \frac{C}{2} + \cot \frac{C}{2} \cot \frac{A}{2} = \prod \cot \frac{A}{2} \left(\sum \tan \frac{A}{2} \right) = \frac{4R+r}{r}$$

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$$\tan^2 \frac{A}{2} + \tan^2 \frac{B}{2} + \tan^2 \frac{C}{2} = \left(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2} \right)^2 - 2 = \frac{(4R + r)^2 - 2s^2}{s^2}$$

Aplicando la desigualdad de Cauchy

$$\begin{aligned} \sum \frac{\tan^4 \frac{A}{2}}{m \tan \frac{A}{2} \tan \frac{B}{2} + n \tan \frac{A}{2} \tan \frac{C}{2}} &= \frac{(\sum \tan^2 \frac{A}{2})^2}{(m+n) (\sum \tan \frac{A}{2} \tan \frac{B}{2})} = \\ &= \frac{\left(\frac{(4R+r)^2 - 2s^2}{s^2} \right)^2}{(m+n) \cdot 1} = \frac{((4R+r)^2 - 2s^2)^2}{(m+n)s^4} \end{aligned}$$

(LQOD)

$$b) \frac{\tan \frac{A}{2}}{m+n \tan \frac{B}{2} \tan \frac{C}{2}} + \frac{\tan \frac{B}{2}}{m+n \tan \frac{C}{2} \tan \frac{A}{2}} + \frac{\tan \frac{C}{2}}{m+n \tan \frac{A}{2} \tan \frac{B}{2}} \geq \frac{(4R+r)^2}{s(m(4R+r)+3nr)}$$

Aplicando la desigualdad de Cauchy

$$\begin{aligned} \sum \frac{\tan^2 \frac{A}{2}}{m \tan \frac{A}{2} + n \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} &\geq \frac{(\sum \tan \frac{A}{2})^2}{m (\sum \tan \frac{A}{2}) + 3 \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2}} = \\ &= \frac{\left(\frac{4R+r}{s} \right)^2}{m \left(\frac{4R+r}{s} \right) + 3n \cdot \frac{r}{s}} = \frac{(4R+r)^2}{s(m(4R+r) + 3nr)} \end{aligned}$$

$$c) \frac{\tan^3 \frac{A}{2}}{m \cot \frac{B}{2} + n \cot \frac{C}{2}} + \frac{\tan^3 \frac{B}{2}}{m \cot \frac{C}{2} + n \cot \frac{A}{2}} + \frac{\tan^3 \frac{C}{2}}{m \cot \frac{A}{2} + n \cot \frac{B}{2}} \geq \frac{(4R+r)r}{(m+n)s^2}$$

Aplicando la desigualdad de Cauchy

$$\begin{aligned} \sum \frac{\tan^2 \frac{A}{2}}{m \cot \frac{A}{2} \cot \frac{B}{2} + n \cot \frac{A}{2} \cot \frac{C}{2}} &\geq \frac{(\sum \tan \frac{A}{2})^2}{m (\sum \cot \frac{A}{2} \cot \frac{B}{2}) + n (\sum \cot \frac{A}{2} \cot \frac{C}{2})} = \\ &= \frac{(\sum \tan \frac{A}{2})^2}{(m+n) (\sum \cot \frac{A}{2} \cot \frac{B}{2})} = \frac{\left(\frac{4R+r}{s} \right)^2}{(m+n) \left(\frac{4R+r}{r} \right)} \\ &\rightarrow \sum \frac{\tan^2 \frac{A}{2}}{m \cot \frac{A}{2} \cot \frac{B}{2} + n \cot \frac{A}{2} \cot \frac{C}{2}} \geq \frac{(4R+r)r}{(m+n)s^2} \end{aligned}$$

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$$d) \frac{\tan^{\frac{A}{2}}}{(x+y \tan^{\frac{B}{2}} \tan^{\frac{C}{2}})^m} + \frac{\tan^{\frac{B}{2}}}{(x+y \tan^{\frac{C}{2}} \tan^{\frac{A}{2}})^m} + \frac{\tan^{\frac{C}{2}}}{(x+y \tan^{\frac{A}{2}} \tan^{\frac{B}{2}})^m} \geq \frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$$

La desigualdad propuesta es equivalente

$$\frac{(\tan \frac{A}{2})^{m+1}}{(x \tan \frac{A}{2} + y \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2})^m} + \frac{(\tan \frac{B}{2})^{m+1}}{(x \tan \frac{B}{2} + y \tan \frac{B}{2} \tan \frac{C}{2} \tan \frac{A}{2})^m} + \frac{(\tan \frac{C}{2})^{m+1}}{(x \tan \frac{C}{2} + y \tan \frac{C}{2} \tan \frac{A}{2} \tan \frac{B}{2})^m} \geq \frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m}$$

Aplicando la desigualdad de Radon

$$\begin{aligned} \sum \frac{(\tan \frac{A}{2})^{m+1}}{(x \tan \frac{A}{2} + y \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2})^m} &\geq \frac{(\sum \tan \frac{A}{2})^{m+1}}{(x(\sum \tan \frac{A}{2}) + 3y \tan \frac{A}{2} \tan \frac{B}{2} \tan \frac{C}{2})^m} = \\ &= \frac{(\frac{4R+r}{s})^{m+1}}{(x(\frac{4R+r}{s}) + \frac{3yr}{s})^m} = \frac{(4R+r)^{m+1}}{s(x(4R+r)+3ry)^m} \end{aligned}$$

Solution 2 by Soumitra Mandal-Chandar Nagore-India

$$\tan \frac{A}{2} = \frac{\Delta}{p(p-a)}, \tan \frac{B}{2} = \frac{\Delta}{p(p-b)} \text{ and } \tan \frac{C}{2} = \frac{\Delta}{p(p-c)}$$

$$\begin{aligned} a) \sum_{cyc} \frac{\tan^{\frac{3A}{2}}}{m \tan^{\frac{B}{2}} + n \tan^{\frac{C}{2}}} &\stackrel{HOLDER}{\geq} \frac{(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2})^2}{3(m+n)} \\ &= \frac{\Delta^2}{(p(p-a)(p-b)(p-c))^2} \cdot \frac{(\sum(p-a)(p-b))^2}{3(m+n)} = \frac{r^2(r+4R)^2}{\Delta^2 \cdot 3(m+n)} = \frac{(r+4R)^2}{3p^2(m+n)} \end{aligned}$$

$$b) \sum_{cyc} \frac{\tan^{\frac{A}{2}}}{m+n \cdot \tan^{\frac{B}{2}} \tan^{\frac{C}{2}}} = \sum_{cyc} \frac{\tan^{\frac{2A}{2}}}{m \tan^{\frac{A}{2}} + n \prod \tan^{\frac{A}{2}}}$$

$$\begin{aligned} \stackrel{BERGSTROM}{\geq} \frac{(\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2})^2}{m \sum \tan^{\frac{A}{2}} + 3n \prod \tan^{\frac{A}{2}}} &= \frac{\Delta^2}{(p(p-a)(p-b)(p-c))^2} \cdot \frac{(\sum(p-a)(p-b))^2}{\frac{\Delta}{p \prod(p-a)} m (\sum(p-a)(p-b)) + 3n \frac{\Delta^3}{p^3 \prod(p-a)}} \\ &= \frac{1}{\Delta^2} \cdot \frac{r^2(r+4R)^2}{\frac{m}{\Delta} r(r+4R) + 3n \frac{\Delta}{p^2}} = \frac{\Delta}{p^2} \cdot \frac{(r+4R)^2}{mr(r+4R) + 3n \frac{\Delta^2}{p^2}} = \frac{(r+4R)^2}{p(m(r+4R) + 3nr)} \text{ (proved)} \end{aligned}$$

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$$\begin{aligned}
 \text{c) } \sum_{cyc} \frac{\tan^3 \frac{A}{2}}{m \cot \frac{B}{2} + n \cot \frac{C}{2}} &= \left(\prod_{cyc} \tan \frac{A}{2} \right) \sum_{cyc} \frac{\tan^2 \frac{A}{2}}{m \tan \frac{C}{2} + n \tan \frac{B}{2}} \stackrel{\text{BERGSTROM}}{\geq} \left(\sum_{cyc} \tan \frac{A}{2} \right) \frac{\tan \frac{A}{2} + \tan \frac{B}{2} + \tan \frac{C}{2}}{m+n} \\
 &= \frac{\Delta^3}{p^3 \prod (p-a)} \cdot \frac{\Delta}{p \prod (p-a)} \cdot \frac{1}{m+n} \left(\sum_{cyc} (p-a)(p-b) \right) \\
 &= \frac{\Delta}{p^2} \cdot \frac{1}{\Delta} \cdot \frac{r(r+4R)}{m+n} = \frac{r(r+4R)}{(m+n)p^2} \quad (\text{proved}) \\
 \text{d) } \sum_{cyc} \frac{\tan \frac{A}{2}}{(x+y \tan \frac{B}{2} \tan \frac{C}{2})^m} &= \sum_{cyc} \frac{\tan^{m+1} \frac{A}{2}}{(x \tan \frac{A}{2} + y \prod \tan \frac{A}{2})^m} \\
 &\stackrel{\text{RADON}}{\geq} \frac{\left(\sum \tan \frac{A}{2} \right)^{m+1}}{\left(x \sum_{cyc} \tan \frac{A}{2} + 2y \prod \tan \frac{A}{2} \right)^m} = \frac{\left(\sum_{cyc} \frac{\Delta}{p(p-a)} \right)^{m+1}}{\left(x \sum_{cyc} \frac{\Delta}{p(p-a)} + 3y \frac{\Delta^3}{p^3(p-a)} \right)^m} \\
 &= \frac{\Delta^{m+1}}{(p \prod (p-a))^{m+1}} \cdot \frac{\left(\sum_{cyc} (p-a)(p-b) \right)^{m+1}}{\left(\frac{\Delta}{p \prod (p-a)} x \sum_{cyc} (p-a)(p-b) + 3y \frac{\Delta}{p^2} \right)^m} \\
 &= \frac{1}{\Delta^{m+1}} \cdot \frac{r^{m+1}(r+4R)^{m+1}}{\left(\frac{x}{\Delta} r(r+4R) + \frac{3y\Delta}{p^2} \right)^m} = \frac{1}{\Delta} \cdot \frac{r^{m+1}(r+4R)^{m+1}}{(xr(r+4R) + 3yr^2)^m} = \\
 &= \frac{(r+4R)^{m+1}}{p(x(r+4R) + 3yr)^m} \\
 &\quad (\text{proved})
 \end{aligned}$$

UP.062. Given the equilateral triangle ABC and let P be any point in its plane.

R, R_a, R_b, R_c denote respectively radii of the circumcircles of the triangles ABC, BPC, CPA, APB and x, y, z are respectively distances from P to the sides BC, CA, AB . Prove that

$$xR_a + yR_b + zR_c \geq \frac{3}{2}R^2.$$

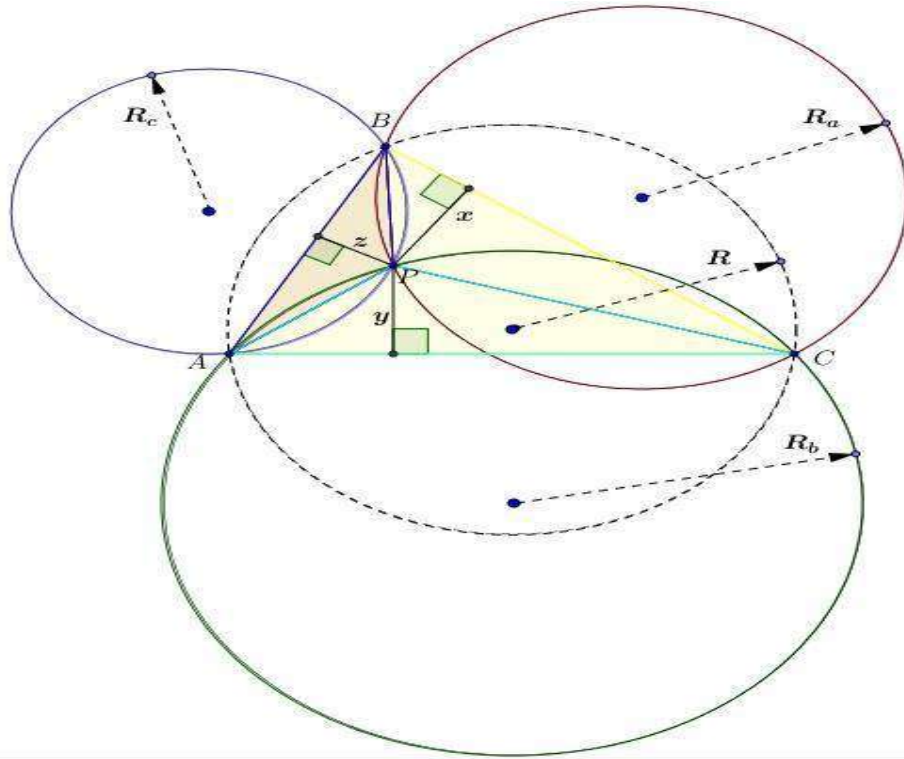
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Solution by Kevin Soto Palacios – Huarmey – Peru



Dado un triángulo equilátero ABC y sea P un punto en este plano. Siendo R, R_a, R_b, R_c , respectivamente los radios de las circunferencias circunscritas ABC, BPC, CPA, APB , además x, y, z son las distancias de P a los lados BC, CA, AB .

Probar que $xR_a + yR_b + zR_c \geq \frac{3}{2}R^2$

En un ΔABC general, se cumple lo siguiente:

$$S_{\Delta BPC} = \frac{BC \cdot x}{2} = \frac{BP \cdot PC \cdot BC}{4R_a} \Leftrightarrow R_a = \frac{PB \cdot PC}{2x}$$

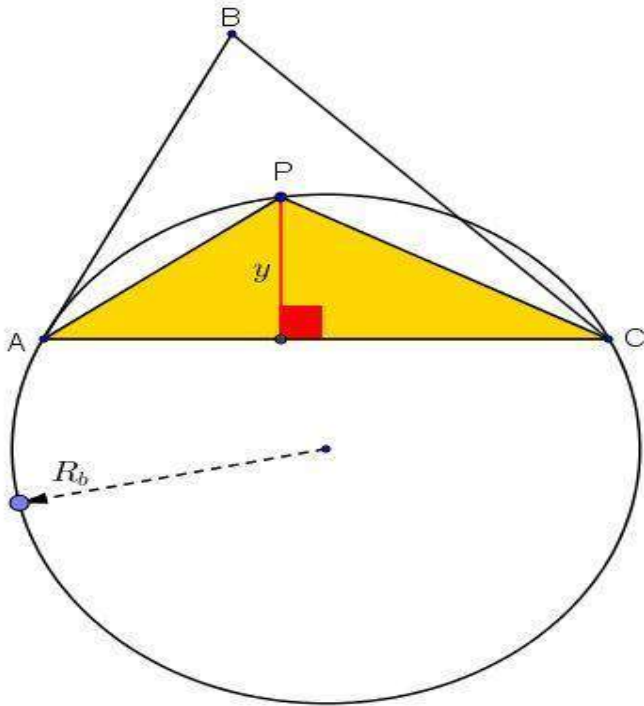
$$S_{\Delta CPA} = \frac{CA \cdot y}{2} = \frac{CP \cdot PA \cdot CA}{4R_b} \Leftrightarrow R_b = \frac{PC \cdot PA}{2y}$$

$$S_{\Delta APB} = \frac{AB \cdot z}{2} = \frac{AP \cdot PB \cdot AB}{4R_c} \Leftrightarrow R_c = \frac{PA \cdot PB}{2z}$$

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$$[APC] = \frac{AP \cdot PC \cdot AC}{4R_b} \dots (I)$$

$$[APC] = \frac{AC \cdot y}{2} \dots (II)$$

De I y II

$$R_b = \frac{AP \cdot PC}{2y}$$

La desigualdad propuesta es equivalente: $PB \cdot PC + PC \cdot PA + PA \cdot PB \geq 3R^2$

Dado que es triángulo equilátero: $BC = CA = AB = l = 2R \sin 60^\circ = \sqrt{3}R$

Es suficiente probar: $PB \cdot PC + BPC \cdot PA + PA \cdot PB \geq l^2$

Para todo, α, β, γ que satisface $\alpha + \beta + \gamma = 360^\circ$ se cumple la siguiente

$$\text{desigualdad} \Rightarrow \cos \alpha + \cos \beta + \cos \gamma \geq -\frac{3}{2}$$

En el triángulo APB, por ley de cosenos tenemos

$$l^2 = PA^2 + PB^2 - 2PA \cdot PB \cdot \cos \alpha \Leftrightarrow \cos \alpha = \frac{PA^2 + PB^2 - l^2}{2PA \cdot PB}$$

En el triángulo BPC, por ley de cosenos tenemos

$$l^2 = PB^2 + PC^2 - 2PB \cdot PC \cdot \cos \beta \Leftrightarrow \cos \beta = \frac{PB^2 + PC^2 - l^2}{2PB \cdot PC}$$

En el triángulo CPA, por ley de cosenos tenemos

$$l^2 = PC^2 + PA^2 - 2PC \cdot PA \cdot \cos \gamma \Leftrightarrow \cos \gamma = \frac{PC^2 + PA^2 - l^2}{2PC \cdot PA}$$

Por lo tanto

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$$\begin{aligned} & \frac{PA^2 + PB^2 - l^2}{2PA \cdot PB} + \frac{PB^2 + PC^2 - l^2}{2PB \cdot PC} + \frac{PC^2 + PA^2 - l^2}{2PC \cdot PA} + \frac{3}{2} \geq 0 \\ \Leftrightarrow & (PA^2 + PB^2 - l^2)PC + (PB^2 + PC^2 - l^2)PA + (PC^2 + PA^2 - l^2)PB + \\ & + 3(PA \cdot PB \cdot PC) \geq 0 \\ \Leftrightarrow & PA \cdot PB(PA + PB) + PB \cdot PC(PB + PC) + PC \cdot PA(PC + PA) + \\ & + 3PA \cdot PB \cdot PC - l^2(PA + PB + PC) \geq 0 \\ \Leftrightarrow & (PA + PB + PC)(PA \cdot PB + PB \cdot PC + PC \cdot PA) - l^2(PA + PB + PC) \geq 0 \\ \Leftrightarrow & (PA + PB + PC)(PA \cdot PB + PB \cdot PC + PC \cdot PA - l^2) \geq 0 \\ & \text{Donde se deduce} \rightarrow PA \cdot PB + PB \cdot PC + PC \cdot PA \geq l^2 \end{aligned}$$

UP.063. If $a, b, c, d > 0$ such that :

$$a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 = 6.$$

then:

$$\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d} \geq 4.$$

Proposed by Nguyen Viet Hung – Hanoi – Vietnam

Solution by Kevin Soto Palacios – Huarmey – Peru

Sean " a, b, c, d " números R^+ de tal manera que:

$$a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2 = 6$$

$$\text{Probar que: } \frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d} \geq 4$$

Elevando al cuadrado la expresión, se tiene lo siguiente:

$$\begin{aligned} \left(\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d}\right)^2 &= \left(\frac{bcd}{a} + \frac{cda}{b}\right)^2 + \left(\frac{dab}{c} + \frac{abc}{d}\right)^2 + 2\left(\frac{bcd}{a} + \frac{cda}{b}\right)\left(\frac{dab}{c} + \frac{abc}{d}\right) \\ \left(\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d}\right)^2 &= \left(\frac{b^2c^2d^2}{a^2} + \frac{c^2d^2a^2}{b^2} + 2c^2d^2\right) + \left(\frac{d^2a^2b^2}{c^2} + \frac{a^2b^2c^2}{d^2} + 2a^2b^2\right) + \\ &+ 2(b^2d^2 + d^2a^2 + b^2c^2 + a^2c^2) \end{aligned}$$

Ordenando la expresión convenientemente:

$$\begin{aligned} \left(\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d}\right)^2 &= \frac{b^2c^2d^2}{a^2} + \frac{c^2d^2a^2}{b^2} + \frac{d^2a^2b^2}{c^2} + \frac{a^2b^2c^2}{d^2} + \\ &+ 2(a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2) \end{aligned}$$

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$$\left(\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d}\right)^2 = \frac{b^2c^2d^2}{a^2} + \frac{c^2d^2a^2}{b^2} + \frac{d^2a^2b^2}{c^2} + \frac{a^2b^2c^2}{d^2} + 12 \quad (A)$$

Desde que: $a, b, c, d > 0$. Por: $MA \geq MG$

$$\frac{1}{a^4} + \frac{1}{b^4} \geq \frac{2}{a^2b^2} \quad (I),$$

$$\frac{1}{b^4} + \frac{1}{c^4} \geq \frac{2}{b^2c^2} \quad (II),$$

$$\frac{1}{c^4} + \frac{1}{d^4} \geq \frac{2}{c^2d^2} \quad (III)$$

$$\frac{1}{d^4} + \frac{1}{a^4} \geq \frac{2}{d^2a^2} \quad (IV)$$

$$\frac{1}{a^4} + \frac{1}{c^4} \geq \frac{2}{a^2c^2} \quad (V)$$

$$\frac{1}{b^4} + \frac{1}{d^4} \geq \frac{2}{b^2d^2} \quad (VI)$$

Sumando: (I) + (II) + (III) + (IV) + (V) + (VI):

$$\Rightarrow \frac{3}{a^4} + \frac{3}{b^4} + \frac{3}{c^4} + \frac{3}{d^4} \geq \frac{2}{a^2b^2} + \frac{2}{b^2c^2} + \frac{2}{c^2d^2} + \frac{2}{d^2a^2} + \frac{2}{a^2c^2} + \frac{2}{b^2d^2}$$

Multiplicando $\times (abcd)^2 \dots$

$$\Rightarrow 3 \left(\frac{b^2c^2d^2}{a^2} + \frac{c^2d^2a^2}{b^2} + \frac{d^2a^2b^2}{c^2} + \frac{a^2b^2c^2}{d^2} \right) \geq 2(c^2d^2 + d^2a^2 + a^2b^2 + b^2c^2 + b^2d^2 + a^2c^2)$$

$$\Rightarrow \frac{b^2c^2d^2}{a^2} + \frac{c^2d^2a^2}{b^2} + \frac{d^2a^2b^2}{c^2} + \frac{a^2b^2c^2}{d^2} \geq \frac{2}{3}(a^2b^2 + b^2c^2 + c^2d^2 + d^2a^2 + a^2c^2 + b^2d^2) =$$

$$= \frac{2}{3}(6) = 4. \text{ Finalmente tenemos en ... (A):}$$

$$\left(\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d}\right)^2 = \frac{b^2c^2d^2}{a^2} + \frac{c^2d^2a^2}{b^2} + \frac{d^2a^2b^2}{c^2} + \frac{a^2b^2c^2}{d^2} + 12 \geq 4 + 12 = 16$$

$$\Rightarrow \frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d} \geq 4 \quad (LQQD)$$

UP.064. Let a, b, c, d be non-negative real numbers such that

$$a + b + c + d = 4.$$

Prove that

$$ab(a + b) + cd(c + d) + 4(a + b)(c + d) \leq$$

$$\leq \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d} + 16$$

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Solution by Kevin Soto Palacios – Huarmey – Peru

Siendo a, b, c, d números reales no negativos de tal manera que

$$a + b + c + d = 4. \text{ Probar que}$$

$$ab(a + b) + cd(c + d) + 4(a + b)(c + d) \leq \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d} + 16$$

Tener en cuenta la siguiente identidad

$$(x + y)^3 = x^3 + y^3 + 3xy(x + y), \text{ donde } x = a + b, y = c + d$$

$$(a + b + c + d)^3 = (a + b)^3 + (c + d)^3 + 3(a + b)(c + d)(a + b + c + d)$$

$$(a + b + c + d)^3 = a^3 + b^3 + 3ab(a + b) + c^3 + d^3 + 3cd(c + d) + 12(a + b)(c + d)$$

La desigualdad propuesta es equivalente

$$\Leftrightarrow 3ab(a + b) + 3ab(a + b) + 12(a + b)(c + d) \leq 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}) + 48$$

$$\Leftrightarrow (a + b + c + d)^3 \leq a^3 + b^3 + c^3 + d^3 + 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}) + 48$$

$$\Leftrightarrow 64 \leq a^3 + b^3 + c^3 + d^3 + 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}) + 48$$

$$\Leftrightarrow 16 \leq a^3 + b^3 + c^3 + d^3 + 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d})$$

Como $a, b, c, d \geq 0$. Aplicando $MA \geq MG$

$$a^3 + \sqrt[3]{a} + \sqrt[3]{a} + \sqrt[3]{a} \geq 4a,$$

$$b^3 + \sqrt[3]{b} + \sqrt[3]{b} + \sqrt[3]{b} \geq 4b,$$

$$c^3 + \sqrt[3]{c} + \sqrt[3]{c} + \sqrt[3]{c} \geq 4c,$$

$$d^3 + \sqrt[3]{d} + \sqrt[3]{d} + \sqrt[3]{d} \geq 4d$$

$$\Rightarrow a^3 + b^3 + c^3 + d^3 + 3(\sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d}) \geq 4(a + b + c + d) = 16$$

UP.065. Let $SABC$ be a tetrahedron and let M be any point inside the triangle ABC . The lines through M parallel with the planes SBC, SCA, SAB intersect SA, SB, SC at X, Y, Z respectively. Prove that

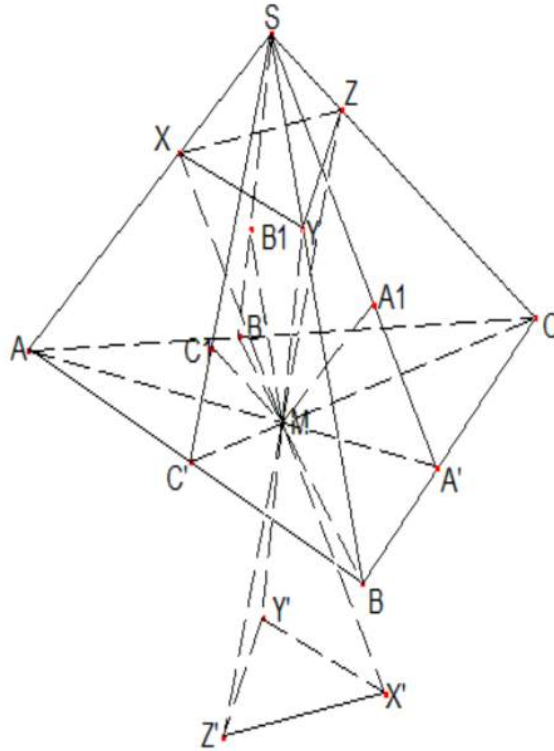
$$Vol(MXYZ) \leq \frac{2}{27} Vol(SABC)$$

Determine position of the point M such that the equality holds.

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Solution by proposer

Let X', Y', Z' be respectively images of X, Y, Z with the central symmetry M . Then

$$\text{Vol}(MXYZ) = \text{Vol}(MX'Y'Z').$$

Let A', B', C' be respectively intersections of the line AM with the sides BC, CA, AB . On

the rays SA', SB', SC' we take points A_1, B_1, C_1 , respectively such that

$SA_1 = XM, SB_1 = YM, SC_1 = ZM$. Then the translation by vector \overrightarrow{MS} transforms the

tetrahedron $MX'Y'Z'$ into the tetrahedron $SA_1B_1C_1$. Therefore

$$\text{Vol}(MX'Y'Z') = \text{Vol}(SA_1B_1C_1)$$

Thus we have $\text{Vol}(MXYZ) = \text{Vol}(SA_1B_1C_1)$. Furthermore

$$\frac{\text{Vol}(SA_1B_1C_1)}{\text{Vol}(SA'B'C')} = \frac{SA_1}{SA'} \cdot \frac{SB_1}{SB'} \cdot \frac{SC_1}{SC'} = \frac{XM}{SA'} \cdot \frac{YM}{SB'} \cdot \frac{ZM}{SC'} = \frac{AM}{AA'} \cdot \frac{BM}{BB'} \cdot \frac{CM}{CC'}$$

From these above we deduce that: $\frac{\text{Vol}(MXYZ)}{\text{Vol}(SA'B'C')} = \left(1 - \frac{MA'}{AA'}\right) \left(1 - \frac{MB'}{BB'}\right) \left(1 - \frac{MC'}{CC'}\right)$

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Using the AM-GM inequality and note that: $\frac{MA'}{AA'} + \frac{MB'}{BB'} + \frac{MC'}{CC'} = 1$, we obtain

$$\left(1 - \frac{MA'}{AA'}\right) \left(1 - \frac{MB'}{BB'}\right) \left(1 - \frac{MC'}{CC'}\right) \leq \left(\frac{3 - \frac{MA'}{AA'} - \frac{MB'}{BB'} - \frac{MC'}{CC'}}{3}\right)^3 = \frac{8}{27}$$

Thus $\frac{\text{Vol}(MXYZ)}{\text{Vol}(SA'B'C')} \leq \frac{8}{27}$. On the other hand: $\frac{\text{Vol}(SA'B'C')}{\text{Vol}(SABC)} = \frac{\text{Area}(A'B'C')}{\text{Area}(ABC)} \leq \frac{1}{4}$

Multiplying up these two inequalities we get: $\frac{\text{Vol}(MXYZ)}{\text{Vol}(SABC)} \leq \frac{2}{27}$, which is the desired result. The equality holds when the point M is the centroid of the triangle ABC . Now we will prove a result that has just been used above as $\frac{\text{Area}(A'B'C')}{\text{Area}(ABC)} \leq \frac{1}{4}$. Indeed, this is

equivalent to

$$\frac{\text{Area}(AB'C')}{\text{Area}(ABC)} + \frac{\text{Area}(BC'A')}{\text{Area}(ABC)} + \frac{\text{Area}(CA'B')}{\text{Area}(ABC)} \geq \frac{3}{4}, \text{ or } \frac{AB'}{AC} \cdot \frac{AC'}{AB} + \frac{BC'}{BA} \cdot \frac{BA'}{BC} + \frac{CA'}{CB} \cdot \frac{CB'}{CA} \geq \frac{3}{4}.$$

Setting $\frac{A'B}{A'C} = x$, $\frac{B'C}{B'A} = y$, $\frac{C'A}{C'B} = z$. By the Ceva's theorem, we have $xyz = 1$. Then our

inequality becomes: $\frac{1}{1+y} \cdot \frac{z}{1+z} + \frac{1}{1+z} \cdot \frac{x}{1+x} + \frac{1}{1+x} \cdot \frac{y}{1+y} \geq \frac{3}{4}$, or $\frac{x(1+y)+y(1+z)+z(1+x)}{(1+x)(1+y)(1+z)} \geq \frac{3}{4}$, or

$$4(x + y + z + xy + yz + zx) \geq 3(1+x)(1+y)(1+z), \text{ or } x + y + z + xy + yz + zx \geq 6$$

The last inequality is true by $x + y + z \geq 3\sqrt[3]{xyz} = 3$, and

$$xy + yz + zx \geq 3\sqrt[3]{x^2y^2z^2} = 3. \text{ The proof is complete and we are done.}$$

UP.066. Evaluate:

$$\sum_{n=1}^{\infty} \sum_{k=1}^n \left(\frac{1}{n^3(2k-1)} \right)$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Mohammed Hijazi-Amman-Jordania

$$\text{Find } \sum_{k=1}^{\infty} \sum_{n=1}^k \frac{1}{k^3(2n-1)}$$

$$= \sum_{k=1}^{\infty} \frac{1}{k^3} \left(\sum_{n=1}^{2k} \frac{1}{n} - \sum_{n=1}^k \frac{1}{2n} \right) = \sum_{k=1}^{\infty} \frac{1}{k^3} \left(H_{2k} - \frac{1}{2} H_k \right)$$

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$$\begin{aligned}
 &= 8 \sum_{k=1}^{\infty} \frac{H_{2k}}{(2k)^3} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k}{k^3} = 8 \sum_{k=1}^{\infty} \frac{H_{2k}}{(2k)^3} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k}{k^3} = 8 \sum_{k=1}^{\infty} \frac{H_k}{k^3} \cdot \frac{(1 + (-1)^k)}{2} - \frac{1}{2} \sum_{k=1}^{\infty} \frac{H_k}{k^3} \\
 &= \frac{7}{2} \sum_{k=1}^{\infty} \frac{H_k}{k^3} + 4 \sum_{k=1}^{\infty} (-1)^k \frac{H_k}{k^3}
 \end{aligned}$$

note that the blue one is well – known from (1995)

$$= \frac{7}{2} \cdot \frac{\pi^4}{72} + 4 \sum_{k=1}^{\infty} (-1)^k \frac{H_k}{k^3} \dots\dots\dots (1)$$

now we want to find the red one:

$$\begin{aligned}
 \sum_{k=1}^{\infty} (-1)^k \frac{H_k}{k^3} &= \sum_{k=1}^{\infty} (-1)^k H_k \frac{1}{2} \int_0^1 \ln^2 x x^{k-1} dx = \frac{1}{2} \int_0^1 \ln^2(x) \left[\sum_{k=1}^{\infty} H_k (-x)^k \right] \frac{dx}{x} \\
 &= \frac{1}{2} \int_0^1 \ln^2(x) \left[\frac{-\ln(1+x)}{1+x} \right] \frac{dx}{x} = -\frac{1}{2} \int_0^1 \frac{\ln^2 x \ln(1+x)}{x(1+x)} dx \\
 &= \frac{1}{2} \int_0^1 \frac{\ln^2 x \ln(1+x)}{(1+x)} dx - \frac{1}{2} \int_0^1 \frac{\ln^2 x \ln(1+x)}{x} dx \\
 &= \frac{1}{6} \int_0^1 \frac{3 \ln^2(x) \ln(1+x) - 3 \ln(x) \ln^2(1+x)}{1+x} dx + \frac{1}{2} \int_0^1 \frac{\ln x \ln^2(1+x)}{1+x} dx + \frac{1}{2} \int_0^{-1} Li_2'(-x) \ln^2(-x) dx \\
 &= \frac{1}{6} \int_0^1 \frac{\ln^3(x)}{1+x} dx - \frac{1}{6} \int_0^1 \frac{\ln^3(1+x)}{1+x} dx - \frac{1}{6} \int_0^1 \ln^3\left(\frac{x}{1+x}\right) \frac{dx}{1+x} - \frac{1}{6} \int_0^1 \frac{\ln^3(1+x)}{x} dx - \\
 &\quad - \int_0^{-1} Li_3'(x) \ln(-x) dx \\
 &= \frac{-1}{6} \int_0^{-1} \frac{\ln^3(-x)}{1-x} dx - \frac{1}{24} \ln^4(2) - \frac{1}{6} \int_0^{\frac{1}{2}} \frac{\ln^3(x)}{1-x} dx + \frac{1}{6} \int_1^2 \frac{\ln^3(x)}{1-x} dx + \int_0^{-1} Li_4(x) dx \dots (B)
 \end{aligned}$$

Now using I.B.P 3 times for $\int \frac{\ln^3(\pm x)}{1-x} dx$ we will get

$$\int \frac{\ln^3(\pm x)}{1-x} dx = -\ln(1-x) \ln^3(\pm x) - 3Li_2(x) \ln^2(\pm x) + 6Li_3(x) \ln(\pm x) - 6Li_4(x) \dots (A)$$

from (A) and (B) we will have

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$$\sum_{k=1}^{\infty} (-1)^k \frac{H_k}{k^3} = \frac{11}{360} \pi^4 - \frac{\pi^2 \ln^2(2)}{12} + \frac{\ln^4(2)}{12} + 2Li_4\left(\frac{1}{2}\right) + \frac{7}{4} \ln(2) \zeta(3) \dots (2)$$

so from (1) and (2) we have:

$$\sum_{k=1}^{\infty} \sum_{n=1}^k \frac{1}{k^3(2n-1)} = \frac{-53\pi^4}{720} - \frac{1}{3} \ln^2(2) \pi^2 + \frac{1}{3} \ln^4(2) + \frac{1}{8} Li_4\left(\frac{1}{2}\right) + 7 \ln(2) \zeta(3)$$

Solution 2 by Khalef Ahmad El Ruhemi-Jarash-Jordania

$$I := \sum_{n=1}^{\infty} \sum_{k=1}^{k=n} \frac{1}{n^3(2k-1)}$$

$$\text{Define } I(x) := \sum_{n=1}^{\infty} \sum_{k=1}^{k=n} \frac{x^{2k-1}}{n^3(2k-1)} \Rightarrow I'(x) = \sum_{n=1}^{\infty} \sum_{k=1}^{k=n} \frac{x^{2k-2}}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^3} \cdot \sum_{k=1}^{k=n} x^{2k-2}$$

$$= \sum_{n=1}^{\infty} \frac{1}{n^3} \cdot \left(\frac{1-x^{2n}}{1-x^2} \right) = \left(\frac{1}{1-x^2} \right) \cdot \sum_{n=1}^{\infty} \left(\frac{1-x^{2n}}{n^3} \right)$$

$$= \left(\frac{1}{1-x^2} \right) \cdot \left[\sum_{n=1}^{\infty} \frac{1}{n^3} - \sum_{n=1}^{\infty} \frac{x^{2n}}{n^3} \right] = \frac{1}{(1-x^2)} [Li_3(1) - Li_3(x^2)]$$

$$\text{Since } I(0) = 0, \text{ then } I = I(1) = \int_0^1 I'(x) dx$$

$$\therefore I = \int_0^1 \frac{(Li_3(1) - Li_3(x^2))}{(1-x^2)} dx \text{ integrate by parts}$$

$$= -\frac{1}{2} (Li_3(1) - Li_3(x^2)) \ln \left(\frac{1-x}{1+x} \right) \Big|_0^1 + \frac{1}{2} \cdot \int_0^1 \left(\ln \left(\frac{1-x}{1+x} \right) \right) x - \frac{Li_2(x^2)}{x^2} x 2x dx$$

$$= \int_0^1 \frac{\ln(1+x)}{x} \cdot Li_2(x^2) dx - \int_0^1 \frac{\ln(1-x)}{x} \cdot Li_2(x^2) dx$$

$$= \int_0^1 \frac{2 \ln(1+x)}{x} \cdot (Li_2(x) + Li_2(-x)) dx - \int_0^1 \frac{2 \ln(1-x)}{x} (Li_2(x) + Li_2(-x)) dx$$

$$= 2 \cdot \int_0^1 Li_2(-x) \frac{\ln(1+x)}{x} dx - 2 \cdot \int_0^1 \frac{\ln(1-x) x}{x} \cdot Li_2(x) dx +$$

$$+ 2 \cdot \int_0^1 \frac{\ln(1+x)}{x} Li_2(x) dx - 2 \cdot \int_0^1 \frac{\ln(1-x)}{x} \cdot Li_2(-x) dx = I$$

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$$\begin{aligned}
 \therefore I &= -\left(\text{Li}_2(-x)\right)^2 \Big|_0^1 + \left(\text{Li}_2(x)\right)^2 \Big|_0^1 \\
 &+ 2 \cdot \int_0^1 \frac{\ln(1+x)}{x} \cdot \text{Li}_2(x) dx - 2 \int_0^1 \frac{\ln(1-x)}{x} \text{Li}_2(x) dx \text{ integrate by parts} \\
 &= \left(\text{Li}_2(1)\right)^2 - \left(\text{Li}_2(-1)\right)^2 + 2 \cdot \int_0^1 \frac{\ln(1+x)}{x} \cdot \text{Li}_2(x) dx \\
 &\quad + 2 \left(\text{Li}_2(x)\text{Li}_2(-x) \Big|_0^1 - \int_0^1 \text{Li}_2(x) \times \frac{\ln(1+x)}{-x} dx \right) \\
 &= \left(\text{Li}_2(1)\right)^2 - \left(\text{Li}_2(-1)\right)^2 + 2\text{Li}_2(1)\text{Li}_2(-1) + 4 \int_0^1 \frac{\ln(1+x)\text{Li}_2(x)}{x} \cdot dx = I \quad (*) \\
 &\quad \text{But } \frac{\ln(1+x)}{x} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot x^{k-1}}{k} \\
 \Rightarrow \int_0^1 \frac{\ln(1+x)\text{Li}_2(x)}{x} \cdot dx &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cdot \left(\int_0^1 x^{k-1} \cdot \text{Li}_2(x) dx \right) \text{ integrate by parts} \\
 \int_0^1 x^{k-1} \cdot \text{Li}_2(x) dx &= \frac{x^k}{k} \text{Li}_2(x) \Big|_0^1 - \int_0^1 \frac{x^k}{k} \times -\frac{\ln(1-x)}{x} dx \\
 &= \frac{\text{Li}_2(1)}{k} + \frac{1}{k} \cdot \int_0^1 x^{k-1} \ln(1-x) dx = \frac{\text{Li}_2(1)}{k} - \frac{H_k}{k^2} \\
 \therefore \int_0^1 \frac{\ln(1+x)\text{Li}_2(x)}{x} \cdot dx &= \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \cdot \left(\frac{\text{Li}_2(1)}{k} - \frac{H_k}{k^2} \right) \\
 \therefore \int_0^1 \frac{\ln(1+x)\text{Li}_2(x)}{x} dx &= \text{Li}_2(1) \cdot \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^2} - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} \cdot H_k}{k^3} \\
 &= \varphi(2)\eta(2) - \sum_{k=1}^{\infty} \frac{(-1)^{k-1} H_k}{k^3} \\
 &= \left(\frac{\pi^2}{6} \right) \left(1 - \frac{1}{2} \right) \left(\frac{\pi^2}{6} \right) + \left(-\frac{11\pi^4}{360} + \frac{1}{12} \ln^4(2) - \frac{\pi^2 \ln^2(2)}{12} + 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4} \ln(2) \varphi(3) \right) \\
 &= \frac{\pi^4}{72} - \frac{11\pi^4}{360} + \frac{1}{12} \ln^4(2) - \frac{\pi^2 \cdot \ln^2(2)}{12} + 2\text{Li}_4\left(\frac{1}{2}\right) + \frac{7}{4} \ln(2) \varphi(3) \\
 &\quad \text{going to } (*), \text{ and using } \text{Li}_2(-1) = -\frac{1}{2} \varphi(2) = -\frac{\pi^2}{12}
 \end{aligned}$$

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$$\begin{aligned}
 I &= \frac{\pi^4}{36} - \frac{\pi^4}{144} - \frac{\pi^4}{36} + \frac{\pi^4}{18} - \frac{11\pi^4}{90} + \frac{1}{3} \ln^4(2) - \frac{\pi^2 \cdot \ln^2(2)}{3} \\
 &\quad + 8Li_4\left(\frac{1}{2}\right) + 7 \ln(2) \varphi(3) \\
 &= -\frac{53\pi^4}{720} - \frac{1}{3} \pi^2 \cdot \ln^2(2) + \frac{1}{3} \ln^4(2) + 7 \ln(2) \varphi(3) + 8Li_4\left(\frac{1}{2}\right) \\
 \therefore \sum_{n=1}^{\infty} \sum_{k=1}^{k=n} \frac{1}{n^3(2k-1)} &= -\frac{53}{720} \pi^4 - \frac{1}{3} \pi^2 \cdot \ln^2(2) + \frac{1}{3} \ln^4(2) \\
 &\quad + 7 \ln(2) \varphi(3) + 9Li_4\left(\frac{1}{2}\right)
 \end{aligned}$$

UP.067. Evaluate:

$$S = \sum_{n=1}^{\infty} H_n \left[\zeta(8) - \frac{1}{1^8} - \frac{1}{2^8} - \frac{1}{3^8} \cdots - \frac{1}{n^8} \right]$$

Proposed by Shivam Sharma – New Delhi – India

Solution 1 by Mohammed Hijazi-Amman-Iordania

Using summation by parts for

$$\sum_{n=1}^N a_n b_n = A_N b_N - \sum_{n=1}^{N-1} (b_{n+1} - b_n)$$

where $A_n = a_1 + a_2 + \cdots + a_n$ in this problem with $a_n = H_n$ and $b_n = \zeta(8) - H_n^{(8)}$

$$S = \sum_{n=1}^{\infty} H_n (\zeta(8) - H_n^{(8)}) = \sum_{n=1}^{\infty} [(n+1)H_{n+1} - (n+1)] \frac{1}{(n+1)^8}$$

$$\text{so } S = -\zeta(7) + \sum_{n=1}^{\infty} \frac{H_n}{n^7} = \frac{\pi^8}{4200} - \zeta(3)\zeta(5) - \zeta(7)$$

Solution 2 by Shivam Sharma – New Delhi – India

Applying Abel's summation, with $a_n = H_n$, $b_n = \left(\zeta(8) - \frac{1}{1^8} - \frac{1}{2^8} - \frac{1}{3^8} - \cdots - \frac{1}{n^8}\right)$

$$S = \lim_{n \rightarrow \infty} H_n \left[\zeta(8) - \frac{1}{1^8} - \frac{1}{2^8} - \frac{1}{3^8} - \cdots - \frac{1}{(n+1)^8} \right] + \sum_{n=1}^{\infty} \left[\frac{H_{n+1}}{(n+1)^7} - \frac{1}{(n+1)^7} \right]$$

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$$\begin{aligned} \Rightarrow 0 + \sum_{n=1}^{\infty} \left[\frac{H_{n+1}}{(n+1)^7} - \frac{1}{(n+1)^7} \right] &\Rightarrow \sum_{n=1}^{\infty} \left[\frac{H_{n+1}}{(n+1)^7} - \frac{1}{(n+1)^7} \right] \Rightarrow \sum_{k=2}^{\infty} \left[\frac{H_k}{k^7} - \frac{1}{k^7} \right] \\ &\Rightarrow \sum_{k=1}^{\infty} \frac{H_k}{k^7} - \sum_{k=1}^{\infty} \frac{1}{k^7} \dots \quad (1) \end{aligned}$$

As we know,

$$\sum_{k=1}^{\infty} \frac{H_k}{k^m} = \frac{1}{2} \left[(m+2)\zeta(m+1) - \sum_{n=1}^{m-2} \{(\zeta(m-n))(\zeta(n+1))\} \right]$$

if $m = 7$, we get

$$\sum_{n=1}^{\infty} \frac{H_k}{k^7} = \frac{\pi^8}{4200} - \zeta(3)\zeta(5)$$

Now put this result in equation (1), we get, $S = \frac{\pi^8}{4200} - \zeta(3)\zeta(5) - \zeta(7)$

UP.068. Prove that if $a, b, c \in \mathbb{R}^*$ then:

$$(abc - ab - bc - ca)^2 \leq 4(1 + a^2)(1 + b^2)(1 + c^2)$$

Proposed by Daniel Sitaru – Romania

Solution by Soumava Chakraborty-Kolkata-India

$$\text{Given inequality} \Leftrightarrow 4 \sum a^2 + 4 + 3 \sum a^2 b^2 + 3a^2 b^2 c^2 + 2abc(\sum ab) \stackrel{(1)}{\geq} 2abc(\sum a)$$

$$\text{Now, } \forall m, n, p \in \mathbb{R}^*, \sum m^2 - \sum mn = \frac{1}{2}[(m-n)^2 + (n-p)^2 + (p-m)^2] \geq 0$$

$$\therefore 2 \sum a^2 b^2 \stackrel{(2)}{\geq} 2abc(\sum a)$$

$$(1), (2) \Rightarrow \text{it suffices to prove: } 4 \sum a^2 + 4 + \sum a^2 b^2 + 3a^2 b^2 c^2 + 2abc(\sum ab) \geq 0 \Leftrightarrow$$

$$\Leftrightarrow 3x^2 + (2 \sum ab)x + \left(\sum a^2 b^2 + 4 \sum a^2 + 4 \right) \geq 0 \quad (x = abc)$$

Let $f(x) = 3x^2 + (2 \sum ab)x + (\sum a^2 b^2 + 4 \sum a^2 + 4)$, which is a quadratic in x as

$$x = abc \neq 0 \quad (\because a, b, c \in \mathbb{R}^*)$$

$$\text{Discriminant } \Delta \text{ of } f(x) = 4(\sum ab)^2 - 4 \cdot 3(\sum a^2 b^2 + 4 \sum a^2 + 4) =$$

$$= 4 \left\{ \sum a^2 b^2 + 2abc(\sum a) - 3 \sum a^2 b^2 - 12(\sum a^2 + 1) \right\} =$$

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$$= 4[2\{abc(\sum a) - \sum a^2 b^2\} - 12(\sum a^2 + 1)] < 0 \text{ (using (2) \& \because } 12(\sum a^2 + 1) > 0)$$

$$\therefore f(x) > 0 \text{ (\because } f(x) \text{ never touches } x\text{-axis as roots of } f(x) = 0 \text{ are imaginary)}$$

$$\therefore f(x) \geq 0 \text{ (Done)}$$

UP.069. Prove that if $n \in \mathbb{N}^*$; $a > 1$ then:

$$(n + a - 1)(a - 1)^{n-1} \leq a^n$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$\text{As } a > 1, 0 < \frac{1}{a} < 1 \Rightarrow 0 < 1 - \frac{1}{a} < 1. \text{ Now, } \frac{1 - (1 - \frac{1}{a})^n}{\frac{1}{a}} = \frac{1 - (1 - \frac{1}{a})^n}{1 - (1 - \frac{1}{a})}$$

$$= 1 + \left(1 - \frac{1}{a}\right) + \left(1 - \frac{1}{a}\right)^2 + \dots + \left(1 - \frac{1}{a}\right)^{n-1}$$

$$\geq \underbrace{\left(1 - \frac{1}{a}\right)^{n-1} + \left(1 - \frac{1}{a}\right)^{n-1} + \dots + \left(1 - \frac{1}{a}\right)^{n-1}}_{\text{"n" times}}$$

$$\Rightarrow 1 - \left(1 - \frac{1}{a}\right)^n \geq \frac{n}{a} \left(1 - \frac{1}{a}\right)^{n-1} \Rightarrow \left(1 - \frac{1}{a}\right)^n + \frac{n}{a} \left(1 - \frac{1}{a}\right)^{n-1} \leq 1$$

$$\Rightarrow \left(1 - \frac{1}{a} + \frac{n}{a}\right) \left(1 - \frac{1}{a}\right)^{n-1} \leq 1 \Rightarrow (a - 1 + n)(a - 1)^{n-1} \leq a^n$$

Solution 2 by SK Rejuan-West Bengal-India

$$n \in \mathbb{N}^*, a > 1,$$

Let us consider two positive terms $(a - 1)$ and $(n + a - 1)$ with the associated weights $(n - 1)$ and 1 respectively. Applying weighted AM \geq weighted GM

$$\frac{(a - 1)(n - 1) + (n + a - 1) \cdot 1}{(n - 1) + 1} \geq \{(a - 1)^{n-1} \cdot (n + a - 1)^1\}^{\frac{1}{n-1+1}}$$

$$\Rightarrow \left(\frac{an - a - n + 1 + n + a - 1}{n}\right) \geq \{(a - 1)^{n-1}(n + a - 1)\}^{\frac{1}{n}}$$

$$\Rightarrow \left(\frac{an}{n}\right)^n \geq (n + a - 1)(a - 1)^{(n-1)} \Rightarrow a^n \geq (n + a - 1)(a - 1)^{n-1}$$

$$\Rightarrow (n + a - 1)(a - 1)^{n-1} \leq a^n$$

[Proved]

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Solution 3 by Nguyen Phuc Tang-Hanoi-Vietnam

If $n = 1$ we have equality hold. If $n > 1$, we have

$$RHS - LHS = [(a - 1) + 1]^n - n(a - 1)^{n-1} - (a - 1)^n = \sum_{k=2}^n C_n^k (a - 1)^{n-k} > 0$$

Solution 4 by Soumitra Mandal-Chandar Nagore-India

Let $f(x) = x^n$ for all $x \in [a - 1, a]$ where $n \in \mathbb{N}^*$

Applying Lagrange's Mean Value Theorem,

$$\frac{a^n - (a-1)^n}{a - (a-1)} = n\xi^{n-1} \text{ where } \xi \in [a - 1, a] \text{ i.e. } a - 1 \leq \xi \leq a$$

$$\Rightarrow a^n - (a - 1)^n \geq n(a - 1)^{n-1} \Rightarrow a^n \geq (a - 1)^n + n(a - 1)^{n-1}$$

$$\Rightarrow a^n \geq (a - 1)^{n-1}(a - 1 + n) \quad (\text{proved})$$

UP.070. If $a, b \in \mathbb{R}; a < b; f, g: [a, b] \rightarrow \mathbb{R}$ are continuous functions such that: $f(a + b - x) = fx, ga + b - xgx = 1, \forall x \in [a, b]$ then:

$$\int_a^b \frac{f(x) \cdot g(x)}{1 + g(x)} dx = \frac{1}{2} \int_a^b f(x) dx$$

Proposed by D. M. Bătinețu – Giurgiu; Neculai Stanciu – Romania

Solution1 by Ravi Prakash-New Delhi-India

Let

$$I = \int_a^b \frac{f(x)g(x)}{1+g(x)} dx \quad (1)$$

$$= \int_a^b \frac{f(a + b - x)g(a + b - x)}{1 + g(a + b - x)} dx = \int_a^b \frac{f(x) \left(\frac{1}{g(x)} \right)}{1 + \frac{1}{g(x)}} dx$$

$$I = \int_a^b \frac{f(x)}{g(x)+1} dx \quad (2)$$

Adding (1) and (2) we get

$$2I = \int_a^b \frac{f(x)(g(x) + 1)}{g(x) + 1} dx = \int_a^b f(x) dx \Rightarrow I = \frac{1}{2} \int_a^b f(x) dx$$

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Solution 2 by SK Rejuan-West Bengal-India

$a, b \in \mathbb{R}$ and $a < b$

$f, g: [a, b] \rightarrow \mathbb{R}$ are continuous functions

$f(a + b - n) = f(n), g(a + b - n)g(n) = 1 \quad \forall n \in [a, b] \Rightarrow g(n) \neq 0, \quad \forall n \in [a, b]$

Let,

$$\begin{aligned}
 I &= \int_a^b \frac{f(n) \cdot g(n)}{1 + g(n)} dn = \int_a^b \frac{f(a + b - n)g(a + b - n)}{1 + g(a + b - n)} dn \\
 &= \int_a^b \frac{f(n)g(a+b-n)}{1+g(a+b-n)} dn \quad [as, f(a + b - x) = f(n)] \\
 &= \int_a^b \frac{\{f(n)g(a + b - n)\} \cdot g(n)}{\{1 + g(a + b - n)\}g(n)} dn = \int_a^b \frac{f(n)\{g(a + b - n) \cdot g(n)\}}{g(n) + g(a + b - n)g(n)} dn \\
 &= \int_a^b \frac{f(n)}{g(n)+1} dn \quad (1) \\
 I &= \int_a^b \frac{f(n)}{1+g(n)} dn \quad [from (1)] \\
 &= \int_a^b \frac{f(n) + f(n)f(n) - f(n)g(n)}{1 + g(n)} dx = \int_a^b \frac{f(n)\{1 + g(n)\} - f(n)g(n)}{1 + g(n)} dn \\
 &= \int_a^b f(n) dn - \int_a^b \frac{f(n)g(n)}{1+g(n)} dn = \int_a^b f(n) dn - I \quad [assem...] \\
 \Rightarrow 2I &= \int_a^b f(n) dn \Rightarrow I = \frac{1}{2} \int_a^b f(n) dn \Rightarrow \int_a^b \frac{f(n)g(n)}{1 + g(n)} dn = \frac{1}{2} \int_a^b f(n) dn \\
 & \quad [proved]
 \end{aligned}$$

Solution 3 by Shivam Sharma-New Delhi-India

As we know, the following lemma,

If $f(x)$ is a continuous functions defined on $[a, b]$; then,

$$\int_a^b f(x) dx = \int_a^b f(a + b - x) dx$$

Using this, we get,

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$$I \Rightarrow \int_a^b \frac{f(a+b-x)g(a+b-x)}{1+g(a+b-x)} dx$$

$$\text{Given} \Rightarrow f(a+b-x) = f(x)$$

$$g(a+b-x) = \frac{1}{g(x)}$$

Using this and putting these values, we get,

$$\Rightarrow \int_a^b \frac{f(x) \cdot \left(\frac{1}{g(x)}\right)}{1 + \left(\frac{1}{g(x)}\right)} dx \Rightarrow \int_a^b \frac{f(x)}{1+g(x)} dx$$

$$\text{Then, } 2I = \int_a^b \frac{f(x)}{1+g(x)} [1+g(x)] dx$$

$$2I = \int_a^b f(x) dx$$

$$\text{Then, } I = \frac{1}{2} \int_a^b f(x) dx \text{ (Proved)}$$

UP.071. Evaluate:

$$\int_0^1 \ln \left[\left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \right] \frac{x dx}{1-x^2}$$

Proposed by Shivam Sharma-New Delhi-India

Solution 1 by Khalef Ruhemi-Jarash-Iordania

$$I := \int_0^1 \ln \left[\left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \right] \cdot \frac{x dx}{1-x^2}$$

$$\text{Let } x = \sin \theta, dx = \cos \theta d\theta$$

$$\therefore I = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \ln \left(\frac{\sin \theta + \cos \theta}{\sin \theta - \cos \theta} \right)^2 \cdot \frac{\sin \theta \cos \theta}{\cos^2 \theta} d\theta = \int_0^{\frac{\pi}{2}} \ln \left(\frac{\tan \theta + 1}{\tan \theta - 1} \right)^2 \cdot \tan(\theta) d\theta$$

$$\text{let } \tan \theta = x; \theta = \tan^{-1}(x); d\theta = \frac{dx}{1+x^2}$$

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$$\therefore I = \int_0^{\infty} \frac{x}{1+x^2} \cdot \ln \left(\frac{1+x}{1-x} \right)^2 \cdot dx = \int_0^{\infty} \frac{2x}{1+x^2} \cdot \ln \left| \frac{1+x}{1-x} \right| \cdot dx, \text{ let } x = \frac{1}{y}, dx = -\frac{dy}{y^2}$$

$$\therefore I = \int_0^{\infty} \frac{\frac{2}{x}}{\left(1 + \frac{1}{x^2}\right) x^2} \ln \left| \frac{1 + \frac{1}{x}}{1 - \frac{1}{x}} \right| \cdot dx = \int_0^{\infty} \frac{2}{x(1+x^2)} \cdot \ln \left| \frac{1+x}{1-x} \right| dx$$

$$= \int_0^{\infty} 2 \ln \left| \frac{1+x}{1-x} \right| \cdot \left(\frac{1}{x} - \frac{x}{1+x^2} \right) \cdot dx$$

$$= \int_0^{\infty} \frac{2 \ln \left| \frac{1+x}{x} \right|}{x} \cdot dx - \int_0^{\infty} \frac{2x}{1+x^2} \ln \left| \frac{1+x}{1-x} \right| dx$$

$$\therefore I = 2 \int_0^{\infty} \frac{\ln \left| \frac{1+x}{1-x} \right| dx}{x} - I \Rightarrow I = \int_0^{\infty} \frac{\ln \left| \frac{1+x}{1-x} \right| dx}{x} \quad (*)$$

integrating by parts

$$I = \lim_{a \rightarrow 0^+} \ln \left| \frac{1+x}{1-x} \right| \cdot \ln x \Big|_a^{\infty} - \int_0^{\infty} (\ln x) \cdot \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx$$

$$= \lim_{a \rightarrow 0^+} \left(\ln \left(\frac{\left(1 + \frac{1}{a}\right)}{\left(1 - \frac{1}{a}\right)} \right) \right) \ln \left(\frac{1}{a} \right) - \ln \left(\left| \frac{1+a}{1-a} \right| \ln(a) \right)$$

$$- \int_0^{\infty} \ln x \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx.$$

$$= \lim_{a \rightarrow 0^+} -2 \ln \left| \frac{1+a}{1-a} \right| \cdot \ln(a) - \int_0^{\infty} \ln x \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx$$

$$\text{But } \lim_{a \rightarrow 0^+} -2 \ln(a) \ln \left| \frac{1+a}{1-a} \right| = -2 \lim_{a \rightarrow 0^+} (\ln(a) \ln(1+a) - \ln(a) \ln(1-a))$$

$$= -2(0 - 0) = 0$$

$$\therefore I = - \int_0^{\infty} \ln(x) \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx$$

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$$\begin{aligned} \therefore -I &= \int_0^1 (\ln x) \cdot \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx + \int_1^\infty \ln(x) \underbrace{\left(\frac{1}{1+x} + \frac{1}{1-x} \right)}_{\text{let } x=\frac{1}{y} \Rightarrow dx = -\frac{dy}{y^2}} dx \\ \therefore -I &= \int_0^1 \frac{\ln x}{1+x} dx + \int_0^1 \frac{\ln x}{1-x} dx + \int_0^1 -\ln(x) \cdot \left(\frac{1}{1+x} - \frac{1}{1-x} \right) \cdot \frac{dx}{x} \\ -I &= \int_0^1 \frac{\ln x}{1+x} dx + \int_0^1 \frac{\ln x}{1-x} dx + \int_0^1 \ln x \left(\frac{1}{1+x} + \frac{1}{1-x} \right) dx \\ -I &= \int_0^1 \frac{\ln x}{1+x} dx + \int_0^1 \frac{\ln x}{1-x} dx + \int_0^1 \frac{\ln x}{1+x} dx + \int_0^1 \frac{\ln x}{1-x} dx \\ \therefore -I &= 2 \cdot \int_0^1 \frac{\ln x}{1+x} dx + 2 \int_0^1 \frac{\ln x}{1-x} dx \quad \text{integrating by parts} \\ &\Rightarrow -\frac{1}{2}I = \ln x \ln(1+x) \Big|_0^1 - \int_0^1 \frac{\ln(1+x)}{x} dx \\ &\quad - \ln x \ln(1-x) \Big|_0^1 + \int_0^1 \frac{\ln(1-x)}{x} dx \\ &\therefore -\frac{1}{2}I = -Li_2(x) \Big|_0^1 + Li_2(-x) \Big|_0^1 \\ &= -Li_2(1) + Li_2(-1) = \varphi(2) - \eta(2) \\ &= -\frac{\pi^2}{6} - \frac{1}{2} \left(\frac{\pi^2}{6} \right) = -\frac{\pi^2}{6} - \frac{\pi^2}{12} = -\frac{3\pi^2}{12} = -\frac{\pi^2}{4} \\ \therefore -\frac{1}{2}I &= -\frac{\pi^2}{4} \Rightarrow I = \frac{\pi^2}{2}, \quad (\text{Note: } \eta(2) = \frac{1}{2}\varphi(2) = \frac{\pi^2}{12}) \\ &\therefore \int_0^1 \ln \left[\left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \right] \cdot \frac{x \cdot dx}{1-x^2} = \frac{\pi^2}{2} \end{aligned}$$

Solution 2 by Mohammed Hijazi-Amman-Iordania

Let $\tanh u = x$. So

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$$I = \int_0^{\infty} \ln \left[\left(\frac{\tanh u + \operatorname{sech} u}{\tanh u - \operatorname{sech} u} \right)^2 \right] \tanh u \, du$$

$$I = 2 \int_0^{\infty} \ln \left[\left(\frac{\sinh u + 1}{\sinh u - 1} \right) \right] \frac{\sinh u}{\cosh u} \, du$$

Let $\sinh u = t$

$$I = 2 \int_0^{\infty} \ln \left| \frac{1+t}{t-1} \right| \frac{t}{1+t^2} \, du \quad \text{let } t = \frac{1}{y}$$

$$2I = 2 \int_0^{\infty} \frac{\ln \left| \frac{1+y}{1-y} \right|}{y} \, dy$$

hence $I = \int_0^1 \frac{\ln \left| \frac{1+y}{1-y} \right|}{y} \, dy + \int_0^{\infty} \frac{\ln \left| \frac{1+y}{1-y} \right|}{y} \, dy$ for the red one let $y = \frac{1}{m}$

$$\text{so } I = \int_0^1 \frac{\ln \left| \frac{1+y}{1-y} \right|}{y} \, dy + \int_0^1 \frac{\ln \left| \frac{1+\frac{1}{m}}{1-\frac{1}{m}} \right|}{\frac{1}{m}} \cdot \frac{1}{m^2} \, dm$$

$$I = \int_0^1 \frac{\ln \left| \frac{1+y}{1-y} \right|}{y} \, dy + \int_0^1 \frac{\ln \left| \frac{m+1}{m-1} \right|}{m} \, dm \quad \text{the integral in the red is same as the blue one using}$$

the power series:

$$I = 2 \int_0^1 2 \sum_{n=0}^{n=\infty} \frac{y^{2n}}{(2n+1)} \, dy = 4 \sum_{n=0}^{n=\infty} \frac{1}{(2n+1)^2} = 4 \frac{3}{4} \xi(2) = \frac{\pi^2}{2}$$

Solution 3 by Atteiah Yahya Ahmed Atiya Al Zahrani-Jeddah-Kingdom Of Saudi Arabia

$$\begin{aligned} I &= \int_0^1 \ln \left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \cdot \frac{x}{1-x^2} \, dx = \\ &= -\frac{1}{2} \cdot \ln(1-x^2) \cdot \ln \left(\frac{x + \sqrt{1-x^2}}{x - \sqrt{1-x^2}} \right)^2 \Bigg|_{x=0}^{x \rightarrow \infty} - 2 \cdot PV \int_0^1 \frac{\ln(1-x^2)}{(2 \cdot x^2 - 1)\sqrt{1-x^2}} \, dx \\ I &= -2 \cdot PV \int_0^1 \frac{\ln(1-x^2)}{(2x^2 - 1)\sqrt{1-x^2}} \, dx \end{aligned}$$

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$$x = \frac{y}{\sqrt{y^2 + 1}} \quad dx = \frac{1}{(y^2 + 1)\sqrt{y^2 + 1}} \quad y = \frac{x}{\sqrt{1 - x^2}} \quad x: (0, 1) \quad y: (0, \infty)$$

$$I = 2 \cdot PV \int_0^{\infty} \frac{\ln(y^2 + 1)}{y^2 - 1} dy = -2 \cdot PV \int_0^1 \int_0^{\infty} \frac{\partial}{\partial \alpha} \frac{\ln(\alpha(y^2 - 1) + 2)}{y^2 - 1} dy d\alpha$$

$$I = \int_0^1 \int_0^{\infty} \frac{2}{\alpha(y^2 - 1) + 2} dy d\alpha = \int_0^1 \frac{1}{\alpha} \int_0^{\infty} \frac{2}{y^2 + \left(\sqrt{\frac{2}{\alpha}} - 1\right)^2} dy d\alpha$$

$$I = \pi \cdot \int_0^1 \frac{1}{\sqrt{\alpha \cdot (2 - \alpha)}} d\alpha = 2 \cdot \pi \int_0^{\frac{1}{\sqrt{2}}} \frac{1}{\sqrt{1 - \beta^2}} d\beta = 2 \cdot \pi \cdot \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) = \frac{\pi^2}{2}$$

UP.072. If $x, y, z > 0$ and $b \geq a > 0$ then:

$$\frac{x}{z} \ln \frac{x^2 + bz}{x^2 + az} + \frac{y}{x} \ln \frac{y^2 + bx}{y^2 + ax} + \frac{z}{y} \ln \frac{z^2 + by}{z^2 + ay} \leq \frac{3}{4} \ln \frac{b}{a} + \frac{b-a}{4} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$$

Proposed by Mihály Bencze – Romania

Solution by proposer

$$\text{For } x, z, t > 0 \text{ we have: } \frac{x}{x^2 + tz} \leq \frac{1}{4} \left(\frac{1}{t} + \frac{1}{z} \right) \Leftrightarrow t(x - z)^2 + z(x - t)^2 \geq 0$$

$$\text{therefore } \int_a^b \frac{x dt}{x^2 + tz} \leq \frac{1}{4} \int_a^b \left(\frac{1}{t} + \frac{1}{z} \right) dt \Rightarrow \frac{x}{z} \ln \frac{x^2 + bz}{x^2 + az} \leq \frac{1}{4} \ln \frac{b}{a} + \frac{b-a}{4z} \Rightarrow$$

$$\Rightarrow \sum_{\text{cyclic}} \frac{x}{z} \ln \frac{x^2 + bz}{x^2 + az} \leq \sum_{\text{cyclic}} \left(\frac{1}{4} \ln \frac{b}{a} + \frac{b-a}{4z} \right) = \frac{3}{4} \ln \frac{b}{a} + \frac{b-a}{4} \left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z} \right)$$

UP.073. Let be $a, b, c \in \mathbb{C}$. Solve the following equation:

$$x^3 - (a + b + c)x^2 + (ab + bc + ca - 1)x + b - abc = 0$$

Proposed by Daniel Sitaru – Romania

Solution 1 by Ravi Prakash-New Delhi-India

$$x^3 - (a + b + c)x + (ab + bc + ca - 1)x + b - abc = 0 \quad (1)$$

$$\Rightarrow x^3 - (a + b + c)x + (ab + bc + ca)x - abc = x - b$$

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$$\Rightarrow (x-a)(x-b)(x-c) = x-b \Rightarrow x-b = 0 \text{ or } (x-a)(x-c) = 1$$

$$\Rightarrow x = b \text{ or } x^2 - (a+c)x + ac - 1 = 0. \text{ Now, } x^2 - (a+c)x + ac - 1 = 0$$

$$D = (a+c)^2 - 4(ac-1) = (a-c)^2 + 4 \therefore x = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4}}{2}$$

Thus, roots of (1) has $b, \frac{1}{2}[(a+c) \pm \sqrt{(a-c)^2 + 4}]$

Solution 2 by SK Rejuan-West Bengal-India

$a, b, c \in \mathbb{C}$. The given equation is,

$$x^3 - (a+b+c)x^2 + (ab+bc+ca-1)x + b - abc = 0$$

$$\Rightarrow x^3 - bx^2 - (a+c)x^2 + b(a+c)x + (ac-1)x + b - abc = 0$$

$$\Rightarrow x^2(x-b) - (a+c)x(x-b) + (ac-1)(x-b) = 0$$

$$\Rightarrow (x-b)\{x^2 - (a+c)x + (ac-1)\} = 0$$

Either $x = b$

$$\text{Or, } x = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4(ac-1)}}{2} = \frac{(a+c) \pm \sqrt{(a+c)^2 - 4ac + 4}}{2} = \frac{(a+c) \pm \sqrt{(a-c)^2 + 4}}{2}$$

$$\therefore \text{Solutions of the given equation } x = b, \frac{(a+c) \pm \sqrt{(a-c)^2 + 4}}{2}$$

UP.074. Let $(a_n)_{n \geq 1}$ be positive real sequences with $\lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*$ and

$f \in \mathbb{R}[x], f(x) \in \mathbb{R}_+^*, \forall x \in \mathbb{R}_+^*; u, v \in \mathbb{R}$ such that $u + v = 1$. Find the following limit

$$\lim_{n \rightarrow \infty} \left((n+1)^u \sqrt[n+1]{(a_{n+1}f(n+1))^v} - n^u \sqrt[n]{(a_n f(n))^v} \right).$$

Proposed by D.M. Bătinețu-Giurgiu, Neculai Stanciu – Romania

Solution by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{na_n} = a \in \mathbb{R}_+^*, u + v = 1 \text{ then}$$

$$\lim_{n \rightarrow \infty} \left((n+1)^u \sqrt[n+1]{(a_{n+1}f(n+1))^v} - n^u \sqrt[n]{(a_n f(n))^v} \right)$$

$$= \lim_{n \rightarrow \infty} \left(\sqrt[n]{\left(\frac{a_n f(n)}{n^n}\right)^v} \cdot \frac{w_n - 1}{\ln w_n} \cdot \ln w_n^n \right) \text{ where } w_n = \left(1 + \frac{1}{n}\right)^u \frac{\sqrt[n+1]{(a_{n+1}f(n+1))^v}}{\sqrt[n]{(a_n f(n))^v}}$$

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$$\text{now, } \lim_{n \rightarrow \infty} \sqrt[n]{\left(\frac{a_n f(n)}{n^n}\right)^v} \stackrel{D'ALEMBERT}{=} \lim_{n \rightarrow \infty} \left(\frac{a_{n+1}}{n a_n} \cdot \frac{f(n+1)}{f(n)} \cdot \frac{n}{n+1} \right)^v = \left(\frac{a}{e}\right)^v$$

$$\lim_{n \rightarrow \infty} w_n = \lim_{n \rightarrow \infty} \frac{n+1}{n} \cdot \frac{\sqrt[n+1]{\left(\frac{a_{n+1} f(n+1)}{(n+1)^{n+1}}\right)^v}}{\sqrt[n]{\left(\frac{a_n f(n)}{n^n}\right)^v}} = 1, \text{ so, } \lim_{n \rightarrow \infty} \frac{w_n - 1}{\ln w_n} = 1$$

$$\lim_{n \rightarrow \infty} w_n^n = \lim_{n \rightarrow \infty} \left(1 + \frac{1}{n}\right)^{nu} \cdot \left(\frac{a_{n+1}}{n a_n} \cdot \frac{f(n+1)}{f(n)} \cdot \frac{1}{\sqrt[n+1]{\left(\frac{a_{n+1} f(n+1)}{(n+1)^{n+1}}\right)^v}} \cdot \frac{n}{n+1} \right)^v = e$$

$$\begin{aligned} \therefore \lim_{n \rightarrow \infty} \left((n+1)^{u^{n+1}} \sqrt[n+1]{(a_{n+1} f(n+1))^v} - n^u \sqrt[n]{(a_n f(n))^v} \right) \\ = \left(\frac{a}{e}\right)^v \cdot 1 \cdot \ln e = \left(\frac{a}{e}\right)^v \text{ (Proved)} \end{aligned}$$

UP.075. Let $(a_n)_{n \geq 1}, (b_n)_{n \geq 1}$, be real positive sequences with

$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a \in \mathbb{R}^*$. If $\lim_{n \rightarrow \infty} (n(a_n - a)) = b \in \mathbb{R}$ and

$\lim_{n \rightarrow \infty} (n(b_n - a)) = c \in \mathbb{R}$, evaluate

$$\lim_{n \rightarrow \infty} \left(a_{n+1}^{n+1} \sqrt[n+1]{(n+1)!} - b_n^n \sqrt[n]{n!} \right).$$

Proposed by D.M. Băținețu-Giurgiu, Neculai Stanciu – Romania

Solution 1 by Soumitra Mandal-Chandar Nagore-India

$$\lim_{n \rightarrow \infty} \frac{\sqrt[n]{n!}}{n} = \lim_{n \rightarrow \infty} \sqrt[n]{\frac{n!}{n^n}} \stackrel{D'ALEMBERT}{=} \lim_{n \rightarrow \infty} \left(\frac{(n+1)!}{(n+1)^{n+1}} \cdot \frac{n^n}{n!} \right) = \lim_{n \rightarrow \infty} \left(\frac{n}{n+1} \cdot \frac{1}{\left(1 + \frac{1}{n}\right)^n} \right) = \frac{1}{e}$$

$$\text{we know, } \lim_{n \rightarrow \infty} \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) = \frac{1}{e}$$

$$\lim_{n \rightarrow \infty} \left(a_{n+1}^{n+1} \sqrt[n+1]{(n+1)!} - b_n^n \sqrt[n]{n!} \right)$$

$$= \lim_{n \rightarrow \infty} \left((a_{n+1} - a)^{n+1} \sqrt[n+1]{(n+1)!} - (b_n - a)^n \sqrt[n]{n!} + a \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right) \right)$$

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$$= \lim_{n \rightarrow \infty} \left(n(a_{n+1} - a_n) \frac{\sqrt[n+1]{(n+1)!}}{n+1} \left(1 + \frac{1}{n}\right) - n(b_n - a) \frac{\sqrt[n]{n!}}{n} + \left(\sqrt[n+1]{(n+1)!} - \sqrt[n]{n!}\right) \right)$$

$$= \frac{a+b-c}{e} \text{ (Proved)}$$

Solution 2 by Shivam Sharma-New Delhi-India

Applying Stolz – Cesaro theorem, we get, $\frac{a_{n+1}-a_n}{n+1-n} = a_{n+1} - a_n$. Similarly, $b_n = b_n - a$

&, $\frac{(a_{n+1})^{\frac{1}{n+1}} - (a_n)^{\frac{1}{n}}}{n+1-n} = (a_{n+1})^{\frac{1}{n+1}} - (a_n)^{\frac{1}{n}}$. Using these all, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\frac{[n(a_{n+1} - a_n)]}{n+1} \cdot \frac{(n+1)}{n} \sqrt[n+1]{(n+1)!} - \frac{[n(b_n - a)]}{n} \sqrt[n]{n!} + \sqrt[n+1]{(n+1)!} - \sqrt[n]{n!} \right]$$

As we know, the Stirling's formula, $n! = \left(\frac{n}{e}\right)^n \sqrt{2\pi n}$. Using this, we get,

$$\Rightarrow \lim_{n \rightarrow \infty} \left[\frac{[n(a_{n+1} - a_n)]}{n+1} \cdot \frac{(n+1)}{n} \left(\left(\frac{n+1}{e}\right)^{n+1} \sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}} - \frac{[n(b_n - a)]}{n} \left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{n}} + \left(\left(\frac{n+1}{e}\right)^{n+1} \sqrt{2\pi(n+1)}\right)^{\frac{1}{n+1}} - \left(\left(\frac{n}{e}\right)^n \sqrt{2\pi n}\right)^{\frac{1}{n}} \right]$$

As we can see many terms are cancelling, so now, applying Ratoin test,

$\frac{n+1}{n} \rightarrow 1$, as $n \rightarrow \infty$ so, our limit equals. As we use the given data,

$$\lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = a \quad (I)$$

&

$$\lim_{n \rightarrow \infty} (n(a_n - a)) = b \quad (II)$$

&

$$\lim_{n \rightarrow \infty} (n(b_n - a)) = c \quad (III)$$

Using (I), (II), (III), we get,

$$L = \frac{1}{e} [a + b - c]$$

(Answer)