*Number 5* *Summer 2017*

# R M M

## ROMANIAN MATHEMATICAL MAGAZINE

Founding Editor
DANIEL SITARU

*Available online*
*www.ssmrmh.ro*

ISSN-L 2501-0099

# PROBLEMS FOR JUNIORS

**JP.061.** Let $a, b, c$ be positive real numbers such that $a+b+c=3$. Prove that
$$\frac{1}{a^3+b^3+c^3}+\frac{8}{ab+bc+ca}\geq 3$$
*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**JP.062.** Prove that if $a, b, c, d \in [1, \infty)$ then:
$$3a+3b+2c+d\leq 6+ab(1+c+cd)$$
*Proposed by Daniel Sitaru - Romania*

**JP.063.** Let $x, y, z$ be non-negative real numbers satisfying $\sqrt[3]{x}+\sqrt[3]{y}+\sqrt[3]{z}=3$. Find the minimum possible value of
$$\sqrt[3]{x+2y+5z}+\sqrt[3]{y+2z+5x}+\sqrt[3]{z+2x+5y}$$
*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**JP.064.** Let $a, b, c$ be non-negative real numbers. Prove that:
$$\sqrt[3]{1+a^3}+\sqrt[3]{1+b^3}+\sqrt[3]{1+c^3}\geq\frac{\sqrt{a+b}+\sqrt{b+c}+\sqrt{c+a}}{\sqrt[6]{2}}$$
*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**JP.065.** Let $ABC$ be an equilateral triangle inscribed in the circle $(O)$ whose radius $R$. Prove that for an arbitrary point $P$ lies on $(O)$,
$$6\sqrt{2}<\frac{PA^3+PB^3+PC^3}{R^3}<3\sqrt[4]{216}$$
*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**JP.066.** Prove that in any triangle
$$\sum\frac{m_a^2}{h_bh_c}\geq n\cdot\frac{R}{r}+(3-2n)\cdot\frac{s}{3r\sqrt{3}}, n\leq\frac{3}{2}$$
*Proposed by Marin Chirciu - Romania*

**JP.067. Prove that in any triangle**

$$n \cdot \frac{s^2+r^2}{Rr} + \sqrt[k]{\frac{2r}{R}} \geq 14n+1, n \geq \frac{1}{2}, k \in N, k \geq 2$$

*Proposed by Marin Chirciu - Romania*

**JP.068. Let $a, b$ and $c$ be the side lenghts of a triangle $ABC$, with circumradius $R$ and inradius $r$. Prove that**

$$\frac{\tan\frac{A}{2}+\tan\frac{B}{2}}{a+b} + \frac{\tan\frac{B}{2}+\tan\frac{C}{2}}{b+c} + \frac{\tan\frac{C}{2}+\tan\frac{A}{2}}{c+a} \leq \frac{1}{r} - \frac{1}{R}$$

*Proposed by George Apostolopoulos - Messolonghi - Greece*

**JP.069. Let $a, b$ be positive real numbers such that $a^2+ab+b^2 = 9$. Find the maximal value of expression:**

$$(a+b)^6 + (ab)^5 + 2(ab)^3 + (ab)^2 - 17$$

*Proposed by George Apostolopoulos - Messolonghi - Greece*

**JP.070. Let $a, b$ be positive real numbers with $a^2+ab+b^2 = k^2, k > 0$. Prove that**

$$\sqrt{a+b} + \sqrt[4]{ab} \leq \frac{\sqrt{2}+1}{\sqrt[4]{3}} \cdot \sqrt{k}.$$

*Proposed by George Apostolopoulos - Messolonghi - Greece*

**JP.071. Let $ABC$ be a triangle with circumradius $R$ and inradius $r$, and let $w_a, w_b, w_c$ be the lengths of the internal bisectors of the angle opposite of the sides of lenghts $a, b, c$ respectively. Prove that**

$$\left(\frac{w_a}{a}\right)^2 \cdot \tan\frac{A}{2} + \left(\frac{w_a}{b}\right)^2 \cdot \tan\frac{B}{2} + \left(\frac{w_c}{c}\right)^2 \cdot \tan\frac{C}{2} \leq \frac{3\sqrt{3}}{8} \cdot \frac{R}{r}$$

*Proposed by George Apostolopoulos - Messolonghi - Greece*

**JP.072. Let $a, b$ and $c$ denote, as usual, the lengths of the sides $BC, CA$, and $AB$, respectively, in $\Delta ABC$. Let $R$ be the circumradius, $r$ the inradius of $\Delta ABC$, and $r_a, r_b$ and $r_c$ the exradii to $A, B$ and $C$, respectively. Prove that**

**(a)** $\frac{r_a}{a^3} + \frac{r_b}{b^3} + \frac{r_c}{c^3} \leq \frac{\sqrt{3}}{8r^2}$,

**(b)** $\frac{3R}{2r} \geq \sqrt{\frac{r_a}{r_b} + \frac{r_b}{r_c} + \frac{r_c}{r_a} + 6}$

*Proposed by George Apostolopoulos - Messolonghi - Greece*

**JP.073. If $a, b, c > 0; n \geq 1$ then:**

$$\frac{3n(a^4+b^4+c^4)}{(a^2+b^2+c^2)^2} + \frac{ab+bc+ca}{a^2+b^2+c^2} \geq n+1$$

*Proposed by Marin Chirciu - Romania*

**JP.074. If $a, b, c, n > 0; n(ab+bc+ca)+2abc = n^3$ then:**

$$\frac{1}{a+b+2n} + \frac{1}{b+c+2n} + \frac{1}{c+a+2n} \leq \frac{1}{n}$$

*Proposed by Marin Chirciu - Romania*

**JP.075. Let $R$ and $r$ be the circumradius and the inradius of a triangle $ABC$ respectively. Prove that**

$$\csc A + \csc B + \csc C \geq 3\sqrt{3}\frac{R}{R+r}$$

*Proposed by Martin Lukarevski – Stip – Macedonia*

# PROBLEMS FOR SENIORS

**SP.061. Let $x_1, x_2, \dots, x_n$ be non-negative real numbers satisfying**

$$\frac{x_1}{1+x_1} + \frac{2x_2}{1+x_2} + \dots + \frac{nx_n}{1+x_n} = 1.$$

**Find the maximum possible value of**

$$P = x_1 x_2^2 \dots x_n^n.$$

*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**SP.062. If $a, b, c \in \mathbb{C}$ then:**

$$|a^3 + b^3 + c^3 - 3abc| \le |a+b+c|(|a| + |b| + |c|)^2$$

*Proposed by Mihály Bencze - Romania*

**SP.063. Prove that for any triangle $ABC$,**

$$\cos\frac{A}{2}\cos\frac{B}{2} + \cos\frac{B}{2}\cos\frac{C}{2} + \cos\frac{C}{2}\cos\frac{A}{2} \ge$$
$$\ge \sqrt{3}\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2} + \frac{1}{2}\left(\cos^2\frac{A}{2} + \cos^2\frac{B}{2} + \cos^2\frac{C}{2}\right)$$

*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**SP.064. If $x, y, z \in (0,1)$ then:**

$$\left(\frac{yz}{1-x^2}\right)^{2n} + \left(\frac{zx}{1-y^2}\right)^{2n} + \left(\frac{xy}{1-z^2}\right)^{2n} \ge \frac{3^{3n+1}}{4^n}(xyz)^{2n}$$

**for all $n \in \mathbb{N}$.**

*Proposed by Mihály Bencze - Romania*

**SP.065. If $a, b, c > 0$ and $n \in \mathbb{N}^*$ then:**

$$2^n(a^n + b^n + c^n) \ge (a+b)^{n-1}(a+c) + (b+c)^{n-1}(b+a) + (c+a)^{n-1}(c+b)$$

*Proposed by Mihály Bencze - Romania*

**SP.066. Let $t \in \mathbb{R}^+$ and $f : \mathbb{R}_+^* \to \mathbb{R}_+^*$ such that**

$$\lim_{x\to\infty} \frac{f(x+1)}{x^t f(x)} \in \mathbb{R}_+$$

**and**

$$\lim_{x\to\infty} \left(f(x)\right)^{\frac{1}{x}} \cdot \frac{1}{x^t} \in \mathbb{R}_+$$

**Prove that**

$$\lim_{x\to\infty} \left(f(x)\right)^{\frac{1}{x}} \cdot \frac{1}{x^t} = \frac{1}{e^t} \cdot \lim_{x\to\infty} \frac{f(x+1)}{x^t f(x)}$$

*Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania*

**SP.067. If $x, y, z > 0$ then:**

$$\frac{1}{(x^2+yz)(3x^2+2y^2+z^2)} + \frac{1}{(y^2+zx)(3y^2+2z^2+x^2)} + \frac{1}{(z^2+xy)(3z^2+2x^2+y^2)} \le \frac{x^2+y^2+z^2+xy+yz+zx}{24x^2y^2z^2}$$

*Proposed by Mihály Bencze - Romania*

**SP.068. Find:**

$$\Omega = \lim_{n\to\infty} \left( \sqrt[mn+m]{(2n+1)!!} - \sqrt[mn]{(2n-1)!!} \right) \cdot n^{\frac{m-1}{m}}$$

*Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania*

**SP.069. If $x, y, z > 0$ and $b \ge a > 0$ then:**

$$\ln \frac{b^2+xy}{a^2+xy} + \ln \frac{b^2+yz}{a^2+yz} + \ln \frac{b^2+zx}{a^2+zx} \le (b-a)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$$

*Proposed by Mihály Bencze - Romania*

**SP.070. Prove that if $a, b, c \in \mathbb{R}$ then:**

$$(2-a-b-c+abc)^2 \le (a^2+2)(b^2+2)(c^2+2)$$

*Proposed by Daniel Sitaru - Romania*

**SP.071. Prove that if $a, b, c \in \mathbb{R}$ then:**

$$(3abc - a^3 - b^3 - c^3)^2 \le (a^2+b^2+c^2)^3$$

*Proposed by Daniel Sitaru - Romania*

**SP.072. If $a, b, c > 0; n \in \mathbb{N}^*$ then:**

$$\left(\frac{2na}{b+(2n-1)c}\right)^{\frac{2}{3}} + \left(\frac{2nb}{c+(2n-1)a}\right)^{\frac{2}{3}} + \left(\frac{2nc}{a+(2n-1)b}\right)^{\frac{2}{3}} \ge 3$$

*Proposed by Marin Chirciu - Romania*

**SP.073. If $x, y, z > 0$ then:**

$$\log\left(1+\frac{1}{x}\right)+\log\left(1+\frac{1}{y}\right)+\log\left(1+\frac{1}{z}\right)\geq 3\log\left(1+\frac{3}{x+y+z}\right)$$

*Proposed by Marin Chirciu - Romania*

**SP.074. Let $x, y, z$ be positive real numbers such that: $x+y+z=3$. Find the minimum of expression:**

$$P=\frac{x^4}{y^4\cdot\sqrt[3]{4z(x^5+1)}}+\frac{y^4}{z^4\cdot\sqrt[3]{4x(y^5+1)}}+\frac{z^4}{x^4\cdot\sqrt[3]{4y(z^5+1)}}$$

*Proposed by Hoang Le Nhat Tung - Hanoi - Vietnam*

**SP.075. Let $x, y, z$ be positive real numbers such that: $xyz=1$. Find the minimum of expression:**

$$P=2(x+y+z)+\frac{x}{y^3+z^3+1}+\frac{y}{z^3+x^3+1}+\frac{z}{x^3+y^3+1}$$

*Proposed by Hoang Le Nhat Tung – Hanoi – Vietnam*

# UNDERGRADUATE PROBLEMS

**UP.061. Prove that in all triangle $ABC$ with usual notations holds the following inequalities:**

(a) $$\frac{\tan^3\frac{A}{2}}{m\cdot\tan\frac{B}{2}+n\cdot\tan\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\cdot\tan\frac{C}{2}+n\cdot\tan\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\cdot\tan\frac{A}{2}+n\cdot\tan\frac{B}{2}}\geq\frac{\left((4R+r)^2-2s^2\right)^2}{(m+n)s^4};$$

(b) $$\frac{\tan\frac{A}{2}}{m+n\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2}}+\frac{\tan\frac{B}{2}}{m+n\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2}}+\frac{\tan\frac{C}{2}}{m+n\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2}}\geq\frac{(4R+r)^2}{s\left(m(4R+r)+3nr\right)};$$

(c) $$\frac{\tan^3\frac{A}{2}}{m\cdot\cot\frac{B}{2}+n\cdot\cot\frac{C}{2}}+\frac{\tan^3\frac{B}{2}}{m\cdot\cot\frac{C}{2}+n\cdot\cot\frac{A}{2}}+\frac{\tan^3\frac{C}{2}}{m\cdot\cot\frac{A}{2}+n\cdot\cot\frac{B}{2}}\geq\frac{(4R+r)r}{(m+n)s^2};$$

(d) $$\frac{\tan\frac{A}{2}}{(x+y\cdot\tan\frac{B}{2}\cdot\tan\frac{C}{2})^m}+\frac{\tan\frac{B}{2}}{(x+y\cdot\tan\frac{C}{2}\cdot\tan\frac{A}{2})^m}+\frac{\tan\frac{C}{2}}{(x+y\cdot\tan\frac{A}{2}\cdot\tan\frac{B}{2})^m}\geq\frac{(4R+r)^{m+1}}{s\left(x(4R+r)+3ry\right)^m}$$

**for any positive real numbers $m, n, x, y$**

*Proposed by D.M Bătineţu-Giurgiu, Neculai Stanciu - Romania*

**UP.062. Given the equilateral triangle $ABC$ and let $P$ be any point in its plane. $R, R_a, R_b, R_c$ denote respectively radii of the circumcircles of the triangles $ABC, BPC, CPA, APB$ and $x, y, z$ are respectively distances from $P$ to the sides $BC, CA, AB$. Prove that**

$$xR_a+yR_b+zR_c\geq\frac{3}{2}R^2.$$

*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**UP.063. Let $a, b, c, d$ be real numbers such that**

$$a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2 = 6.$$

**Prove that**

$$\frac{bcd}{a} + \frac{cda}{b} + \frac{dab}{c} + \frac{abc}{d} \geq 4.$$

*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**UP.064. Let $a, b, c, d$ be non-negative real numbers such that**

$$a + b + c + d = 4.$$

**Prove that**

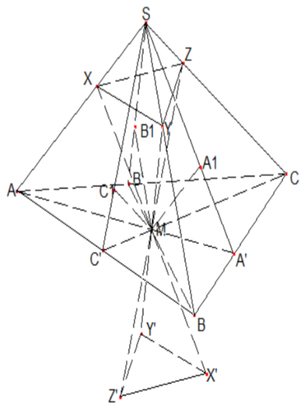
$$ab(a+b) + cd(c+d) + 4(a+b)(c+d) \leq \sqrt[3]{a} + \sqrt[3]{b} + \sqrt[3]{c} + \sqrt[3]{d} + 16$$

*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**UP.065. Let $SABC$ be a tetrahedron and let $M$ be any point inside the triangle $ABC$. The lines through $M$ parallel with the planes $SBC, SCA, SAB$ intersect $SA, SB, SC$ at $X, Y, Z$, respectively. Prove that**

$$\text{Vol}(MXYZ) \leq \frac{2}{27}\text{Vol}(SABC).$$

**Determine position of the point $M$ such that the equality holds.**




*Proposed by Nguyen Viet Hung - Hanoi - Vietnam*

**UP.066. Evaluate:**

$$\sum_{n=1}^{\infty}\sum_{k=1}^{n}\left(\frac{1}{n^3(2k-1)}\right)$$

*Proposed by Shivam Sharma - New Delhi - India*

**UP.067. Evaluate:**

$$S=\sum_{n=1}^{\infty}H_n\left[\zeta(8)-\frac{1}{1^8}-\frac{1}{2^8}-\frac{1}{3^8}\cdot\dots\cdot\frac{1}{n^8}\right]$$

*Proposed by Shivam Sharma - New Delhi - India*

**UP.068. Prove that if $a,b,c,d\in\mathbb{R}^*$ then:**

$$(abc-ab-bc-ac)^2\le 4(1+a^2)(1+b^2)(1+c^2)$$

*Proposed by Daniel Sitaru - Romania*

**UP.069. Prove that if $n\in\mathbb{N}^*;a>1$ then:**

$$(n+a-1)(a-1)^{n-1}\le a^n$$

*Proposed by Daniel Sitaru - Romania*

**UP.070. If $a,b\in\mathbb{R};a<b;f,g:[a,b]\to\mathbb{R}$ are continuous functions such that:**

$$f(a+b-x)=f(x),g(a+b-x)g(x)=1;(\forall)x\in[a,b]$$

**then:**

$$\int_a^b\frac{f(x)\cdot g(x)}{1+g(x)}dx=\frac{1}{2}\int_a^b f(x)dx$$

*Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania*

**UP.071. Evaluate:**

$$\int_0^1\ln\left[\left(\frac{x+\sqrt{1-x^2}}{x-\sqrt{1-x^2}}\right)^2\right]\frac{xdx}{1-x^2}$$

*Proposed by Shivam Sharma - New Delhi - India*

**UP.072. If $x,y,z>0$ and $b\ge a>0$ then:**

$$\frac{x}{z}\ln\frac{x^2+bz}{x^2+az}+\frac{y}{x}\ln\frac{y^2+bx}{y^2+ax}+\frac{z}{y}\ln\frac{z^2+by}{z^2+ay}\le$$

$$\le\frac{3}{4}\ln\frac{b}{a}+\frac{b-a}{4}\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)$$

*Proposed by Mihály Bencze - Romania*

**UP.073. Let be $a,b,c\in\mathbb{C}$. Solve the following equation:**

$$x^3-(a+b+c)x^2+(ab+bc+ca-1)x+b-abc=0$$

*Proposed by Daniel Sitaru - Romania*

**UP.074. Let $(a_n)_{n\geq 1}$ be positive real sequences with**

$$\lim_{n\to\infty}\frac{a_{n+1}}{na_n}=a\in R_+^* \text{ and } f\in R[x], f(x)\in R_+^*, \forall x\in R_+^*; u,v\in R$$

**such that $u+v=1$. Find the following limit**

$$\lim_{n\to\infty}\Big((n+1)^u\sqrt[n+1]{(a_{n+1}f(n+1))^v}-n^u\sqrt[n]{(a_nf(n)^v}\Big).$$

*Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania*

**UP.075. Let $(a_n)_{n\geq 1},(b_n)_{n\geq 1}$, be real positive sequences with**

$$\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n=a\in R_+^*. \text{ If}$$

$$\lim_{n\to\infty}\Big(n(a_n-a)\Big)=b\in R \text{ and } \lim_{n\to\infty}\Big(n(b_n-a)\Big)=c\in R,$$

$$\text{evaluate } \lim_{n\to\infty}\Big(a_{n+1}\sqrt[n+1]{(n+1)!}-b_n\sqrt[n]{n!}\Big)$$

*Proposed by D.M. Bătineţu - Giurgiu; Neculai Stanciu - Romania*

Mathematics Department, "Theodor Costescu" National Economic, College Drobeta Turnu - Severin, MEHEDINTI, ROMANIA